Research Article

Classes of I -Convergent Double Sequences over n -Normed Spaces

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I introduce some new classes of $\mathcal{I}$ -convergent double sequences defined by a sequence of moduli over n -normed space. Study of their algebraic and topological properties and some inclusion relations has also been done.

1. Introduction

The notion of $\mathcal{I}$ -convergence was introduced by Kostyrko et al. in [1]. It is known that $\mathcal{I}$ -convergence is generalization of the statistical convergence which was introduced by Fast [2]. It was further studied by Demirci [3], Das et al. [4], Šalát et al. [5], and many others.

For a nonempty set X , the family of sets $\mathcal{F} \subset 2^X$, the power set of X , is said to be an ideal if

- (1) $\phi \in \mathcal{F}$;
- (2) $\mathcal{F}$ is additive; that is, $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$;
- (3) $\mathcal{F}$ is hereditary; that is, $A \in \mathcal{F}, B \subseteq A \Rightarrow B \in \mathcal{F}$.

A nontrivial ideal $\mathcal{F}$ is called admissible if $\{\{x\} : x \in X\} \subseteq \mathcal{F}$. $\mathcal{F}$ is maximal if there cannot exist any nontrivial ideal $J \neq \mathcal{F}$ containing $\mathcal{F}$ as a subset.

Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of natural, real, and complex numbers, respectively. A double sequence of complex numbers is defined as a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{C}$. A number $a \in \mathbb{C}$ is called a limit of a double sequence (x_{ij}) if for every $\epsilon > 0$ there exists some $N = N(\epsilon) \in \mathbb{N}$ such that $|x_{ij} - a| < \epsilon$, $\forall i, j \geq N$. The set of all double sequences is denoted by ${}_2\omega$. Any subset of the ${}_2\omega$ is called double sequence space. A sequence $(x_{ij}) \in {}_2\omega$ is said to be $\mathcal{I}$ -convergent to a number L if, for every $\epsilon > 0$, $\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in \mathcal{F}$. In this case we write $\mathcal{I} - \lim x_{ij} = L$.

A double sequence space E is said to be solid or normal if $(x_{ij}) \in E$ implies $(\alpha_{ij}x_{ij}) \in E$ for all sequences of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all $i, j \in \mathbb{N}$. For more details please see [6–8].

Example 1. Let $\mathcal{I}_2(P)$ be the class of all subsets of $\mathbb{N} \times \mathbb{N}$ such that $D \in \mathcal{I}_2(P)$ implies that there exists $n_0, k_0 \in \mathbb{N}$ such that $D \subset \mathbb{N} \times \mathbb{N} \setminus \{(n, k) \in \mathbb{N} \times \mathbb{N} : n \geq n_0, k \geq k_0\}$.

Then $\mathcal{I}_2(P)$ is an ideal of $\mathbb{N} \times \mathbb{N}$ in the usual Pringsheim sense of convergence of double sequences. If $\mathcal{I}_2(P)$ is replaced by $\mathcal{F}(f)$, the class of finite subsets of $\mathbb{N}$, then we get the usual regular convergence of double sequences.

The theory of 2-normed spaces was first introduced by Gähler [9] in 1964. Later on it was extended to n -normed spaces by Misiak [10]. Since then many mathematicians have worked in this field and obtained many interesting results; for instance see Gunawan [11, 12], Gunawan and Mashadi [13], Mursaleen and Mohiuddine [14], Şahiner et al. [15, 16], and Yamancı and Gürdal [17]. Let $n \in \mathbb{N}$ and let X^n be a linear metric space over the field $\mathbb{K}$ of real or complex numbers of dimension d , where $d \geq n \geq 2$. A real valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four conditions:

- (1) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent;
- (2) $\|x_1, x_2, \dots, x_n\|$ is invariant under permutation;
- (3) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$ for any $\alpha \in \mathbb{K}$;
- (4) $\|x + x', x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$

is called an n -norm on X and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n -normed space over the field $\mathbb{K}$.

Example 2. If we take $X = \mathbb{R}^n$, equipped with Euclidean n -norm $\|x_1, x_2, \dots, x_n\|_E = \text{Vol}(n - \text{dim parallelopiped})$ spanned by vectors $(x_1, x_2, \dots, x_n)$, then the n -norm may be given by the formula $\|x_1, x_2, \dots, x_n\|_E = |x_{ij}|$, where $(x_{ij}) = (x_{i1}, x_{i2}, \dots, x_{in})$ for $i = 1, 2, 3, \dots, n$.

The standard n -norm on X is defined as

$$\|x_1, x_2, \dots, x_n\|_E = \left| \begin{array}{ccc} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \cdots & \vdots \\ \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle \end{array} \right|^{1/2}, \quad (1)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on X . If $X = \mathbb{R}^n$, then this n -norm is exactly the same as the Euclidean n -norm $\|x_1, x_2, \dots, x_n\|_E$ mentioned earlier. For $n = 1$ this n -norm is the usual norm $\|x\| = \langle x_1, x_1 \rangle^{1/2}$.

A sequence (x_k) in an n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to converge to some $L \in \mathbb{K}$ if

$$\lim_{k \rightarrow \infty} \|x_k - L, z_1, \dots, z_{n-1}\| = 0 \quad (2)$$

for every $z_1, z_2, \dots, z_{n-1} \in X$.

A sequence (x_k) in an n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k, p \rightarrow \infty} \|x_k - x_p, z_1, \dots, z_{n-1}\| = 0 \quad (3)$$

for every $z_1, z_2, \dots, z_{n-1} \in X$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n -norm. Any complete n -normed space is said to be an n -Banach space.

The concept of modulus function was introduced by Nakano [18] in the year 1953. It was further studied by [7, 8, 19–21] and many more. It is defined as a function $f: [0, \infty) \rightarrow [0, \infty)$ satisfying the following conditions:

- (1) $f(t) = 0$ if and only if $t = 0$,
- (2) $f(t + u) \leq f(t) + f(u)$ for all $t, u \geq 0$,
- (3) f is increasing,
- (4) f is continuous from the right at zero.

Ruckle [22] used the idea of a modulus function f to construct the sequence space

$$X(f) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty \right\}. \quad (4)$$

The space $X(f)$ is closely related to the space l_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle [23, 24] proved that, for any modulus f ,

$$\begin{aligned} X(f) &\subset l_1, \\ X(f)^\alpha &= l_\infty. \end{aligned} \quad (5)$$

The space $X(f)$ is a Banach space with respect to the norm

$$\|x\| = \sum_{k=1}^{\infty} f(|x_k|) < \infty. \quad (6)$$

After then Kolk [25, 26] gave an extension of $X(f)$ by considering a sequence of modulus functions called the sequence of moduli $F = (f_k)$ and defined the sequence space:

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}. \quad (7)$$

From the above four properties of modulus function it can be clearly seen that f must be continuous everywhere on $[0, \infty)$. For a sequence of moduli, we have further two properties:

- (5) $\sup_k f_k(t) < \infty$ for all $t > 0$;
- (6) $\lim_{t \rightarrow 0} f_k(t) = 0$ uniformly in X and for $k \geq 1$.

Example 3. Let f be a function from $[0, \infty)$ to $[0, \infty)$. If we take $f(x) = x/(x + 1)$, then the function f is a bounded modulus function and if we take $f(x) = x^p$, $0 < p < 1$, then f is an unbounded modulus function.

By a lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, \dots$, where $k_0 = 0$, we mean an increasing sequence of nonnegative integers $h_r = (k_r - k_{r-1}) \rightarrow \infty$ as $(r \rightarrow \infty)$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $k_r/(k_r - 1)$ will be denoted by q_r . The space of lacunary strongly convergent sequence N_θ was defined by Freedman et al. [27] as follows:

$$\begin{aligned} N_\theta &= \left\{ x = (x_k) : \lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| \right. \\ &\quad \left. = 0, \text{ for some } L \right\}. \end{aligned} \quad (8)$$

The double lacunary sequence was defined by Savaş and Patterson [28]. A double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that

$$\begin{aligned} k_0 &= 0, \\ h_r &= k_r - k_{r-1} \rightarrow \infty \quad \text{as } r \rightarrow \infty, \\ l_0 &= 0, \\ h_s &= l_s - l_{s-1} \rightarrow \infty \quad \text{as } s \rightarrow \infty. \end{aligned} \quad (9)$$

The following interval is determined by θ :

$$I_{r,s} = \{(k, l) : k_{r-1} < k < k_r, l_{s-1} < l < l_s\}. \quad (10)$$

The space of double lacunary strongly convergent sequence is defined as follows:

$$\begin{aligned} N_{\theta_{r,s}} &= \left\{ x = (x_{kl}) : \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} |x_{kl} - L| \right. \\ &\quad \left. = 0, \text{ for some } L \right\}. \end{aligned} \quad (11)$$

2. New Classes of Double Sequences

Now, we will define the new classes of double sequences.

Let $\mathcal{J}$ be an admissible ideal, let $F = (f_{ij})$ be a sequence of moduli, let $(X, \|\cdot, \dots, \cdot\|)$ be an n -normed space, let $p = (p_{ij})$ be a sequence of positive real numbers, let $u = (u_{ij})$ be a sequence of strictly positive real numbers, and let ${}_2w(n - X)$ be the space of all double sequences defined over the n -normed space $(X, \|\cdot, \dots, \cdot\|)$; then for some $L \in \mathbb{K}$ and every $z_i \in X$, we define

- (i) $[{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J} = \{x = (x_{ij}) \in {}_2w(n - X) : [(r, s) \in \mathbb{N} \times \mathbb{N} : (1/h_{r,s}) \sum_{(i,j) \in I_{r,s}} u_{ij} [F(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \geq \epsilon] \in \mathcal{J}\},$
- (ii) $[{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_s]_0^\mathcal{J} = \{x = (x_{ij}) \in {}_2w(n - X) : [(r, s) \in \mathbb{N} \times \mathbb{N} : (1/h_{r,s}) \sum_{(i,j) \in I_{r,s}} u_{ij} [F(\|x_{ij}, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \geq \epsilon] \in \mathcal{J}\}.$

Case 1. If $F(x) = x$, then we get

- (i) $[{}_2N_\theta, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J} = \{x = (x_{ij}) \in {}_2w(n - X) : [(r, s) \in \mathbb{N} \times \mathbb{N} : (1/h_{r,s}) \sum_{(i,j) \in I_{r,s}} u_{ij} (\|x_{ij} - L, z_1, z_2, \dots, z_{n-1}\|)^{p_{ij}} \geq \epsilon] \in \mathcal{J}\},$
- (ii) $[{}_2N_\theta, u, p, \|\cdot, \dots, \cdot\|, X_s]_0^\mathcal{J} = \{x = (x_{ij}) \in {}_2w(n - X) : [(r, s) \in \mathbb{N} \times \mathbb{N} : (1/h_{r,s}) \sum_{(i,j) \in I_{r,s}} u_{ij} (\|x_{ij}, z_1, z_2, \dots, z_{n-1}\|)^{p_{ij}} \geq \epsilon] \in \mathcal{J}\}.$

Case 2. If $p = (p_{ij}) = 1$, then we get

- (i) $[{}_2N_\theta, F, u, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J} = \{x = (x_{ij}) \in {}_2w(n - X) : [(r, s) \in \mathbb{N} \times \mathbb{N} : (1/h_{r,s}) \sum_{(i,j) \in I_{r,s}} u_{ij} [F(\|x_{ij} - L, z_1, z_2, \dots, z_{n-1}\|)] \geq \epsilon] \in \mathcal{J}\},$
- (ii) $[{}_2N_\theta, F, u, \|\cdot, \dots, \cdot\|, X_s]_0^\mathcal{J} = \{x = (x_{ij}) \in {}_2w(n - X) : [(r, s) \in \mathbb{N} \times \mathbb{N} : (1/h_{r,s}) \sum_{(i,j) \in I_{r,s}} u_{ij} [F(\|x_{ij}, z_1, z_2, \dots, z_{n-1}\|)] \geq \epsilon] \in \mathcal{J}\}.$

Case 3. If $p = (p_{ij}) = 1$ and $u = (u_{ij}) = 1$, then we get

- (i) $[{}_2N_\theta, F, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J} = \{x = (x_{ij}) \in {}_2w(n - X) : [(r, s) \in \mathbb{N} \times \mathbb{N} : (1/h_{r,s}) \sum_{(i,j) \in I_{r,s}} [F(\|x_{ij} - L, z_1, z_2, \dots, z_{n-1}\|)] \geq \epsilon] \in \mathcal{J}\},$
- (ii) $[{}_2N_\theta, F, \|\cdot, \dots, \cdot\|, X_s]_0^\mathcal{J} = \{x = (x_{ij}) \in {}_2w(n - X) : [(r, s) \in \mathbb{N} \times \mathbb{N} : (1/h_{r,s}) \sum_{(i,j) \in I_{r,s}} [F(\|x_{ij}, z_1, z_2, \dots, z_{n-1}\|)] \geq \epsilon] \in \mathcal{J}\}.$

The following inequality will be used throughout the paper. If $0 \leq p_{ij} \leq \sup p_{ij} = H, K = \max(1, 2^{H-1})$, then we have

$$|a_{ij} + b_{ij}|^{p_{ij}} \leq K \left\{ |a_{ij}|^{p_{ij}} + |b_{ij}|^{p_{ij}} \right\} \quad (12)$$

for all $a_{ij}, b_{ij} \in \mathbb{C}$ and $(i, j) \in \mathbb{N} \times \mathbb{N}$. Also $|a|^{p_{ij}} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

3. Main Results

Theorem 4. The sets $[{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J}$ and $[{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_s]_0^\mathcal{J}$ are linear spaces over the field of complex numbers $\mathbb{C}$.

Proof. Let $x = (x_{ij}), y = (y_{ij}) \in [{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J}$, $\alpha, \beta \in \mathbb{C}$, and $C = \max\{1, (|\alpha|/(|\alpha| + |\beta|))^H, (|\beta|/(|\alpha| + |\beta|))^H\}$; then for every $z_i \in X$ we can write

$$\begin{aligned} & \frac{1}{h_{r,s} \sum_{(i,j) \in I_{r,s}} u_{ij}} [F(\|x_{ij} - L_1, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \\ & \leq \frac{\epsilon}{2KC}, \\ & \frac{1}{h_{r,s} \sum_{(i,j) \in I_{r,s}} u_{ij}} [F(\|y_{ij} - L_2, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \\ & \leq \frac{\epsilon}{2KC}. \end{aligned} \quad (13)$$

By the use of inequality (12), we have the following inequality:

$$\frac{1}{h_{r,s} \sum_{(i,j) \in I_{r,s}} u_{ij}} [F(\|((\alpha x_{ij} + \beta y_{ij}) - (\alpha L_1 + \beta L_2)), z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \leq KC \left(\frac{\epsilon}{2KC} \right) < \epsilon. \quad (14)$$

This inequality says to us that the inclusion

$$\begin{aligned} & \left[(r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \right. \\ & \cdot \sum_{(i,j) \in I_{r,s}} [F(\|(\alpha x + \beta y) - (\alpha L_1 + \beta L_2)), z_1, z_2, \dots, z_{n-1}\|)] \\ & \left. \geq \epsilon \right] \subseteq \left[(r, s) \in \mathbb{N} \times \mathbb{N} : KC \frac{1}{h_{r,s}} \right. \\ & \cdot \sum_{(i,j) \in I_{r,s}} u_{ij} [F(\|(x - L_1), z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \geq \epsilon \Big] \\ & \cup \left[(r, s) \in \mathbb{N} \times \mathbb{N} : KC \frac{1}{h_{r,s}} \right. \\ & \cdot \sum_{(i,j) \in I_{r,s}} u_{ij} [F(\|(y - L_2), z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \geq \epsilon \Big] \end{aligned} \quad (15)$$

holds. From here, since the right side belongs to $\mathcal{J}$, the left side also belongs to $\mathcal{J}$. This completes the proof. $\square$

Lemma 5. Let f be a modulus function and let $0 < \delta < 1$. Then for each $x > \delta$, one has

$$f(x) \leq 2f(1)\delta^{-1}x. \quad (16)$$

Theorem 6. Let $F = (f_{ij})$ be a sequence of moduli and $0 \leq \inf_{i,j} p_{ij} = h \leq \sup_{i,j} p_{ij} = H < \infty$. Then the following statements hold:

- (i) $[{}_2N_\theta, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J} \subset [{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J}$,
- (ii) $[{}_2N_\theta, u, p, \|\cdot, \dots, \cdot\|, X_s]_0^\mathcal{J} \subset [{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_s]_0^\mathcal{J}$.

Proof. For some $\delta > 0$, choose $\delta_0 > 0$ such that $\max\{\delta_0^h, \delta_0^H\} < \delta$. By the continuity of $F = (f_{ij})$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$, we can choose some $\epsilon \in (0, 1)$ such that for every t with $0 < t \leq \epsilon$ we have

$$F(t) < \delta_0 \quad \forall (i, j) \in \mathbb{N} \times \mathbb{N}. \quad (17)$$

Let $x = (x_{ij}) \in [{}_2N_\theta, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J}$; then for some $L > 0$, $\delta > 0$ and for every $z_i \in X$, $i = 1, 2, 3, \dots, (n-1)$, we have

$$A = \left\{ (r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \geq \epsilon^H \right\} \in \mathcal{J}. \quad (18)$$

Therefore for $(r, s) \notin A$, we have

$$\begin{aligned} \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} &< \epsilon^H, \\ \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} &\leq \epsilon. \end{aligned} \quad (19)$$

So, by inequality (17), we can write

$$\begin{aligned} F \left[\frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} (\|x - L, z_1, z_2, \dots, z_{n-1}\|) \right]^{p_{ij}} &\leq \delta_0, \\ \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} [F(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} &< \max\{\delta_0^h, \delta_0^H\} < \delta, \end{aligned}$$

$$\left\{ (r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [F(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \geq \delta \right\} \in \mathcal{J}. \quad (20)$$

This implies that $x \in [{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J}$. Hence $[{}_2N_\theta, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J} \subset [{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J}$. The inclusion is strict as for the reverse inclusion we need the condition given in the next theorem. The other part can be proved similarly. $\square$

Theorem 7. Let F be a sequence of moduli. If $\lim_t \sup (f_{ij}(t)/t) = A > 0$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$, then

- (i) $[{}_2N_\theta, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J} = [{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J}$,
- (ii) $[{}_2N_\theta, u, p, \|\cdot, \dots, \cdot\|, X_s]_0^\mathcal{J} = [{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_s]_0^\mathcal{J}$.

Proof. (i) To prove $[{}_2N_\theta, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J} = [{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J}$, it is sufficient to show that $[{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J} \subset [{}_2N_\theta, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J}$. Let $x \in [{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{J}$; then, by the definition, we get

$$\frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [F(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} < \delta. \quad (21)$$

By the given condition $\lim_t \sup (f_{ij}(t)/t) = A > 0$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$, we have $f_{ij}(t) \geq At$ for all (i, j) ; that is,

$$\begin{aligned} F(\|x - L, z_1, z_2, \dots, z_{n-1}\|) &\geq A(\|x - L, z_1, z_2, \dots, z_{n-1}\|), \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [F(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} &\geq A^H \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} (\|x - L, z_1, z_2, \dots, z_{n-1}\|)^{p_{ij}}. \end{aligned} \quad (23)$$

From inequalities (21) and (23), we get

$$\begin{aligned} \delta &> \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [F(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \\ &\geq A^H \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} (\|x - L, z_1, z_2, \dots, z_{n-1}\|)^{p_{ij}} \end{aligned} \quad (24)$$

which consequently implies that

$$\frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} (\|x - L, z_1, z_2, \dots, z_{n-1}\|)^{p_{ij}} < \delta. \quad (25)$$

That is,

$$\left\{ (r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \geq \delta \right\} \in \mathcal{F}. \quad (26)$$

This implies that $x \in [{}_2N_\theta, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{F}$.

$$\begin{aligned} & [{}_2N_\theta, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{F} \\ & \subseteq [{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{F}. \end{aligned} \quad (27)$$

Hence, from the previous theorem and inclusion (27), we get the required result. The other part can be proved similarly. $\square$

Corollary 8. Let $F' = (f'_{ij})$ and $F'' = (f''_{ij})$ be sequences of moduli. If $\lim_t \sup (f'_{ij}(t)/f''_{ij}(t)) < \infty$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$, then

- (i) $[{}_2N_\theta, F', u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{F} = [{}_2N_\theta, F'', u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{F}$,
- (ii) $[{}_2N_\theta, F', u, p, \|\cdot, \dots, \cdot\|, X_s]_0^\mathcal{F} = [{}_2N_\theta, F'', u, p, \|\cdot, \dots, \cdot\|, X_s]_0^\mathcal{F}$.

Theorem 9. Let $(X, \|\cdot, \dots, \cdot\|_{X_S})$ and $(X, \|\cdot, \dots, \cdot\|_{X_E})$ be the standard and the Euclidean n -norm spaces, respectively. Then

$$\begin{aligned} & [{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_S]^\mathcal{F} \\ & \cap [{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|, X_E]^\mathcal{F} \\ & \subseteq [{}_2N_\theta, F, u, p, \|\cdot, \dots, \cdot\|_{X_S} + \|\cdot, \dots, \cdot\|_{X_E}]^\mathcal{F}. \end{aligned} \quad (28)$$

Proof. The proof of this result is easy, so it is omitted. $\square$

Theorem 10. Let $F' = (f'_{ij})$ and $F'' = (f''_{ij})$ be sequences of moduli; then

- (i) $[{}_2N_\theta, F'', u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{F} \subseteq [{}_2N_\theta, F' \circ F'', u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{F}$,
- (ii) $[{}_2N_\theta, F'', u, p, \|\cdot, \dots, \cdot\|, X_s]_0^\mathcal{F} \subseteq [{}_2N_\theta, F' \circ F'', u, p, \|\cdot, \dots, \cdot\|, X_s]_0^\mathcal{F}$,
- (iii) $[{}_2N_\theta, F', u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{F} \cap [{}_2N_\theta, F'', u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{F} \subseteq [{}_2N_\theta, F' + F'', u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{F}$,
- (iv) $[{}_2N_\theta, F', u, p, \|\cdot, \dots, \cdot\|, X_s]_0^\mathcal{F} \cap [{}_2N_\theta, F'', u, p, \|\cdot, \dots, \cdot\|, X_s]_0^\mathcal{F} \subseteq [{}_2N_\theta, F' + F'', u, p, \|\cdot, \dots, \cdot\|, X_s]_0^\mathcal{F}$.

Proof. (i) For some $\epsilon > 0$, we choose $\epsilon_0 > 0$ such that $\max \epsilon_0^H < \epsilon$. Now as F' is a sequence of modulus functions

which are always continuous, we can choose $\delta \in (0, 1)$ such that, for every $t \in (0, \delta)$, we get $F'(t) < \epsilon_0$:

$$x = (x_{ij}) \in [{}_2N_\theta, F'', u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{F}. \quad (29)$$

Then, by the definition, we have

$$\begin{aligned} A = \left\{ (r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \cdot \sum_{(i,j) \in I_{r,s}} u_{ij} [F''(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \geq \delta^H \right\} \in \mathcal{F}. \end{aligned} \quad (30)$$

Thus, for $(r, s) \notin A$, we get

$$\begin{aligned} & \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [F''(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} < \delta^H, \\ & \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [F''(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \leq \delta. \end{aligned} \quad (31)$$

Now, by the continuity of F' , we have

$$\begin{aligned} & F' \left[\frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [F''(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \right] \\ & < \epsilon, \end{aligned} \quad (32)$$

which further implies that

$$\begin{aligned} & \left[\frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [F' (F''(\|x - L, z_1, z_2, \dots, z_{n-1}\|))]^{p_{ij}} \right] \\ & < \max \epsilon^H < \epsilon, \end{aligned} \quad (33)$$

which implies

$$\begin{aligned} & \left\{ (r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \cdot \sum_{(i,j) \in I_{r,s}} u_{ij} [F' \circ F''(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \geq \epsilon \right\} \in \mathcal{F}. \end{aligned} \quad (34)$$

Therefore $x \in [{}_2N_\theta, F' \circ F'', u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{F}$. This completes the proof. $\square$

(iii) Again consider

$$\begin{aligned} & x \in [{}_2N_\theta, F', u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{F} \\ & \cap [{}_2N_\theta, F'', u, p, \|\cdot, \dots, \cdot\|, X_s]^\mathcal{F}. \end{aligned} \quad (35)$$

Then by the definition of both the spaces, we get

$$\begin{aligned} \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [F'(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} &< \epsilon, \\ \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [F''(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} &< \epsilon. \end{aligned} \quad (36)$$

Using the fact that $(F' + F'')(x) \leq KF'(x) + KF''(x)$, we have

$$\begin{aligned} \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [(F' + F'')(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \\ \leq K \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [F'(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \\ + K \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [F''(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \\ \leq K(\epsilon + \epsilon) = 2K\epsilon = \epsilon' \text{ (say)}. \end{aligned} \quad (37)$$

So we get

$$\left\{ (r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} u_{ij} [(F' + F'')(\|x - L, z_1, z_2, \dots, z_{n-1}\|)]^{p_{ij}} \geq \epsilon \right\} \in \mathcal{F}. \quad (38)$$

Therefore we have

$$x \in [{}_2N_\theta, F' + F'', u, p, \|\cdot, \dots, \cdot\|, X_S]^\mathcal{F}. \quad (39)$$

Hence we have

$$\begin{aligned} [{}_2N_\theta, F', u, p, \|\cdot, \dots, \cdot\|, X_S]^I \\ \cap [{}_2N_\theta, F'', u, p, \|\cdot, \dots, \cdot\|, X_S]^\mathcal{F} \\ \subseteq [{}_2N_\theta, F' + F'', u, p, \|\cdot, \dots, \cdot\|, X_S]^\mathcal{F}. \end{aligned} \quad (40)$$

4. Conclusion

A detailed study of some new classes of I -convergent double sequences over n -normed spaces has been done. Some algebraic and topological properties and inclusion relations have been proved with supported examples.

Competing Interests

The author declares that he has no competing interests.

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