Research Article
A Subclass of Harmonic Functions Related to a Convolution Operator

Saqib Hussain, 1 Akhter Rasheed, 1 and Maslina Darus 2

1 Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad Campus, Abbottabad, Pakistan
2 School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia

Correspondence should be addressed to Saqib Hussain; saqib_math@yahoo.com

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We introduce a new subclass of harmonic functions by using a certain linear operator. For this class we derive coefficient bounds, extreme points, and inclusion results and also show that this class is closed under an integral operator.

1. Introduction

Harmonic functions have long been used in the representation of minimal surface; for example, Heinz [1] in 1952 used such mappings in the study of the Gaussian curvature of nonparametric minimal surface over the unit disc. Such mappings have vast application in the field of engineering, physics, electronics, medicine, operations research, aerodynamics, and other branches of applied mathematics (see [2]).

\[ f(z) = u(x, y) + iv(x, y) \]

is said to be complex valued harmonic function if both \( u \) and \( v \) are continuous and real harmonic; that is, \( u_{xx} + u_{yy} = 0 \) and \( v_{xx} + v_{yy} = 0 \). In simply connected domain [3], we can write

\[ f(\zeta) = h(\zeta) + g(\zeta) \]

where \( h \) and \( g \) are analytic in \( E = \{ z : |z| < 1 \} \). We call \( h \) the analytic part and \( g \) the coanalytic part of \( f \). The necessary and sufficient condition for a function \( f(z) = h(z) + \overline{g(z)} \) to be univalent and sense preserving in \( E \) is \( |h'(z)| > |g'(z)| \) (see [3]). A function \( f(z) = h(z) + \overline{g(z)} \) is in class \( \mathcal{S}_H \) if it is harmonic, univalent, and sense preserving in \( E \), where \( h \) and \( g \) have the following series:

\[ h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \]

\[ g(z) = \sum_{k=1}^{\infty} b_k z^k, \]

\( |b_1| < 1 \). (1)

Hence

\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k \]

\( |b_1| < 1 \). (2)

Note that if the coanalytic part \( g \) is identically zero, then \( \mathcal{S}_H \) reduces to well-known class of normalized univalent analytic functions \( \mathcal{S} \). For this class, the function \( f(z) \) may be expressed as

\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \] (3)

A function \( f(z) \) given by (2) is said to be harmonic starlike of order \( \beta \) \((0 \leq \beta < 1)\) for \( |z| = r < 1 \), if

\[ \frac{\partial}{\partial \theta} (\arg(f(re^{i\theta}))) > \beta, \]

or equivalently

\[ \Re\left(\frac{zh'(z) - zg'(z)}{h(z) + g(z)}\right) > \beta, \quad z = re^{i\theta}. \] (5)

The class of all harmonic starlike functions of order \( \beta \) is denoted by \( \mathcal{S}_{H}^{\beta} \). This class was studied by Jahangiri [4]. The cases \( \beta = 0 \) and \( b_1 = 0 \) were studied by Silverman and Silvia [5] and Silverman [6].
For \( F(z) = z + \sum_{k=2}^{\infty} A_k z^k \) and \( f(z) \) given by (2), the convolution is denoted by \( f * F \) and defined as
\[
f * F = z + \sum_{k=2}^{\infty} a_k A_k z^k + \sum_{k=1}^{\infty} b_k B_k z^k.
\] (6)

Let \( \alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \beta_2, \ldots, \beta_q \) be positive real parameter such that
\[
1 + \sum_{k=1}^{\infty} B_k - \sum_{k=1}^{\infty} A_k \geq 0.
\] (7)

The Wright generalized hypergeometric function [7] is defined as
\[
l_\Psi \left[(\alpha_1, A_1), \ldots, (\alpha_p, A_p); (\beta_1, B_1), \ldots, (\beta_q, B_q); z \right]
\] (8)

which is defined by
\[
l_\Psi \left[(\alpha_1, A_1), \ldots, (\alpha_p, A_p); (\beta_1, B_1), \ldots, (\beta_q, B_q); z \right] = \sum_{k=2}^{\infty} \sum_{i=0}^{\min(p, q)} \frac{\Gamma(\alpha_i) \Gamma(\beta_i)}{\Gamma(\alpha_i + kA_i) \Gamma(\beta_i + kB_i)} \frac{z^k}{k!}. \] (9)

If \( A_i = 1 \) \( (i = 1, 2, \ldots, p) \) and \( B_j = 1 \) \( (j = 1, 2, \ldots, q) \), we have the relationship
\[
\Omega_l \Psi \left[(\alpha_1, A_1), \ldots, (\alpha_p, A_p); (\beta_1, B_1), \ldots, (\beta_q, B_q); z \right]
\] (10)

\[
= \sum_{k=2}^{\infty} \sum_{i=0}^{\min(p, q)} \frac{(\alpha_i)_k \cdots (\alpha_p)_k}{(\beta_i)_k \cdots (\beta_q)_k} \frac{z^k}{k!}.
\]

where \( \Omega_l \Psi \) is the generalized hypergeometric function [8], where \( N \) denotes the set of all positive integers and \( (\alpha)_k \) is the Pochhammer symbol and
\[
\Omega_l = \frac{\Pi_{i=1}^{p} \Gamma(\alpha_i)}{\Pi_{i=1}^{q} \Gamma(\beta_i)}. \] (11)

Dziok and Srivastava [8] introduced a linear operator which is generalization of Dziok-Srivastava operator [8–10], Carlson-Shaffer operator [11], and the generalized Bernardi’s integral operator [12].

Dziok and Raina [13] considered the linear operator \( W[(\alpha, A_1)_1, \beta_1, B_1] : \mathcal{S} \to \mathcal{S} \) defined by
\[
W[(\alpha, A_1)_1, \beta_1, B_1](f(z)) = \sum_{k=1}^{\infty} \sigma_k(\alpha, A_1) a_k z^k,
\]
where \( \sigma_k(\alpha, A_1) \) is given by
\[
\sigma_k(\alpha, A_1) = \frac{\Gamma(\alpha_1 + A_1(k - 1)) \cdots \Gamma(\alpha_p + A_p(k - 1))}{\Gamma(\beta_1 + \beta_1(k - 1)) \cdots \Gamma(\beta_q + \beta_q(k - 1))}. \] (15)

For our convenience we write \( W[(\alpha, A_1)_1, \beta_1, B_1] = W[\alpha_1] \).

Motivated by the work of [4, 14–21], we extend the work of Chandrasekar et al. [22] by introducing some new subclasses of \( \mathcal{S} \) using the generalized hypergeometric function.

Definition 1. A function \( f \in \mathcal{S} \) is in class \( \mathcal{G}(\alpha, A_1, \beta_1, B_1, y) \) if
\[
\text{Re} \left( 1 + (1 + e^{i\phi}) \frac{z^2 (W[\alpha] h(z))^2 + 2(W[\alpha] g(z))' + z^2 (W[\alpha] g(z))''}{z(W[\alpha] h(z)') - z(W[\alpha] g(z))'} \right) \geq y.
\] (16)

Throughout this paper, we shall assume \( 0 \leq r < 1, f = h + g, \Omega, \) and \( \sigma_k(\alpha_1) \) as given in (11) and (15), respectively, unless otherwise mentioned.

2. Main Results

In Theorem 3, we shall present a sufficient condition for \( f \in \mathcal{S} \) to be in class \( \mathcal{S} \).
Theorem 3. Let \( f = h + \bar{g} \) be given by (2). If
\[
\sum_{k=1}^{\infty} k \left( \frac{2k-1}{1 - \gamma} |a_k| + \frac{2k+1}{1 - \gamma} |b_k| \right) \sigma_k(\alpha_1) \Omega \leq 2,
\]
then \( f \in \mathcal{F}_\mathcal{H}(\alpha_1, A_1, \beta_1, B_1, \gamma) \).

Proof. When inequality (18) holds for the coefficients of \( f = h + \bar{g} \) given in (2), we have to show that inequality (16) is satisfied. Arranging the left side inequality (16), we have
\[
\Re \left\{ \frac{z(W[\alpha_1]h(z))' + (1 + \epsilon^\phi) z^2(W[\alpha_1]g(z))'' + (1 + 2\epsilon^\phi) z(W[\alpha_1]g(z))'}{z(W[\alpha_1]h(z))' - z(W[\alpha_1]g(z))'} \right\} \geq \gamma
\]
by hypothesis in (18), which implies that \( f \in \mathcal{F}_\mathcal{H}(\alpha_1, A_1, \beta_1, B_1, \gamma) \).

Now we obtain the necessary and sufficient condition for the function \( f = h + \bar{g} \) given by (16) to be in \( \mathcal{F}_\mathcal{H}(\alpha_1, A_1, \beta_1, B_1, \gamma) \).

Theorem 4. Let \( f = h + \bar{g} \) be given by (16). Then \( f \in \mathcal{F}_\mathcal{H}(\alpha_1, A_1, \beta_1, B_1, \gamma) \) if and only if
\[
\sum_{k=1}^{\infty} k \left( \frac{2k-1}{1 - \gamma} |a_k| + \frac{2k+1}{1 - \gamma} |b_k| \right) \sigma_k(\alpha_1) \Omega \leq 2.
\]

Proof. Since \( \mathcal{F}_\mathcal{H}(\alpha_1, A_1, \beta_1, B_1, \gamma) \subset \mathcal{F}_\mathcal{H}(\alpha_1, A_1, \beta_1, B_1, \gamma) \), we only have to prove the necessary part of theorem. Assume that \( f \in \mathcal{F}_\mathcal{H}(\alpha_1, A_1, \beta_1, B_1, \gamma) \), and then by virtue of (16), we obtain
\[
\Re \left\{ (1 - \gamma) + (1 + \epsilon^\phi) \right\} \geq 0.
\]
This condition must hold for all values of \( z \in E \) and for real \( \phi \), so that, by taking \( z = r < 1 \) and \( \phi = 0 \), the above inequality reduces to

\[
(1 - \gamma) - \left[ \sum_{k=2}^{\infty} k \left( 2k - 1 - \gamma \right) \sigma_k(\alpha_1) \Omega |a_k| r^{k-1} + \left( \frac{z}{z} \right) \sum_{k=1}^{\infty} k \left( 2k + 1 + \gamma \right) \sigma_k(\alpha_1) \Omega |b_k| r^{k-1} \right] 
\geq 0. 
\] (25)

Letting \( r \to 1^- \) through real values, we obtain condition (22). This completes the proof. \( \square \)

We determine the extreme points of closed convex hulls of \( \mathcal{T}_\mathcal{F}((\alpha_1, A_1, \beta_1, B_1, \gamma)) \) denoted by \( \text{clco} \mathcal{T}_\mathcal{F}((\alpha_1, A_1, \beta_1, B_1, \gamma)) \).

**Theorem 5.** A function \( f(z) \in \text{clco} \mathcal{T}_\mathcal{F}((\alpha_1, A_1, \beta_1, B_1, \gamma)) \) if and only if

\[
f(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_k(z)),
\] (26)

where

\[
h_k(z) = z - \frac{1 - \gamma}{k \left( 2k - 1 - \gamma \right) \sigma_k(\alpha_1) \Omega} z^k, \quad (k \geq 2),
\] (27)

\[
g_k(z) = z - \frac{1 - \gamma}{k \left( 2k + 1 + \gamma \right) \sigma_k(\alpha_1) \Omega} z^k, \quad (k \geq 2),
\]

\[\sum_{k=1}^{\infty} (X_k + Y_k) = 1, \quad X_k \geq 0, \quad Y_k \geq 0. \] In particular, the extreme points of \( \mathcal{T}_\mathcal{F} ((\alpha_1, A_1, \beta_1, B_1, \gamma)) \) are \( \{h_k\} \) and \( \{g_k\} \).

**Proof.** First, we consider

\[
f(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_k(z))
\]

\[
= \sum_{k=1}^{\infty} (X_k + Y_k) z
\]

\[
- \sum_{k=2}^{\infty} \frac{1 - \gamma}{k \left( 2k - 1 - \gamma \right) \sigma_k(\alpha_1) \Omega} X_k z^k
\]

\[
- \sum_{k=1}^{\infty} \frac{1 - \gamma}{k \left( 2k + 1 + \gamma \right) \sigma_k(\alpha_1) \Omega} Y_k z^k,
\]

\[
f(z) = z - \sum_{k=1}^{\infty} A_k z^k - \sum_{k=1}^{\infty} B_k z^k,
\] (28)

where \( A_k = ((1 - \gamma)/k(2k - 1 - \gamma)) \sigma_k(\alpha_1) \Omega X_k \), and \( B_k = ((1 - \gamma)/k(2k + 1 + \gamma)) \sigma_k(\alpha_1) \Omega Y_k \).

Using (22) for the coefficients in (29), we have

\[
\sum_{k=2}^{\infty} k \left( 2k - 1 - \gamma \right) \sigma_k(\alpha_1) \Omega A_k
\]

\[
+ \sum_{k=1}^{\infty} k \left( 2k + 1 + \gamma \right) \sigma_k(\alpha_1) \Omega B_k = \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k
\]

\[
= 1 - x_1 \leq 1,
\]

and hence \( f(z) \in \text{clco} \mathcal{T}_\mathcal{F}((\alpha_1, A_1, \beta_1, B_1, \gamma)) \).

Conversely, suppose that \( f(z) \in \text{clco} \mathcal{T}_\mathcal{F}((\alpha_1, A_1, \beta_1, B_1, \gamma)) \), and set

\[
X_k = \frac{k \left( 2k - 1 - \gamma \right) \sigma_k(\alpha_1) \Omega}{1 - \gamma} A_k, \quad (k \geq 2),
\]

\[
Y_k = \frac{k \left( 2k + 1 + \gamma \right) \sigma_k(\alpha_1) \Omega}{1 - \gamma} B_k, \quad (k \geq 1),
\]

where \( \sum_{k=1}^{\infty} (X_k + Y_k) = 1 \). Then

\[
f(z) = z - \sum_{k=2}^{\infty} A_k z^k - \sum_{k=1}^{\infty} B_k z^k, \quad A_k \geq 0, \quad B_k \geq 0
\]

\[
= z - \sum_{k=2}^{\infty} \frac{1 - \gamma}{k \left( 2k - 1 - \gamma \right) \sigma_k(\alpha_1) \Omega} X_k z^k
\]

\[
- \sum_{k=1}^{\infty} \frac{1 - \gamma}{k \left( 2k + 1 + \gamma \right) \sigma_k(\alpha_1) \Omega} Y_k z^k
\]

\[
= z + \sum_{k=1}^{\infty} \left( h_k(z) - z \right) X_k + \sum_{k=1}^{\infty} \left( g_k(z) - z \right) Y_k
\]

\[
= \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_k(z)),
\]

which is the required result. \( \square \)

Next we show that \( \mathcal{T}_\mathcal{F} ((\alpha_1, A_1, \beta_1, B_1, \gamma)) \) is closed under convex combinations of its members.

**Theorem 6.** The family \( \mathcal{T}_\mathcal{F} ((\alpha_1, A_1, \beta_1, B_1, \gamma)) \) is closed under convex combination.

**Proof.** For \( i = 1, 2, \ldots \), suppose that \( f_i \in \mathcal{T}_\mathcal{F} ((\alpha_1, A_1, \beta_1, B_1, \gamma)) \), where

\[
f_i(z) = z - \sum_{k=2}^{\infty} a_{ik} z^k - \sum_{k=2}^{\infty} b_{ik} z^k.
\] (33)
For coefficients in (22) the relation given in (33) takes the form
\[
\sum_{k=2}^{\infty} k (2k - 1 - \gamma) \sigma_k \Omega a_{k,j} + \sum_{k=1}^{\infty} \frac{k (2k + 1 + \gamma) \sigma_k \Omega b_{k,j}}{1 - \gamma} \leq 1.
\] (34)

We note that \(|A_k| \leq 1\) and \(|B_k| \leq 1\). Therefore
\[
\sum_{k=2}^{\infty} \frac{k (2k - 1 - \delta) \sigma_k (\alpha_i) \Omega}{1 - \delta} |a_k| |A_k| + \sum_{k=1}^{\infty} \frac{k (2k + 1 + \delta) \sigma_k (\alpha_i) \Omega}{1 - \delta} |b_k| |B_k| \leq \sum_{k=2}^{\infty} \frac{k (2k - 1 - \delta) \sigma_k (\alpha_i) \Omega}{1 - \delta} |a_k| + \sum_{k=1}^{\infty} \frac{k (2k + 1 + \delta) \sigma_k (\alpha_i) \Omega}{1 - \delta} |b_k| \leq 1,
\] (40)
by using (22), since \(f(z) \in \mathcal{T}_\theta([\alpha_1, \alpha_1, \beta_1, \beta_1], \gamma)\) and \(0 \leq \delta \leq \gamma \leq 1\). This proves that \(f(z) * F(z) \in \mathcal{T}_\theta([\alpha_1, \alpha_1, \beta_1, \beta_1], \delta)\).

Now for the class \(\mathcal{T}_\theta([\alpha_1, \alpha_1, \beta_1, \beta_1], \gamma)\) the closure property under the generalized Bernardi-Livingston integral operator \(L_c(f)\) is examined which is defined by
\[
L_c(f) = \frac{c + 1}{z^c} \int_0^z t^{-1} f(t) \, dt, \quad c > -1.
\] (41)

**Theorem 7.** For \(0 \leq \delta \leq \gamma \leq 1\), let \(f(z) \in \mathcal{T}_\theta([\alpha_1, \alpha_1, \beta_1, \beta_1], \gamma)\) and \(F(z) \in \mathcal{T}_\theta([\alpha_1, \alpha_1, \beta_1, \beta_1], \delta)\). Then
\[
f(z) * F(z) \in \mathcal{T}_\theta([\alpha_1, \alpha_1, \beta_1, \beta_1], \gamma),
\] (37)
\[
c \in \mathcal{T}_\theta([\alpha_1, \alpha_1, \beta_1, \beta_1], \delta).
\]

**Proof.** Let
\[
f(z) = z - \sum_{k=2}^{\infty} a_k z^k - \sum_{k=1}^{\infty} b_k z^k \in \mathcal{T}_\theta([\alpha_1, \alpha_1, \beta_1, \beta_1], \gamma),
\] (38)
\[
F(z) = z - \sum_{k=2}^{\infty} A_k z^k - \sum_{k=1}^{\infty} B_k z^k \in \mathcal{T}_\theta([\alpha_1, \alpha_1, \beta_1, \beta_1], \delta),
\]
and then
\[
f(z) * F(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k + \sum_{k=1}^{\infty} b_k B_k z^k.
\] (39)

**Theorem 8.** Let \(f(z) \in \mathcal{T}_\theta([\alpha_1, \alpha_1, \beta_1, \beta_1], \gamma)\) and then \(L_c(f) \in \mathcal{T}_\theta([\alpha_1, \alpha_1, \beta_1, \beta_1], \gamma)\).

**Proof.** Consider the generalized Bernardi-Livingston integral operator \(L_c(f)\) given in (41):
\[
L_c(f) = \frac{c + 1}{z^c} \int_0^z t^{-1} [h(t) + \tilde{g}(t)] \, dt
\]
where
\[
A_k = \frac{c + 1}{c + n} a_k;
\] (43)
\[
B_k = \frac{c + 1}{c + n} b_k.
\]
\[
L_c(F) = \frac{c + 1}{z^c} \left( \int_0^z t^{-1} \left( t - \sum_{k=2}^{\infty} a_k t^k \right) \, dt - \int_0^z t^{-1} \left( \sum_{k=1}^{\infty} b_k t^k \right) \right)
\]
\[
= z - \sum_{k=2}^{\infty} A_k z^k - \sum_{k=1}^{\infty} B_k z^k,
\] (42)
and then
\[
f(z) * F(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k + \sum_{k=1}^{\infty} b_k B_k z^k.
\] (39)
Therefore
\[
\sum_{k=1}^{\infty} \left( \frac{2k-1-\gamma}{1-\gamma} \left( \frac{c+1}{c+n} \right) |a_k| + \frac{2k+1+\gamma}{1-\gamma} \left( \frac{c+1}{c+n} \right) |b_k| \right) \sigma_k(a_1) \Omega 
\]
\[
\leq \sum_{k=1}^{\infty} \left( \frac{2k-1-\gamma}{1-\gamma} |a_k| + \frac{2k+1+\gamma}{1-\gamma} |b_k| \right) \sigma_k(a_1) \Omega \leq 2 (1-\gamma) .
\]

Since \( f(z) \in \mathcal{T}_H([\alpha_1, A_1, B_1], \gamma) \), therefore, by Theorem 4, \( L_c(f) \in \mathcal{T}_H([\alpha_1, A_1, B_1], \gamma) \).

\[\square\]

Competing Interests
The authors declare that they have no competing interests.

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