Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2016, Article ID 7123907, 6 pages
http://dx.doi.org/10.1155/2016/7123907

Research Article

A Subclass of Harmonic Functions Related to a Convolution Operator

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Received 23 May 2016; Accepted 21 July 2016

Academic Editor: Wilfredo Urbina

We introduce a new subclass of harmonic functions by using a certain linear operator. For this class we derive coefficient bounds, extreme points, and inclusion results and also show that this class is closed under an integral operator.

1. Introduction

Harmonic functions have long been used in the representation of minimal surface; for example, Heinz [1] in 1952 used such mappings in the study of the Gaussian curvature of nonparametric minimal surface over the unit disc. Such mappings have vast application in the field of engineering, physics, electronics, medicine, operations research, aerodynamics, and other branches of applied mathematics (see [2]).

$f(z) = u(x, y) + iv(x, y)$ is said to be complex valued harmonic function if both u and v are continuous and real harmonic; that is, $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$. In simply connected domain [3], we can write $f(z) = h(z) + \overline{g(z)}$, where h and g are analytic in $E = \{z: |z| < 1\}$. We call h the analytic part and g the coanalytic part of f. The necessary and sufficient condition for a function $f(z) = h(z) + \overline{g(z)}$ to be univalent and sense preserving in E is $|h'(z)| > |g'(z)|$ (see [3]). A function $f(z) = h(z) + \overline{g(z)}$ is in class $\mathcal{S}_{\mathcal{H}}$ if it is harmonic, univalent, and sense preserving in E, where h and g have the following series:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$
$$g(z) = \sum_{k=1}^{\infty} b_k z^k, \qquad |b_1| < 1. \tag{1}$$

Hence

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \quad |b_1| < 1. \tag{2}$$

Note that if the coanalytic part g is identically zero, then $\mathcal{S}_{\mathcal{H}}$ reduces to well-known class of normalized univalent analytic functions $\mathcal{S}$. For this class, the function $f(z)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{3}$$

A function $f(z)$ given by (2) is said to be harmonic starlike of order β $(0 \le \beta < 1)$ for $|z| = r < 1$, if

$$\frac{\partial}{\partial \theta}\left(\arg\left(f\left(re^{i\theta}\right)\right)\right) > \beta, \tag{4}$$

or equivalently

$$\operatorname{Re}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + g(z)}\right) > \beta, \quad z = re^{i\theta}. \tag{5}$$

The class of all harmonic starlike functions of order β is denoted by $\mathcal{S}^*_{\mathcal{H}}(\beta)$. This class was studied by Jahangiri [4]. The cases $\beta = 0$ and $b_1 = 0$ were studied by Silverman and Silvia [5] and Silverman [6].

For $F(z) = z + \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k z^k$ and $f(z)$ given by (2), the convolution is denoted by $f * F$ and defined as

$$f * F = z + \sum_{k=2}^{\infty} a_k A_k z^k + \sum_{k=1}^{\infty} b_k B_k z^k. \tag{6}$$

Let $\alpha_1, A_1, \ldots, \alpha_p, A_p$ and $\beta_1, B_1, \ldots, \beta_q, B_q$ $(p, q \in N = \{1, 2, 3, \ldots\})$ be positive real parameter such that

$$1 + \sum_{k=1}^{\infty} B_k - \sum_{k=1}^{\infty} A_k \geq 0. \tag{7}$$

The Wright generalized hypergeometric function [7] is defined as

$$\begin{aligned} &{}_p\Psi_q\left[(\alpha_1, A_1), \ldots, (\alpha_p, A_p); (\beta_1, B_1), \ldots, (\beta_q, B_q); z\right] \\ &\quad = {}_p\Psi_q\left[(\alpha_i, A_i)_{1,p}; (\beta_i, B_i)_{1,q}; z\right] \end{aligned} \tag{8}$$

which is defined by

$$\begin{aligned} &{}_p\Psi_q\left[(\alpha_i, A_i)_{1,p}; (\beta_i, B_i)_{1,q}; z\right] \\ &\quad = \sum_{k=0}^{\infty} \frac{\Pi_{i=1}^{p} \Gamma(\alpha_i + kA_i)}{\Pi_{i=1}^{q} \Gamma(\beta_i + kB_i)} \frac{z^k}{k!} \quad z \in E. \end{aligned} \tag{9}$$

If $A_i = 1$ $(i = 1, 2, \ldots, p)$ and $B_i = 1$ $(i = 1, 2, \ldots, q)$, we have the relationship

$$\begin{aligned} &\Omega_p\Psi_q\left[(\alpha_i, A_i)_{1,p}; (\beta_i, B_i)_{1,q}; z\right] \\ &\quad = {}_pF_q\left(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z\right) \\ &\quad = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k} \frac{z^k}{k!}. \end{aligned} \tag{10}$$

$(p \leq q+1;\ p, q \in N_0 = N \cup \{0\};\ z \in E)$ is the generalized hypergeometric function [8], where N denotes the set of all positive integers and $(\alpha)_k$ is the Pochhammer symbol and

$$\Omega = \frac{\Pi_{i=1}^{p} \Gamma(\alpha_i)}{\Pi_{i=1}^{q} \Gamma(\beta_i)}. \tag{11}$$

Dziok and Srivastava [8] introduced a linear operator which is generalization of Dziok-Srivastava operator [8–10], Carlson-Shaffer operator [11], and the generalized Bernardi's integral operator [12].

Dziok and Raina [13] considered the linear operator $W[(\alpha_i, A_i)_{1,p}; (\beta_i, B_i)_{1,q}] : \mathcal{S} \to \mathcal{S}$ defined by

$$\begin{aligned} &W\left[(\alpha_i, A_i)_{1,p}; (\beta_i, B_i)_{1,q}\right] f(z) \\ &\quad = {}_p\Phi_q\left[(\alpha_i, A_i)_{1,p}; (\beta_i, B_i)_{1,q}; z\right] * f(z), \end{aligned} \tag{12}$$

where ${}_p\Phi_q[(\alpha_i, A_i)_{1,p}; (\beta_i, B_i)_{1,q}; z]$ is given by

$$\begin{aligned} &{}_p\Phi_q\left[(\alpha_i, A_i)_{1,p}; (\beta_i, B_i)_{1,q}; z\right] \\ &\quad = \Omega z_p\Psi_q\left[(\alpha_i, A_i)_{1,p}; (\beta_i, B_i)_{1,q}; z\right]. \end{aligned} \tag{13}$$

For $f(z) \in \mathcal{S}$, we have

$$\begin{aligned} &W\left[(\alpha_i, A_i)_{1,p}; (\beta_i, B_i)_{1,q}\right] f(z) \\ &\quad = z + \sum_{k=2}^{\infty} \Omega \sigma_k(\alpha_1) a_k z^k, \end{aligned} \tag{14}$$

where $\sigma_k(\alpha_1)$ is given by

$$\begin{aligned} &\sigma_k(\alpha_1) \\ &\quad = \frac{\Gamma(\alpha_1 + A_1(k-1)) \cdots \Gamma(\alpha_p + A_p(k-1))}{\Gamma(\beta_1 + \beta_1(k-1)) \cdots \Gamma(\beta_q + \beta_q(k-1))(k-1)!}. \end{aligned} \tag{15}$$

For our convenience we write $W[(\alpha_i, A_i)_{1,p}; (\beta_i, B_i)_{1,q}] = W[\alpha_1]$.

Motivated by the work of [4, 14–21], we extend the work of Chandrashekar et al. [22] by introducing some new subclasses of $\mathcal{S}_{\mathcal{H}}$ using the generalized hypergeometric function.

Definition 1. A function $f \in \mathcal{S}_{\mathcal{H}}$ is in class $\mathcal{G}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$ if

$$\operatorname{Re}\left\{1 + \left(1 + e^{i\phi}\right) \frac{\left[z^2 (W[\alpha_1] h(z))'' + \overline{2(W[\alpha_1] g(z))' + z^2 (W[\alpha_1] g(z))''}\right]}{z(W[\alpha_1] h(z))' - \overline{z(Wg(z))'}}\right\} \geq \gamma. \tag{16}$$

Definition 2. Let $\mathcal{T}_{\mathcal{H}}$ denote the class of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k - \overline{\sum_{k=1}^{\infty} |b_k| z^k}, \quad |b_1| < 1, \tag{17}$$

and $\mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma) = \mathcal{G}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma) \cap \mathcal{T}_{\mathcal{H}}$.

Throughout this paper, we shall assume $0 \leq r < 1$, $f = h + \overline{g}$, Ω, and $\sigma_k(\alpha_1)$ as given in (11) and (15), respectively, unless otherwise mentioned.

2. Main Results

In Theorem 3, we shall present a sufficient condition for $f \in S_H$ to be in class $\mathcal{G}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$.

Theorem 3. *Let $f = h + \overline{g}$ be given by (2). If*

$$\sum_{k=1}^{\infty} k\left(\frac{(2k-1-\gamma)}{1-\gamma}|a_k| + \frac{(2k+1+\gamma)}{1-\gamma}|b_k|\right)\sigma_k(\alpha_1)\Omega \leq 2, \tag{18}$$

then $f \in \mathcal{G}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$.

Proof. When inequality (18) holds for the coefficients of $f = h + \overline{g}$ given in (2), we have to show that inequality (16) is satisfied. Arranging the left side inequality (16), we have

$$\operatorname{Re}\left\{\frac{z(W[\alpha_1]h(z))' + (1+e^{i\phi})z^2(W[\alpha_1]h(z))'' + (1+2e^{i\phi})\overline{z(W[\alpha_1]g(z))'} + (1+e^{i\phi})\overline{z^2(W[\alpha_1]g(z))''}}{z(W[\alpha_1]h(z))' - \overline{z(W[\alpha_1]g(z))'}}\right\} \geq \gamma = \operatorname{Re}\frac{A(z)}{B(z)}. \tag{19}$$

As we know $\operatorname{Re}(w) \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it is sufficient to show that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0. \tag{20}$$

Substituting the values $A(z)$ and $B(z)$ in left side of (20), we obtain

$$\begin{aligned}
&|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq (2-\gamma)|z| - \sum_{k=2}^{\infty} k(2k-\gamma)|a_k|\sigma_k(\alpha_1)\Omega|z|^k \\
&- \sum_{k=1}^{\infty} k(2k+\gamma)|b_k|\sigma_k(\alpha_1)\Omega|z|^k - \gamma|z| \\
&- \sum_{k=2}^{\infty} k(2k-2-\gamma)|a_k|\sigma_k(\alpha_1)\Omega|z|^k \\
&- \sum_{k=1}^{\infty} k(2k+2+\gamma)|a_k|\sigma_k(\alpha_1)\Omega|z|^k \geq 2(1-\gamma) \\
&\cdot\left\{1 - \sum_{k=2}^{\infty} k\frac{(2k-1-\gamma)}{1-\gamma}|a_k||\sigma_k(\alpha_1)||\Omega| + \sum_{k=1}^{\infty} k\frac{(2k+1+\gamma)}{1-\gamma}|b_k|\sigma_k(\alpha_1)\Omega\right\} = 2(1-\gamma)\left\{2 - \left(\sum_{k=1}^{\infty} k\frac{(2k-1-\gamma)}{1-\gamma}|a_k| + \sum_{k=1}^{\infty} k\frac{(2k+1+\gamma)}{1-\gamma}|b_k|\right)\right. \\
&\left.\cdot\sigma_k(\alpha_1)\Omega\right\} \geq 0
\end{aligned} \tag{1, 21}$$

by hypothesis in (18), which implies that $f \in \mathcal{G}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$. □

Now we obtain the necessary and sufficient condition for the function $f = h + \overline{g}$ given by (16) to be in $\mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$.

Theorem 4. *Let $f = h + \overline{g}$ be given by (16). Then $f \in \mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$ if and only if*

$$\sum_{k=1}^{\infty} k\left\{\frac{(2k-1-\gamma)}{1-\gamma}|a_k| + \frac{(2k+1+\gamma)}{1-\gamma}|b_k|\right\}\sigma_k(\alpha_1)\Omega \leq 2. \tag{22}$$

Proof. Since $\mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma) \subset \mathcal{G}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$, we only have to prove the necessary part of theorem. Assume that $f \in \mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$, and then by virtue of (16), we obtain

$$\operatorname{Re}\left\{(1-\gamma) + (1+e^{i\phi})\frac{\left[z^2(W[\alpha_1]h(z))'' + \overline{2(W[\alpha_1]g(z))' + z^2(W[\alpha_1]g(z))''}\right]}{z(W[\alpha_1]h(z))' - \overline{z(W[\alpha_1]g(z))'}}\right\} \geq 0. \tag{23}$$

This is equivalent to

$$\begin{aligned}
&\operatorname{Re}\left\{\frac{(1-\gamma)z - \left(\sum_{k=2}^{\infty} k\left[k(1+e^{i\phi}) - \gamma - e^{i\phi}\right]\sigma_k(\alpha_1)\Omega|a_k|z^k + \sum_{k=1}^{\infty} k\left[k(1+e^{i\phi}) + \gamma + e^{i\phi}\right]\sigma_k(\alpha_1)\Omega|b_k|\overline{z}^k\right)}{z - \sum_{k=2}^{\infty} k\sigma_k(\alpha_1)\Omega|a_k|z^k - \sum_{k=1}^{\infty} k\sigma_k(\alpha_1)\Omega|b_k|\overline{z}^k}\right\} \\
&= \operatorname{Re}\left\{\frac{(1-\gamma) - \sum_{k=2}^{\infty} k\left[k(1+e^{i\phi}) - \gamma - e^{i\phi}\right]\sigma_k(\alpha_1)\Omega|a_k|z^{k-1} - (\overline{z}/z)\sum_{k=1}^{\infty} k\left[k(1+e^{i\phi}) + \gamma + e^{i\phi}\right]\sigma_k(\alpha_1)\Omega|b_k|\overline{z}^{k-1}}{1 - \sum_{k=2}^{\infty} k\sigma_k(\alpha_1)\Omega|a_k|z^{k-1} + (\overline{z}/z)\sum_{k=1}^{\infty} k\sigma_k(\alpha_1)\Omega|b_k|\overline{z}^{k-1}}\right\} \\
&\geq 0.
\end{aligned} \tag{24}$$

This condition must hold for all values of $z \in E$ and for real ϕ, so that, by taking $z = r < 1$ and $\phi = 0$, the above inequality reduces to

$$\frac{(1-\gamma) - \left[\sum_{k=2}^{\infty} k(2k-1-\gamma)\sigma_k(\alpha_1)\Omega|a_k|r^{k-1} + (\overline{z}/z)\sum_{k=1}^{\infty} k(2k+1+\gamma)\sigma_k(\alpha_1)\Omega|b_k|r^{k-1}\right]}{1 - \sum_{k=2}^{\infty} k\sigma_k(\alpha_1)\Omega|a_k|r^{k-1} + (\overline{z}/z)\sum_{k=1}^{\infty} k\sigma_k(\alpha_1)\Omega|b_k|r^{k-1}} \geq 0. \tag{25}$$

Letting $r \to 1^-$ through real values, we obtain condition (22). This completes the proof. □

We determine the extreme points of closed convex hulls of $\mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$ denoted by $\mathrm{clco}\mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$.

Theorem 5. *A function $f(z) \in clco\mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty} \left(X_k h_k(z) + Y_k g_k(z)\right), \tag{26}$$

where

$$\begin{aligned} h_1(z) &= z, \\ h_k(z) &= z - \frac{1-\gamma}{k(2k-1-\gamma)\sigma_k(\alpha_1)\Omega} z^k; \quad (k \geq 2), \\ g_k(z) &= z - \frac{1-\gamma}{k(2k+1+\gamma)\sigma_k\Omega} \overline{z}^k; \quad (k \geq 2), \end{aligned} \tag{27}$$

$\sum_{k=1}^{\infty}(X_k + Y_k) = 1$, $X_k \geq 0$, *and* $Y_k \geq 0$. *In particular, the extreme points of* $\mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$ *are* $\{h_k\}$ *and* $\{g_k\}$.

Proof. First, we consider

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} \left(X_k h_k(z) + Y_k g_k(z)\right) \\ &= \sum_{k=1}^{\infty} (X_k + Y_k) z \\ &\quad - \sum_{k=2}^{\infty} \frac{1-\gamma}{k(2k-1-\gamma)\sigma_k(\alpha_1)\Omega} X_k z^k \\ &\quad - \sum_{k=1}^{\infty} \frac{1-\gamma}{k(2k+1+\gamma)\sigma_k(\alpha_1)\Omega} Y_k \overline{z}^k, \end{aligned} \tag{28}$$

$$f(z) = z - \sum_{k=1}^{\infty} A_k z^k - \overline{\sum_{k=1}^{\infty} B_k z^k}, \tag{29}$$

where $A_k = ((1-\gamma)/k(2k-1-\gamma)\sigma_k(\alpha_1)\Omega)X_k$, and $B_k = ((1-\gamma)/k(2k+1+\gamma)\sigma_k(\alpha_1)\Omega)Y_k$.

Using (22) for the coefficients in (29), we have

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{k(2k-1-\gamma)\sigma_k(\alpha_1)\Omega}{1-\gamma} A_k \\ &\quad + \sum_{k=1}^{\infty} \frac{k(2k+1+\gamma)\sigma_k(\alpha_1)\Omega}{1-\gamma} B_k = \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k \\ &\quad = 1 - x_1 \leq 1, \end{aligned} \tag{30}$$

and hence $f(z) \in \mathrm{clco}\mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$.

Conversely, suppose that $f(z) \in \mathrm{clco}\mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$, and set

$$\begin{aligned} X_k &= \frac{k(2k-1-\gamma)\sigma_k(\alpha_1)\Omega}{1-\gamma} A_k, \quad (k \geq 2), \\ Y_k &= \frac{k(2k+1+\gamma)\sigma_k(\alpha_1)\Omega}{1-\gamma} B_k, \quad (k \geq 1), \end{aligned} \tag{31}$$

where $\sum_{k=1}^{\infty}(X_k + Y_k) = 1$. Then

$$\begin{aligned} f(z) &= z - \sum_{k=2}^{\infty} A_k z^k - \sum_{k=1}^{\infty} B_k \overline{z}^k, \quad A_k \geq 0,\ B_k \geq 0 \\ &= z - \sum_{k=2}^{\infty} \frac{1-\gamma}{k(2k-1-\gamma)\sigma_k(\alpha_1)\Omega} X_k z^k \\ &\quad - \sum_{k=1}^{\infty} \frac{1-\gamma}{k(2k+1+\gamma)\sigma_k(\alpha_1)\Omega} Y_k \overline{z}^k \\ &= z + \sum_{k=2}^{\infty} (h_k(z) - z) X_k + \sum_{k=1}^{\infty} (g_k(z) - z)_k Y_k \\ &= \sum_{k=1}^{\infty} \left(X_k h_k(z) + Y_k g_k(z)\right), \end{aligned} \tag{32}$$

which is the required result. □

Next we show that $\mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$ is close under convex combinations of its members.

Theorem 6. *The family* $\mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$ *is closed under convex combination.*

Proof. For $i = 1, 2, \ldots$, suppose that $f_i \in \mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$, where

$$f_i(z) = z - \sum_{k=2}^{\infty} a_{i,k} z^k - \overline{\sum_{k=2}^{\infty} b_{i,k} z^k}. \tag{33}$$

For coefficients in (22) the relation given in (33) takes the form

$$\sum_{k=2}^{\infty}\frac{k(2k-1-\gamma)\sigma_k\Omega}{1-\gamma}a_{i,k}+\sum_{k=1}^{\infty}\frac{k(2k+1+\gamma)\sigma_k\Omega}{1-\gamma}b_{i,k} \leq 1. \tag{34}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{i,k}\right)z^k - \overline{\sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i b_{i,k}\right)}z^k. \tag{35}$$

From inequality (22) for (35),

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{k(2k-1-\gamma)\sigma_k(\alpha_1)\Omega}{1-\gamma}\left(\sum_{i=1}^{\infty} t_i a_{i,k}\right)\\&+\sum_{k=1}^{\infty}\frac{k(2k+1+\gamma)\sigma_k(\alpha_1)\Omega}{1-\gamma}\left(\sum_{i=1}^{\infty} t_i a_{i,k}\right)\\&\cdot\sum_{i=1}^{\infty} t_i\left(\sum_{k=2}^{\infty}\frac{k(2k-1-\gamma)\sigma_k(\alpha_1)\Omega}{1-\gamma}a_{i,k}\right.\\&\left.+\sum_{k=1}^{\infty}\frac{k(2k+1+\gamma)\sigma_k(\alpha_1)\Omega}{1-\gamma}b_{i,k}\right)\leq\sum_{t=1}^{\infty} t_i = 1,\end{aligned} \tag{36}$$

and therefore $\sum_{t=1}^{\infty} t_i f_i(z) \in \mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$. □

Theorem 7. *For $0 \leq \delta \leq \gamma \leq 1$, let $f(z) \in \mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$ and $F(z) \in \mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \delta)$. Then*

$$\begin{aligned}f(z) * F(z) &\in \mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)\\&\subset \mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \delta).\end{aligned} \tag{37}$$

Proof. Let

$$\begin{aligned}f(z) &= z - \sum_{k=2}^{\infty} a_k z^k - \overline{\sum_{k=1}^{\infty} b_k z^k}\\&\in \mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma),\\F(z) &= z - \sum_{k=2}^{\infty} A_k z^k - \overline{\sum_{k=1}^{\infty} B_k z^k}\\&\in \mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \delta),\end{aligned} \tag{38}$$

and then

$$f(z) * F(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k + \overline{\sum_{k=1}^{\infty} b_k B_k z^k}. \tag{39}$$

We note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Therefore

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{k(2k-1-\delta)\sigma_k(\alpha_1)\Omega}{1-\delta}|a_k||A_k|\\&+\sum_{k=1}^{\infty}\frac{k(2k+1+\delta)\sigma_k(\alpha_1)\Omega}{1-\delta}|b_k||B_k|\\&\leq\sum_{k=2}^{\infty}\frac{k(2k-1-\delta)\sigma_k(\alpha_1)\Omega}{1-\delta}|a_k|\\&+\sum_{k=1}^{\infty}\frac{k(2k+1+\delta)\sigma_k(\alpha_1)\Omega}{1-\delta}b_k\\&\leq\sum_{k=2}^{\infty}\frac{k(2k-1-\gamma)\sigma_k(\alpha_1)\Omega}{1-\gamma}|a_k|\\&+\sum_{k=1}^{\infty}\frac{k(2k+1+\gamma)\sigma_k(\alpha_1)\Omega}{1-\gamma}|b_k|\leq 1,\end{aligned} \tag{40}$$

by using (22), since $f(z) \in \mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$ and $0 \leq \delta \leq \gamma \leq 1$. This proves that $f(z) * F(z) \in \mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \delta)$. □

Now for the class $\mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$ the closure property under the generalized Bernardi-Livingston integral operator $L_c(f)$ is examined which is defined by

$$L_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\,dt, \quad c > -1. \tag{41}$$

Theorem 8. *Let $f(z) \in \mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$ and then $L_c(f) \in \mathcal{T}_{\mathcal{H}}([\alpha_1, A_1, \beta_1, B_1], \gamma)$.*

Proof. Consider the generalized Bernardi-Livingston integral operator $L_c(f)$ given in (41):

$$\begin{aligned}L_c(f) &= \frac{c+1}{z^c}\int_0^z t^{c-1}\left[h(t) + \overline{g(t)}\right]dt\\&= \frac{c+1}{z^c}\left(\int_0^z t^{c-1}\left(t - \sum_{k=2}^{\infty} a_k t^k\right)dt\right.\\&\left.-\overline{\int_0^z t^{c-1}\left(\sum_{k=1}^{\infty} b_k t^k\right)}\right),\\&= z - \sum_{k=2}^{\infty} A_k z^k - \overline{\sum_{k=1}^{\infty} B_k z^k},\end{aligned} \tag{42}$$

where

$$\begin{aligned}A_k &= \frac{c+1}{c+n}a_k;\\B_k &= \frac{c+1}{c+n}b_k.\end{aligned} \tag{43}$$

Therefore

$$\begin{aligned}
&\sum_{k=1}^{\infty} k\left(\frac{(2k-1-\gamma)}{1-\gamma}\left(\frac{c+1}{c+n}\right)|a_k|\right.\\
&\quad\left.+\frac{(2k+1+\gamma)}{1-\gamma}\left(\frac{c+1}{c+n}\right)|b_k|\right)\sigma_k(\alpha_1)\Omega\\
&\quad\le\sum_{k=1}^{\infty} k\left(\frac{(2k-1-\gamma)}{1-\gamma}|a_k|+\frac{(2k+1+\gamma)}{1-\gamma}|b_k|\right)\\
&\quad\cdot\sigma_k(\alpha_1)\Omega\le 2(1-\gamma).
\end{aligned}\tag{44}$$

Since $f(z)\in\mathcal{T}_{\mathcal{H}}([\alpha_1,A_1,\beta_1,B_1],\gamma)$, therefore, by Theorem 4, $L_c(f)\in\mathcal{T}_{\mathcal{H}}([\alpha_1,A_1,\beta_1,B_1],\gamma)$. □

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

This work is supported by AP-2013-009 and GP-K006392, Universiti Kebangsaan Malaysia.

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