Research Article

Existence of Generalized Homoclinic Solutions of Lotka-Volterra System under a Small Perturbation

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This paper investigates Lotka-Volterra system under a small perturbation $v_{xx} = -\mu(1 - a_2u - v)v + \epsilon f(\epsilon, v, v_x, u, u_x)$, $u_{xx} = -(1 - u - a_1v)u + \epsilon g(\epsilon, v, v_x, u, u_x)$. By the Fourier series expansion technique method, the fixed point theorem, the perturbation theorem, and the reversibility, we prove that near $\mu = 0$ the system has a generalized homoclinic solution exponentially approaching a periodic solution.

1. Introduction

The Lotka-Volterra system introduced by Lotka [1] and Volterra [2] is frequently used to describe the dynamics of biological systems and particularly plays an important role in studying population dynamics. Another reason for the importance of such system is that it has been successfully used in economic theory [3]. Many authors have studied the existence and uniqueness of stationary solution for a variety of Lotka-Volterra equations (e.g., Dancer [4] and Gui and Lou [5]). In recent years, it has been broadly studied in many aspects. Stability was discussed in Sakthivel et al. [6]. Travelling wave solutions were studied by Hosono [7] and Lin and Ruan [8]. The multiple periodic solutions were obtained in Wang et al. [9]. The bifurcation structures are given by Kan [10]. Limit cycles bifurcations were discussed by Wu and Liang [11]. The cyclicity of period annulus was analyzed in Li and Llibre [12].

In this paper, we will study the most well-known Lotka-Volterra system

$$\begin{aligned} v_{xx} &= -\mu(1 - a_2u - v)v, \\ u_{xx} &= -(1 - u - a_1v)u, \end{aligned} \quad (1)$$

where a_1, a_2 , and μ are constants and satisfy $\mu > 0, a_1 < 1 < a_2$.

Motivated by [13, 14], we will study the existence of generalized homoclinic solutions of this Lotka-Volterra system. Since the real world always has some small noises or disturbances, here we are specially interested in the solutions of system (1) under small perturbations; that is, we will focus on the following system:

$$\begin{aligned} v_{xx} &= -\mu(1 - a_2u - v)v + \epsilon f(\epsilon, v, v_x, u, u_x), \\ u_{xx} &= -(1 - u - a_1v)u + \epsilon g(\epsilon, v, v_x, u, u_x), \end{aligned} \quad (2)$$

where ϵ is a small parameter and the general nonlinear terms f and g are smooth real functions. When $\epsilon = 0$ this system has four equilibriums $(0, 0, 0, 0)$, $(0, 0, 1, 0)$, $(1, 0, 0, 0)$, and $((a_2 - 1)/(a_1a_2 - 1), 0, (a_1 - 1)/(a_1a_2 - 1), 0)$. When $\mu < 0$ the linearized operator of (2) at $(1, 0, 0, 0)^T$ has two pairs of purely imaginary eigenvalues. When $\mu = 0$ the linearized operator of the system at $(1, 0, 0, 0)^T$ has double real eigenvalues 0 and a pair of purely imaginary eigenvalues. When $\mu > 0$ the linearized operator at $(1, 0, 0, 0)^T$ has a positive eigenvalue, a negative eigenvalue, and a pair of purely imaginary eigenvalues. This yields that the linearized operator of system (2) has a saddle-center near $\mu = 0$ while $\mu > 0$, which means $\mu = 0$ is the bifurcation point for system (2). We guess there might exist a homoclinic solution exponentially approaching a periodic solution (called generalized homoclinic solution) near $\mu = 0$. Using the dynamical system approach, we will

rigorously prove the existence of a generalized homoclinic solution (homoclinic solution exponentially tending to a small periodic solution at infinity) for $\mu > 0$ but close to 0, which corresponds to a generalized solitary wave solution of (2).

This paper is organized as follows. Section 2 derives the homoclinic solution of the dominant system of (2), which exponentially approaches zero at infinity. Section 3 shows that the system of (2) has a periodic solution. In Section 4, the fixed point theorem and the perturbation method yield that when higher order terms are added, the homoclinic solution of the dominant system deforms to a homoclinic solution exponentially approaching the periodic solution obtained in Section 3. This gives the existence of a generalized homoclinic solution of (2).

2. Preliminary

Let $v_1 = v_x$ and $u_1 = u_x$, and we change (2) into

$$\begin{aligned} v_x &= v_1, \\ v_{1x} &= -\mu v + \mu v^2 + a_2 \mu u v + \epsilon f(\epsilon, v, v_1, u, u_1), \\ u_x &= u_1, \\ u_{1x} &= -u + u^2 + a_1 u v + \epsilon g(\epsilon, v, v_1, u, u_1). \end{aligned} \quad (3)$$

In this paper, we assume that f and g satisfy

$$\begin{aligned} f(\epsilon, 0, 0, u, u_1) &= 0, \\ f(\epsilon, v, -v_1, u, -u_1) &= f(\epsilon, v, v_1, u, u_1), \\ g(\epsilon, 0, 0, 0, 0) &= 0, \\ g(\epsilon, v, -v_1, u, -u_1) &= g(\epsilon, v, v_1, u, u_1). \end{aligned} \quad (4)$$

Under assumption (4), system (3) is reversible with a reverser S defined by

$$S(v, v_1, u, u_1) = (v, -v_1, u, -u_1); \quad (5)$$

that is, $S(v, v_1, u, u_1)(-x)$ is also a solution whenever $(v, v_1, u, u_1)(x)$ is a solution of (3). A solution $(v, v_1, u, u_1)(x)$ is reversible if $S(v, v_1, u, u_1)(-x) = (v, v_1, u, u_1)(x)$. This means that $v(x)$ and $u(x)$ are even functions and $v_1(x)$ and $u_1(x)$ are odd functions. The reversibility will play an important role in the existence of the generalized homoclinic solution.

Let

$$\begin{aligned} v &= w + 1, \\ v_1 &= w_1, \\ \tilde{f}(\epsilon, w, w_1, u, u_1) &= f(\epsilon, w + 1, w_1, u, u_1), \\ \tilde{g}(\epsilon, w, w_1, u, u_1) &= g(\epsilon, w + 1, w_1, u, u_1) \end{aligned} \quad (6)$$

and we have from (4)

$$\begin{aligned} \tilde{f}(\epsilon, w, -w_1, u, -u_1) &= \tilde{f}(\epsilon, w, w_1, u, u_1), \\ \tilde{g}(\epsilon, w, -w_1, u, -u_1) &= \tilde{g}(\epsilon, w, w_1, u, u_1). \end{aligned} \quad (7)$$

Note that (6) changes system (3) into

$$\begin{aligned} w_x &= w_1, \\ w_{1x} &= \mu w + \mu w^2 + a_2 \mu u (w + 1) + \epsilon \tilde{f}(\epsilon, w, w_1, u, u_1), \\ u_x &= u_1, \\ u_{1x} &= (a_1 - 1)u + u^2 + a_1 u w + \epsilon \tilde{g}(\epsilon, w, w_1, u, u_1). \end{aligned} \quad (8)$$

System (8) can be written as

$$\frac{dW}{dx} = LW + N_1(W) + N_2(W) + \epsilon R(\epsilon, W), \quad (9)$$

where $W = (w, w_1, u, u_1)^T$ and

$$\begin{aligned} L &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_1 - 1 & 0 \end{pmatrix}, \\ N_1(W) &= \begin{pmatrix} 0 \\ \mu w^2 \\ 0 \\ 0 \end{pmatrix}, \\ N_2(W) &= \begin{pmatrix} 0 \\ a_2 \mu u (w + 1) \\ 0 \\ u^2 + a_1 u w \end{pmatrix}, \\ R(\epsilon, W) &= \begin{pmatrix} 0 \\ \tilde{f}(\epsilon, w, w_1, u, u_1) \\ 0 \\ \tilde{g}(\epsilon, w, w_1, u, u_1) \end{pmatrix}. \end{aligned} \quad (10)$$

From (7), system (9) is still reversible with the reverser S if

$$S(w, w_1, u, u_1) = (w, -w_1, u, -u_1). \quad (11)$$

We write the dominant system of (9) for small u and u_1 as

$$\frac{dW}{dx} = LW + N_1(W), \quad (12)$$

which has a homoclinic solution $H(x)$ given by

$$\begin{aligned} H(x) &= \left(-\frac{3}{2} \operatorname{sech}^2 \left(\frac{\sqrt{\mu}}{2} x \right), \frac{3\sqrt{\mu}}{2} \operatorname{sech}^2 \left(\frac{\sqrt{\mu}}{2} x \right) \right. \\ &\quad \left. \cdot \tanh \left(\frac{\sqrt{\mu}}{2} x \right), 0, 0 \right)^T \end{aligned} \quad (13)$$

approaching zero as $x \rightarrow \pm\infty$. Moreover

$$\begin{aligned} SH(-x) &= H(x), \\ H(0) &= \left(-\frac{3}{2}, 0, 0, 0 \right)^T \end{aligned} \quad (14)$$

and $H(x)$ satisfies the following inequality:

$$|H(x)| \leq Me^{-\sqrt{\mu}|x|} \quad \text{for any } x \in (-\infty, \infty), \quad (15)$$

where M is a positive constant.

In Section 4, we will prove the deformation of the homoclinic solution $H(x)$ for whole system (9). This demonstrates that system (3) at $(1, 0, 0, 0)^T$ has a generalized homoclinic solution.

3. Periodic Solutions

By using the Fourier series expansion technique and the fixed point theorem, we will prove that (9) has a periodic solution which determines the forms of the generalized homoclinic solutions at infinity. Let

$$\begin{aligned} C &= \sqrt{1-a_1}u - iu_1, \\ \tau &= \sqrt{1-a_1}(1+r_1)x, \end{aligned} \quad (16)$$

where r_1 is a small real constant to be determined later. Then system (8) is changed into

$$\begin{aligned} w_\tau &= \frac{1}{\sqrt{1-a_1}(1+r_1)}w_1, \\ w_{1\tau} &= \frac{\mu}{\sqrt{1-a_1}(1+r_1)}w + h_1(\mu, \epsilon, w, w_1, C, \bar{C}), \\ C_\tau &= \frac{i}{1+r_1}C + h_2(\mu, \epsilon, w, w_1, C, \bar{C}), \\ \bar{C}_\tau &= -\frac{i}{1+r_1}\bar{C} - h_2(\mu, \epsilon, w, w_1, C, \bar{C}), \end{aligned} \quad (17)$$

where h_1 is a real function and h_2 is a purely imaginary function and

$$\begin{aligned} h_1(\mu, \epsilon, w, w_1, C, \bar{C}) &= \frac{1}{\sqrt{1-a_1}(1+r_1)} \left[\mu w^2 \right. \\ &\quad + \frac{a_2\mu}{2\sqrt{1-a_1}}(C+\bar{C})(w+1) \\ &\quad \left. + \epsilon \tilde{f}\left(\epsilon, w, w_1, \frac{C+\bar{C}}{2\sqrt{1-a_1}}, \frac{(C-\bar{C})i}{2}\right) \right], \\ h_2(\mu, \epsilon, w, w_1, C, \bar{C}) &= -\frac{i}{\sqrt{1-a_1}(1+r_1)} \left[\frac{(C+\bar{C})^2}{4(1-a_1)} + \frac{a_1(C+\bar{C})w}{2\sqrt{1-a_1}} \right. \\ &\quad \left. + \epsilon \tilde{g}\left(\epsilon, w, w_1, \frac{C+\bar{C}}{2\sqrt{1-a_1}}, \frac{(C-\bar{C})i}{2}\right) \right]. \end{aligned} \quad (18)$$

From (7) and (11), we may define

$$S(w, w_1, C, \bar{C}) = (w, -w_1, \bar{C}, C), \quad (19)$$

such that system (17) is reversible where we avoid again the introduction of a new notation.

Assume that

$$\begin{aligned} w(\tau) &= \sum_n w_n e^{in\tau}, \\ w_1(\tau) &= \sum_n w_{1,n} e^{in\tau}, \\ C(\tau) &= \sum_n C_n e^{in\tau}, \\ \bar{C}(\tau) &= \sum_n \bar{C}_n e^{-in\tau}. \end{aligned} \quad (20)$$

Plugging (20) into (17) and making the coefficient of each term in the Fourier series equal yield

$$\begin{aligned} w_n &= -\frac{\sqrt{1-a_1}(1+r_1)}{n^2(1-a_1)(1+r_1)^2 + \mu} \left[h_1(\mu, \epsilon, w, w_1, C, \bar{C}) \right]_n, \\ w_{1,n} &= -\frac{i(1-a_1)(1+r_1)^2 n}{n^2(1-a_1)(1+r_1)^2 + \mu} \left[h_1(\mu, \epsilon, w, w_1, C, \bar{C}) \right]_n, \\ C_n &= -\frac{i(1+r_1)}{n(1+r_1)-1} \left[h_2(\mu, \epsilon, w, w_1, C, \bar{C}) \right]_n, \\ \bar{C}_n &= -\frac{i(1+r_1)}{n(1+r_1)-1} \left[h_2(\mu, \epsilon, w, w_1, C, \bar{C}) \right]_{-n}, \end{aligned} \quad (21)$$

$n \neq 1,$
 $n \neq 1,$

and for $n = 1$

$$\begin{aligned} r_1 C_1 &= -i(1+r_1) \left[h_2(\mu, \epsilon, w, w_1, C, \bar{C}) \right]_1, \\ r_1 \bar{C}_1 &= -i(1+r_1) \left[h_2(\mu, \epsilon, w, w_1, C, \bar{C}) \right]_{-1}, \end{aligned} \quad (22)$$

where $[h]_k$ denotes the k th Fourier coefficient of h .

Now we activate C_1 and assume $|C_1| = I$. We first solve (21) for $w_n, w_{1,n}, C_n$, and $\bar{C}_n$ ($n \neq 1$) and then solve (22) for r_1 .

Let $H^m(0, 2\pi)$ be a space of periodic functions of τ with a period 2π such that their derivatives up to order m are in $L^m(0, 2\pi)$; the norm is denoted by $\|\cdot\|_m$. Fix C_1 and define two spaces

$$\begin{aligned} H_1^1(0, 2\pi) &= \left\{ f(\tau) = \sum_n f_n e^{in\tau} \in H^1(0, 2\pi) \mid f_1 = 0 \right\}, \\ H_{-1}^1(0, 2\pi) &= \left\{ f(\tau) = \sum_n f_n e^{in\tau} \in H^1(0, 2\pi) \mid f_{-1} = 0 \right\}. \end{aligned} \quad (23)$$

For $A, B \in H^1(0, 2\pi) \times H^1(0, 2\pi)$ and $D \in H_1^1(0, 2\pi)$, use (21) and we define a mapping $\Theta(A, B, D, \bar{D}, \alpha)$ from

$H^1(0, 2\pi) \times H^1(0, 2\pi) \times H_1^1(0, 2\pi) \times H_{-1}^1(0, 2\pi)$ to itself by

$$\Theta(A, B, D, \bar{D}, \alpha) = \begin{pmatrix} \sum_n \frac{-\sqrt{1-a_1}(1+r_1)}{n^2(1-a_1)(1+r_1)^2 + \mu} [h_1(\mu, \epsilon, w, w_1, C, \bar{C})]_n e^{in\tau} \\ \sum_n \frac{-in(1-a_1)(1+r_1)^2}{n^2(1-a_1)(1+r_1)^2 + \mu} [h_1(\mu, \epsilon, w, w_1, C, \bar{C})]_n e^{in\tau} \\ \sum_{n \neq 1} \frac{-i(1+r_1)}{n(1+r_1)-1} [h_2(\mu, \epsilon, w, w_1, C, \bar{C})]_n e^{in\tau} \\ \sum_{n \neq 1} \frac{-i(1+r_1)}{n(1+r_1)-1} [h_2(\mu, \epsilon, w, w_1, C, \bar{C})]_{-n} e^{-in\tau} \end{pmatrix}, \quad (24)$$

where $\alpha = (\mu, \epsilon, r_1, C_1, \bar{C}_1)$. Assume that $B_r(0)$ is a ball with a radius r in the space $H^1(0, 2\pi) \times H^1(0, 2\pi) \times H_1^1(0, 2\pi) \times H_{-1}^1(0, 2\pi)$; we have the following lemma.

Lemma 1. For $(A, B, D, \bar{D}), (A_1, B_1, D_1, \bar{D}_1), (A_2, B_2, D_2, \bar{D}_2) \in \bar{B}_r(0)$ and any small bounded α and r , Θ is smooth in its arguments and satisfies

$$\begin{aligned} \|\Theta(A, B, D, \bar{D}; \alpha)\|_1 &\leq M(\mu + |\epsilon| + \|A\|_1^2 + \|D\|_1^2 \\ &\quad + |C_1|^2), \\ \|\Theta(A_1, B_1, D_1, \bar{D}_1; \alpha) - \Theta(A_2, B_2, D_2, \bar{D}_2; \alpha)\|_1 &\leq M(\mu + |\epsilon| + |C_1| + \|A_1\|_1 + \|A_2\|_1 + \|D_1\|_1 \\ &\quad + \|D_2\|_1)(\|A_1 - A_2\|_1 + \|D_1 - D_2\|_1 \\ &\quad + \|B_1 - B_2\|_1). \end{aligned} \quad (25)$$

Take $r = |C_1|$ and

$$\begin{aligned} \mu &= \mu_1 I^{\beta_1}, \\ \epsilon &= \epsilon_1 I^{\beta_2}, \\ \beta_1 &> 1, \\ \beta_2 &> 1, \end{aligned} \quad (26)$$

where $\mu_1, \epsilon_1, \beta_1$, and β_2 are fixed constants; Lemma 1 yields that Θ is a contraction mapping on $\bar{B}_r(0)$ for small I . Thus, Θ has a unique fixed point which is a smooth function of α ; write this fixed point as

$$(w_p^0, w_{1p}^0, C_p^0, \bar{C}_p^0)(\mu, \epsilon, r_1, C_1, \bar{C}_1)(\tau) \quad (27)$$

which satisfies

$$\|w_p^0\|_1 + \|w_{1p}^0\|_1 + \|C_p^0\|_1 + \|\bar{C}_p^0\|_1 \leq M(\mu + |\epsilon| + I^2). \quad (28)$$

Using the same argument we can show that (27) is in $H^m(0, 2\pi)$ and satisfies (28) with $H^m(0, 2\pi)$ -norm for any integer $m > 0$. We use $(w_p^0, w_{1p}^0, C_p^0, \bar{C}_p^0)(\tau)$ to denote

$$(w_p^0(\tau), w_{1p}^0(\tau), C_p^0(\tau) + C_1 e^{i\tau}, \bar{C}_p^0(\tau) + \bar{C}_1 e^{-i\tau}). \quad (29)$$

Now we solve (22) for r_1 . Substitute (27) into (22) and obtain

$$r_1 = g_1(\mu, \epsilon, r_1, C_1, \bar{C}_1), \quad (30)$$

where

$$\begin{aligned} g_1(\mu, \epsilon, r_1, C_1, \bar{C}_1) &= \frac{i(1+r_1)[h_2(\mu, \epsilon, w, w_1, C, \bar{C})]_1}{C_1} \end{aligned} \quad (31)$$

is smooth when $\mu, \epsilon, r_1, C_1, \bar{C}_1$ are near 0. According to (26) and (18) we get $|g_1| \leq M(\mu + |\epsilon| + I)$, so g_1 is a contraction mapping. By the fixed point theorem, g_1 has a unique fixed point

$$r_1 = r_1(\mu, \epsilon, I) \quad (32)$$

as a smooth real function for small (μ, ϵ, I) satisfying

$$|r_1| \leq M(\mu + |\epsilon| + I). \quad (33)$$

Therefore, (17) has a periodic solution

$$\begin{aligned} &(w_p(\mu, \epsilon, I)(\tau), w_{1p}(\mu, \epsilon, I)(\tau), C_p(\mu, \epsilon, I) \\ &\quad \cdot (\tau), \bar{C}_p(\mu, \epsilon, I)(\tau)) \end{aligned} \quad (34)$$

in $H^m(0, 2\pi)$ if $I \in (0, I_1]$ and (26) holds where I_1 is a fixed small positive constant.

By the relation $\tau = \sqrt{1-a_1}(1+r_1)x$, we write the periodic solution $(w_p, w_{1p}, C_p, \bar{C}_p)(\tau)$ as $(w_p(\mu, \epsilon, I)(x), w_{1p}(\mu, \epsilon, I)(x), C_p(\mu, \epsilon, I)(x), \bar{C}_p(\mu, \epsilon, I)(x))$ with the frequency

$$w_1(\mu, \epsilon, I) = \sqrt{1-a_1}(1+r_1)(\mu, \epsilon, I) \quad (35)$$

for $I \in (0, I_1]$. Letting $C_p = \sqrt{1 - a_1} u_p - i u_{1p}$, we have

$$\begin{aligned} u_p(-x) &= u_p(x), \\ u_{1p}(-x) &= -u_{1p}(x). \end{aligned} \quad (36)$$

Define

$$\begin{aligned} X_{\mu, \epsilon, I}(x) &= (w_p, w_{1p}, u_p, u_{1p})^T(x) \\ &= \left(w_p, w_{1p}, \frac{C_p + \bar{C}_p}{2\sqrt{1 - a_1}}, i \frac{C_p - \bar{C}_p}{2} \right)^T(x), \end{aligned} \quad (37)$$

which is smooth for x and small (μ, ϵ, I) with condition (26). Then, $X_{\mu, \epsilon, I}(x)$ is a reversible periodic solution of (9) under the reserve S with frequency $w_1(\mu, \epsilon, I)$ which from (28) satisfies that for any integer $m > 0$

$$\|X_{\mu, \epsilon, I}(x)\|_m \leq M(\mu + |\epsilon| + I). \quad (38)$$

The Sobolev embedding theorem gives that (38) holds in $C_B^m(\mathbf{R})$ -norm, which is a space of continuously differentiable functions up to the order m with a supreme norm.

4. Existence of Generalized Homoclinic Solutions

In this section we demonstrate that (9) has a generalized homoclinic solution exponentially approaching the periodic solution $X_{\mu, \epsilon, I}$ obtained in Section 3.

Theorem 2. Suppose that assumption (4) holds. There exist constants $I_0 > 0$, $\mu_1 > 0$, and ϵ_1 such that, for $I \in (0, I_0]$, if the small parameters $\mu = \mu_1 I^{3/2}$ and $\epsilon = \epsilon_1 I^{3/2}$, then (3) has a generalized homoclinic solution which is reversible and exponentially approaches the periodic solutions $(1, 0, 0, 0)^T + X_{\mu, \epsilon, I}(x + \theta)$ as $x \rightarrow \infty$ and $(1, 0, 0, 0)^T + SX_{\mu, \epsilon, I}(-x + \theta)$ as $x \rightarrow -\infty$, where the phase shift θ is a continuous function in I and the operator S is defined in (5).

We divide the proof into two steps. Using the relationship between (3) and (9), we will first prove that (9) has a solution for $x \in [0, \infty)$, which exponentially approaches the periodic solution $X_{\mu, \epsilon, I}$ for some phase shift θ as $x \rightarrow \infty$. Then we solve for θ as a function of μ , ϵ , and I . This yields that this solution can be extended to $x \in (-\infty, 0]$ by using the reversibility.

Step 1 (existence of generalized homoclinic solution for $x \in [0, \infty)$). Assume that the solution $U(x)$ of (9) has the following form:

$$U(x) = H(x) + Z(x) + \zeta(x) X_{\mu, \epsilon, I}(x + \theta), \quad (39)$$

where $H(x)$ and $X_{\mu, \epsilon, I}(x + \theta)$ are defined in (13) and (37), respectively, the phase shift $\theta \in S^1 = [-\pi, \pi]$ is a constant, and the cut-off function $\zeta(x)$ is in $C^\infty(\mathbf{R}, \mathbf{R})$ satisfying $0 \leq \zeta(x) \leq 1$ and

$$\zeta(x) = \begin{cases} 1 & |x| \geq 2, \\ 0 & |x| \leq 1, \end{cases} \quad (40)$$

where $Z(x)$ is a perturbation term to be determined, which exponentially tends to 0 as $x \rightarrow \infty$ so that $U(x)$ is a solution of (9) that approaches the periodic solution $X_{\mu, \epsilon, I}(x + \theta)$ as $x \rightarrow \infty$. Plugging (39) into (9) yields

$$\frac{dZ}{dx} = \mathcal{L}(x)Z + \tilde{N}(x, z) + \epsilon \tilde{R}(x, \epsilon, z), \quad (41)$$

where

$$\begin{aligned} \mathcal{L}(x) &= L + dN_1[H(x)] \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ \mu - 3\mu \operatorname{sech}^2\left(\frac{\sqrt{\mu}}{2}x\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & a_1 - 1 & 0 \end{pmatrix}, \end{aligned}$$

$$\tilde{N}(x, z)$$

$$\begin{aligned} &= N_1(H(x) + z(x) + \zeta(x) X_{\mu, \epsilon, I}(x + \theta)) \\ &\quad - N_1(H(x)) - dN_1[H(x)]Z(x) \\ &\quad + N_2(H(x) + z(x) + \zeta(x) X_{\mu, \epsilon, I}(x + \theta)) \\ &\quad - \zeta(x) N_2(X_{\mu, \epsilon, I}(x + \theta)) \\ &\quad - \zeta(x) N_1(X_{\mu, \epsilon, I}(x + \theta)), \end{aligned} \quad (42)$$

$$\tilde{R}(x, \epsilon, z)$$

$$\begin{aligned} &= R(\epsilon, H(x) + z(x) + \zeta(x) X_{\mu, \epsilon, I}(x + \theta)) \\ &\quad - \zeta(x) R(\epsilon, X_{\mu, \epsilon, I}(x + \theta)) \\ &\quad - \frac{1}{\epsilon} \zeta'(x) X_{\mu, \epsilon, I}(x + \theta), \end{aligned}$$

and d means taking the Fréchet derivative.

By (15) and (38) we can obtain the following lemma.

Lemma 3. If μ , ϵ , and I are small and $|Z| + |Z_1| + |Z_2| \leq M_0$ for some positive constant M_0 , then $\tilde{N}$ and $\tilde{R}$ satisfy for $x \geq 0$

$$\begin{aligned} |\tilde{N}(x, Z)| &\leq M \left[(e^{-\sqrt{\mu}x} + |Z|)(\mu + |\epsilon| + I) + |Z|^2 \right], \\ |\tilde{N}(x, Z_2) - \tilde{N}(x, Z_1)| &\leq M [\mu + |\epsilon| + I + |Z_1| + |Z_2|] |Z_2 - Z_1|, \\ |\tilde{R}(x, \epsilon, Z)| &\leq M \left[e^{-\sqrt{\mu}x} + |Z| + \frac{1}{|\epsilon|} (\mu + |\epsilon| + I) e^{-\sqrt{\mu}x} \right], \\ |\tilde{R}(x, \epsilon, Z_2) - \tilde{R}(x, \epsilon, Z_1)| &\leq M |Z_2 - Z_1|. \end{aligned} \quad (43)$$

Lemma 4. The solution of (39) that decays to zero at infinity can be found as

$$\begin{aligned} Z &= \mathcal{F}(Z) \\ &\triangleq \int_0^x \langle \tilde{N}(t, Z) + \epsilon \tilde{R}(t, \epsilon, Z), s_1^*(t) \rangle dt s_1(x) \\ &\quad - \sum_{j=2}^4 \int_x^\infty \langle \tilde{N}(t, Z) + \epsilon \tilde{R}(t, \epsilon, Z), s_j^*(t) \rangle dt s_j(x). \end{aligned} \quad (44)$$

Proof.

$$\frac{dZ(x)}{dx} = \mathcal{L}(x) Z(x) \quad (45)$$

has four linearly independent solutions

$$\begin{aligned} s_1(x) &= \left\{ \frac{3\sqrt{\mu}}{2} \operatorname{sech}^2\left(\frac{\sqrt{\mu}}{2}x\right) \tanh\left(\frac{\sqrt{\mu}}{2}x\right), -\frac{3\mu}{2} \right. \\ &\quad \cdot \operatorname{sech}^2\left(\frac{\sqrt{\mu}}{2}x\right) + \frac{3\mu}{4} \operatorname{sech}^4\left(\frac{\sqrt{\mu}}{2}x\right), 0, 0 \left. \right\}^T, \\ s_2(x) &= \left\{ \frac{4}{3\mu} \operatorname{sech}^2\left(\frac{\sqrt{\mu}}{2}x\right) \tanh\left(\frac{\sqrt{\mu}}{2}x\right) \left(\frac{15\sqrt{\mu}}{16}x \right. \right. \\ &\quad \left. \left. - \coth\frac{\sqrt{\mu}}{2}x + \frac{1}{2} \sinh(\sqrt{\mu}x) + \frac{1}{32} \sinh(2\sqrt{\mu}x) \right), \right. \\ &\quad \left. -\frac{2\sqrt{\mu}}{3\mu} \left(2 \operatorname{sech}^2\left(\frac{\sqrt{\mu}}{2}x\right) - 3 \operatorname{sech}^4\left(\frac{\sqrt{\mu}}{2}x\right) \right) \right. \\ &\quad \cdot \left(\frac{15\sqrt{\mu}}{16}x - \coth\frac{\sqrt{\mu}}{2}x + \frac{1}{2} \sinh(\sqrt{\mu}x) \right. \\ &\quad \left. \left. + \frac{1}{32} \sinh(2\sqrt{\mu}x) \right) + \frac{2\sqrt{\mu}}{3\mu} \operatorname{sech}^2\left(\frac{\sqrt{\mu}}{2}x\right) \right. \\ &\quad \cdot \tanh\left(\frac{\sqrt{\mu}}{2}x\right) \left(\frac{15}{8} + \coth^2\left(\frac{\sqrt{\mu}}{2}x\right) \right. \\ &\quad \left. \left. + \cosh(\sqrt{\mu}x) + \frac{1}{8} \cosh(2\sqrt{\mu}x) \right), 0, 0 \right\}^T, \\ s_3(x) &= \left\{ 0, 0, \cos(\sqrt{1-a_1}x), -\sqrt{1-a_1} \right. \\ &\quad \cdot \sin(\sqrt{1-a_1}x) \left. \right\}^T, \\ s_4(x) &= \left\{ 0, 0, \sin(\sqrt{1-a_1}x), \sqrt{1-a_1} \right. \\ &\quad \cdot \cos(\sqrt{1-a_1}x) \left. \right\}^T, \end{aligned} \quad (46)$$

which satisfy

$$\begin{aligned} |s_1(x)| &\leq Me^{-\sqrt{\mu}x}, \\ |s_2(x)| &\leq Me^{\sqrt{\mu}x}, \\ |s_3(x)| + |s_4(x)| &\leq M, \end{aligned} \quad (47)$$

for $x \in [0, \infty)$, and

$$\begin{aligned} s_1(0) &= \left\{ 0, -\frac{3\mu}{4}, 0, 0 \right\}^T, \\ s_2(0) &= \left\{ \frac{4}{3\mu}, 0, 0, 0 \right\}^T, \\ s_3(0) &= \{0, 0, 1, 0\}^T, \\ s_4(0) &= \{0, 0, 0, \sqrt{1-a_1}\}^T. \end{aligned} \quad (48)$$

The adjoint equation of (45) has four linearly independent solutions given by

$$\begin{aligned} s_1^*(x) &= \left\{ -\frac{2\sqrt{\mu}}{3\mu} \left(2 \operatorname{sech}^2\left(\frac{\sqrt{\mu}}{2}x\right) \right. \right. \\ &\quad \left. \left. - 3 \operatorname{sech}^4\left(\frac{\sqrt{\mu}}{2}x\right) \right) \left(\frac{15\sqrt{\mu}}{16}x - \coth\frac{\sqrt{\mu}}{2}x \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \sinh(\sqrt{\mu}x) + \frac{1}{32} \sinh(2\sqrt{\mu}x) \right) + \frac{2\sqrt{\mu}}{3\mu} \right. \\ &\quad \cdot \operatorname{sech}^2\left(\frac{\sqrt{\mu}}{2}x\right) \tanh\left(\frac{\sqrt{\mu}}{2}x\right) \left(\frac{15}{8} \right. \\ &\quad \left. \left. + \coth^2\left(\frac{\sqrt{\mu}}{2}x\right) + \cosh(\sqrt{\mu}x) \right. \right. \\ &\quad \left. \left. + \frac{1}{8} \cosh(2\sqrt{\mu}x) \right), -\frac{4}{3\mu} \operatorname{sech}^2\left(\frac{\sqrt{\mu}}{2}x\right) \right. \\ &\quad \cdot \tanh\left(\frac{\sqrt{\mu}}{2}x\right) \left(\frac{15\sqrt{\mu}}{16}x - \coth\frac{\sqrt{\mu}}{2}x \right. \\ &\quad \left. \left. + \frac{1}{2} \sinh(\sqrt{\mu}x) + \frac{1}{32} \sinh(2\sqrt{\mu}x) \right), 0, 0 \right\}^T, \\ s_2^*(x) &= \left\{ \frac{3\mu}{2} \operatorname{sech}^2\left(\frac{\sqrt{\mu}}{2}x\right) - \frac{9\mu}{4} \operatorname{sech}^4\left(\frac{\sqrt{\mu}}{2}x\right), \right. \\ &\quad \left. -\frac{3\sqrt{\mu}}{2} \operatorname{sech}^2\left(\frac{\sqrt{\mu}}{2}x\right) \tanh\left(\frac{\sqrt{\mu}}{2}x\right), 0, 0 \right\}^T, \\ s_3^*(x) &= \left\{ 0, 0, \cos(\sqrt{1-a_1}x), -\frac{1}{\sqrt{1-a_1}} \right. \\ &\quad \cdot \sin(\sqrt{1-a_1}x) \left. \right\}^T, \\ s_4^*(x) &= \left\{ 0, 0, \sin(\sqrt{1-a_1}x), \frac{1}{\sqrt{1-a_1}} \right. \\ &\quad \cdot \cos(\sqrt{1-a_1}x) \left. \right\}^T, \end{aligned} \quad (49)$$

which satisfy

$$\begin{aligned} |s_1^*(x)| &\leq Me^{\sqrt{\mu}x}, \\ |s_2^*(x)| &\leq Me^{-\sqrt{\mu}x}, \\ |s_3^*(x)| + |s_4^*(x)| &\leq M \end{aligned} \quad (50)$$

for each $x \in [0, \infty)$; we have

$$\begin{aligned} \langle s_i(x), s_j^*(x) \rangle &= 0, \quad i \neq j \\ \langle s_i(x), s_i^*(x) \rangle &= 1 \quad i, j = 1, 2, 3, 4, \end{aligned} \quad (51)$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbf{R}^4$. From the above conditions we can get the expression of Z in Lemma 4. $\square$

Fix $\nu \in (0, \sqrt{\mu})$ and consider (44) as a fixed point problem in a Banach space

$$E_\nu = \left\{ Z \in C([0, \infty) \times S^1) \mid \sup_{x \in [0, \infty)} \{ |Z(x, \theta)| e^{\nu x} \} < \infty \right\} \quad (52)$$

with the norm

$$\|Z\|_\nu = \sup \{ |Z(x, \theta)| e^{\nu x} \mid x \in [0, \infty), \theta \in S^1 \}. \quad (53)$$

It is easy to obtain the following lemma using Lemma 3, (15), (28), and (38).

Lemma 5. *The function $\mathcal{F}$ satisfies*

$$\begin{aligned} \|\mathcal{F}(Z)\|_\nu &\leq M \left[(1 + \|Z\|_\nu) (\mu + |\epsilon| + I) + \|Z\|_\nu^2 \right], \\ \|\mathcal{F}(Z_1) - \mathcal{F}(Z_2)\|_\nu &\leq M [\mu + |\epsilon| + I + \|Z_1\|_\nu + \|Z_2\|_\nu] \|Z_1 - Z_2\|_\nu, \end{aligned} \quad (54)$$

for $Z, Z_1, Z_2 \in E_\nu$.

For any fixed constant $\rho \in (0, 1)$, we let $r = MI^\rho$ and

$$\begin{aligned} \mu &= \mu_1 I^{\beta_1}, \\ \epsilon &= \epsilon_1 I^{\beta_2}, \\ \beta_i &= \rho + \tilde{\beta}_i > 1, \end{aligned} \quad (55)$$

where $\tilde{\beta}_i$ are positive constants for $i = 1, 2$. Thus, (26) is satisfied, and we can show from Lemma 4 that $\mathcal{F}$ is a contraction on $\bar{B}_r(0) \subset E_\nu$ for small I . Therefore, (44) has a unique solution $Z(x, \theta, \mu, \epsilon, I)$ satisfying

$$|Z(x, \theta, \mu, \epsilon, I)| \leq MI^\rho, \quad x \in [0, \infty). \quad (56)$$

Using the same argument as that for (56) and an extension of a contraction principle [15], we can show that Z is smooth in its argument. Thus the solution Z of (41) exists if x is in a finite interval and an initial condition is given, so that $U(x, \mu, \epsilon, \xi, I)$ defined in (39) exists for $x \geq \tilde{x}_0$ with any fixed $\tilde{x}_0 \in (-\infty, \infty)$.

Step 2 (reversible generalized homoclinic solution for $x \in (-\infty, \infty)$). Using (5), (39), and the relationship between v and w in (6), we may define

$$\bar{U}(x) = \begin{cases} U(x; \theta, \mu, \epsilon, I) + U_1 & \text{for } x \geq 0, \\ S(U(-x; \mu, \epsilon, \xi, I)) + SU_1 & \text{for } x \leq 0, \end{cases} \quad (57)$$

where $U_1 = (1, 0, 0, 0)^T$. In this step, we show that system (3) has a reversible generalized homoclinic solution for $x \in (-\infty, +\infty)$. Depending on the reversibility of systems, this problem is equivalent to solving the following equation:

$$(I - S)\bar{U}(0; \theta, \mu, \epsilon, I) = 0 \quad (58)$$

for θ . By (14), the definition of the cut-off function $\varsigma(x)$ in (40), and $W = (w, w_1, u, u_1)^T$, it is easy to check that (58) is equivalent to

$$w_1(0) = 0, \quad (59)$$

$$u_1(0) = 0. \quad (60)$$

Using (48) and (44) we know that (59) holds automatically. Thus we only need study (60) which can be transformed to

$$\int_0^\infty \langle \bar{N}(t, z) + \epsilon \bar{R}(t, \epsilon, z), s_4^*(t) \rangle dt = 0. \quad (61)$$

Lemma 6. *Under the assumptions of Theorem 2, (61) can be transformed to*

$$\theta = I^{1/2} \varphi(\theta, \mu, \epsilon, I), \quad (62)$$

where φ is differentiable with respect to its arguments and φ and its derivative with respect to θ are uniformly bounded for small bounded μ, ϵ , and I .

The proof of Lemma 6 is given in the appendix.

Using the fixed point theorem and Lemma 3, we can solve (62) for θ as a smooth function of μ, ϵ, I , so (60) is true. Uniqueness of the solution for an initial value problem implies that $\bar{U}$ is a generalized homoclinic solution of (3) and $S\bar{U}(-x) = \bar{U}(x)$, which exponentially approaches the periodic solution $(1, 0, 0, 0)^T + X_{\mu, \epsilon, I}(x + \theta)$ as $x \rightarrow \infty$ and the periodic solution $(1, 0, 0, 0)^T + S(X_{\mu, \epsilon, I}(-x + \theta))$ as $x \rightarrow -\infty$. This completes the proof of Theorem 2.

Appendix

In this section, we will give the proof for Lemma 6. Let

$$\begin{aligned} C_p &= \sqrt{1 - a_1} u_p - i u_{1p}, \\ \tau &= \sqrt{1 - a_1} (1 + r_1) x, \end{aligned} \quad (A.1)$$

where r_1, u_p , and u_{1p} are given in (32) and (37), respectively, which yields

$$\begin{aligned} u_p &= \frac{C_p + \bar{C}_p}{2\sqrt{1 - a_1}}, \\ u_{1p} &= \frac{C_p - \bar{C}_p}{2} i. \end{aligned} \quad (A.2)$$

Thus, $(w_p, w_{1p}, C_p, \bar{C}_p)$ is a 2π -periodic solution of the following system:

$$\begin{aligned} w_{p\tau} &= \frac{1}{\sqrt{1-a_1}(1+r_1)} w_{1p}, \\ w_{1p\tau} &= \frac{\eta}{\sqrt{1-a_1}(1+r_1)} w_p \\ &\quad + h_1(\mu, \epsilon, w_p, w_{1p}, C_p, \bar{C}_p), \\ C_{p\tau} &= \frac{i}{1+r_1} C_p + h_2(\mu, \epsilon, w_p, w_{1p}, C_p, \bar{C}_p), \\ \bar{C}_{p\tau} &= -\frac{i}{1+r_1} \bar{C}_p - h_2(\mu, \epsilon, w_p, w_{1p}, C_p, \bar{C}_p), \end{aligned} \quad (\text{A.3})$$

where h_1 and h_2 are given in (18). We can express $C_p(\tau)$ as

$$C_p(\tau) = e^{i(\tau/(1+r_1))} C_p(0) + \varphi(\tau), \quad (\text{A.4})$$

where

$$\varphi(\tau) = \int_0^\tau e^{i((\tau-s)/(1+r_1))} h_2(\mu, \epsilon, w_p, w_{1p}, C_p, \bar{C}_p) ds. \quad (\text{A.5})$$

We know that the coefficient of $e^{i\tau}$ is $C_1 = I$. Thus,

$$\begin{aligned} I &= \frac{1}{2\pi} \int_0^{2\pi} C_p(s) e^{-is} ds \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{-is} \left(e^{i(s/(1+r_1))} C_p(0) + \varphi(s) \right) ds \\ &= (1 + \tilde{\varphi}(r_1)) C_p(0) + \frac{1}{2\pi} \int_0^{2\pi} e^{-is} \varphi(s) ds, \end{aligned} \quad (\text{A.6})$$

where $\tilde{\varphi}(r_1) = ((1+r_1)/i2\pi r_1)(1 - e^{-i(2\pi r_1/(1+r_1))}) - 1 = O(r_1)$ and $\tilde{\varphi}(0) = 0$, which yields

$$C_p(0) = \frac{1}{1 + \tilde{\varphi}(r_1)} \left(I - \frac{1}{2\pi} \int_0^{2\pi} e^{-is} \varphi(s) ds \right). \quad (\text{A.7})$$

So,

$$\begin{aligned} C_p(\tau) &= \frac{e^{i(\tau/(1+r_1))}}{1 + \tilde{\varphi}(r_1)} \left(I - \frac{1}{2\pi} \int_0^{2\pi} e^{-is} \varphi(s) ds \right) \\ &\quad + \varphi(\tau), \end{aligned} \quad (\text{A.8})$$

or

$$\begin{aligned} C_p(x) &= \frac{e^{i\sqrt{1-a_1}x}}{1 + \tilde{\varphi}(r_1)} \left(I - \frac{1}{2\pi} \int_0^{2\pi} e^{-is} \varphi(s) ds \right) \\ &\quad + \varphi(\sqrt{1-a_1}(1+r_1)x). \end{aligned} \quad (\text{A.9})$$

From (33), (38), (56), and the expression of h_2 in (18), we can get $\varphi(x) = O(\mu + |\epsilon| + I^2)$, so that $C_p(x) = O(I)$ for small I . Then we obtain by (A.2)

$$\begin{aligned} u_p(x) &= \frac{1}{\sqrt{1-a_1}} \operatorname{Re} C_p(x) \\ &= \frac{1}{\sqrt{1-a_1}} \cos(\sqrt{1-a_1}x) I \\ &\quad + U_1(x, \mu, \epsilon, I), \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} u_{1p}(x) &= -\operatorname{Im} C_p(x) \\ &= -\sin(\sqrt{1-a_1}x) I + U_2(x, \mu, \epsilon, I), \end{aligned} \quad (\text{A.11})$$

where $U_1(x) = O(\mu + |\epsilon| + I^2)$ and $U_2(x) = O(\mu + |\epsilon| + I^2)$.

By (13), (37), (42), and $W = (w, w_1, u, u_1)^T$, we know that (61) is changed into

$$\begin{aligned} 0 &= \int_0^\infty \frac{1}{\sqrt{1-a_1}} \left((u(s) + \varsigma(s) u_p(s+\theta))^2 \right. \\ &\quad + a_1 (u(s) + \varsigma(s) u_p(s+\theta)) \left(-\frac{3}{2} \operatorname{sech}^2 \left(\frac{\sqrt{\mu}}{2} s \right) \right. \\ &\quad + w(s) + \varsigma(s) w_p(s+\theta) - \varsigma(s) \\ &\quad \cdot (u_p^2(s+\theta) + a_1 u_p(s+\theta) w_p(s+\theta)) \\ &\quad \cdot \cos \sqrt{1-a_1} s \Big) - \varsigma'(s) u_p(s+\theta) \sin(\sqrt{1-a_1} s) \\ &\quad - \frac{1}{\sqrt{1-a_1}} \varsigma'(s) u_{1p}(s+\theta) \Big) \cos(\sqrt{1-a_1} s) \\ &\quad - \frac{1}{\sqrt{1-a_1}} \left(\epsilon \tilde{g} \left(\epsilon, -\frac{3}{2} \operatorname{sech}^2 \left(\frac{\sqrt{\mu}}{2} s \right) + w(s) \right. \right. \\ &\quad + \varsigma(s) w_p(s+\theta), \frac{3\sqrt{\mu}}{2} \operatorname{sech}^2 \left(\frac{\sqrt{\mu}}{2} s \right) \\ &\quad \cdot \tanh \left(\frac{\sqrt{\mu}}{2} s \right) + w_1(s) + \varsigma(s) w_{1p}(s+\theta), u(s) \\ &\quad + \varsigma(s) u_p(s+\theta) \Big), u_1(s) + \varsigma(s) u_{1p}(s+\theta) \Big) \\ &\quad - \varsigma(s) \\ &\quad \cdot \tilde{g}(\epsilon, w_p(s+\theta), w_{1p}(s+\theta), u_p(s+\theta), u_{1p}(s \\ &\quad + \theta)) \cos \sqrt{1-a_1} s ds. \end{aligned} \quad (\text{A.12})$$

For computational simplicity, we take $\rho = \tilde{\beta}_k = 3/4$ for $k = 1, 2$ such that (55) and (56) are satisfied. Thus, (A.12) is changed into

$$\begin{aligned} 0 &= I \int_0^\infty -\frac{1}{\sqrt{1-a_1}} \varsigma'(s) \left(\cos \sqrt{1-a_1}(s+\theta) \right. \\ &\quad \cdot \sin(\sqrt{1-a_1}s) - \sin(\sqrt{1-a_1}(s+\theta)) \\ &\quad \cdot \cos(\sqrt{1-a_1}s) ds + Q(\theta, \mu, \epsilon, I) = \frac{1}{\sqrt{1-a_1}} \\ &\quad \cdot \sin(\sqrt{1-a_1}\theta) I + Q(\theta, \mu, \epsilon, I), \end{aligned} \quad (\text{A.13})$$

where $Q(\theta, \mu, \epsilon, I) = O(I^{3/2})$, which is equivalent to

$$\theta = I^{1/2} \varphi(\mu, \epsilon, \xi, I), \quad (\text{A.14})$$

where $\varphi(\mu, \epsilon, \beta, I) = -(1/\sqrt{(1-a_1)I})\arcsin(\sqrt{1-a_1}Q(\theta, \mu, \epsilon, I)/I)$ is uniformly bounded for small μ , ϵ , and I . We can check that $\varphi(\theta, \mu, \epsilon, I)$ is differentiable with respect to its arguments, and $\varphi(\theta, \mu, \epsilon, I)$ and its derivatives with respect to θ are uniformly bounded for small bounded (μ, ϵ, I) . This completes the proof of Lemma 6.

Competing Interests

The author declares that there are no competing interests regarding the publication of this paper.

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