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Research Article

Blow-Up Phenomena for Nonlinear Reaction-Diffusion Equations under Nonlinear Boundary Conditions

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This paper deals with blow-up and global solutions of the following nonlinear reaction-diffusion equations under nonlinear boundary conditions: $(g(u))_t = \nabla \cdot (a(u)\nabla u) + f(u)$ in $\Omega \times (0,T)$, $\partial u/\partial n = b(x,u,t)$ on $\partial\Omega \times (0,T)$, $u(x,0) = u_0(x) > 0$, in $\overline{\Omega}$, where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$. We obtain the conditions under which the solutions either exist globally or blow up in a finite time by constructing auxiliary functions and using maximum principles. Moreover, the upper estimates of the "blow-up time," the "blow-up rate," and the global solutions are also given.

1. Introduction

During the past few decades, the blow-up phenomena for the nonlinear reaction-diffusion equations have been studied by a large number of authors, and the reader is referred to [1–8] and the references therein. In this paper, we consider the following nonlinear reaction-diffusion problem under nonlinear conditions:

$$\begin{aligned} (g(u))_t &= \nabla \cdot (a(u)\nabla u) + f(u) \quad \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial n} &= b(x,u,t) \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) > 0 \quad \text{in } \overline{\Omega}, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $\partial u/\partial n$ represents the outward normal derivative on $\partial\Omega$, and T is the maximal existence time of u. Set $\mathbb{R}_+ := (0,+\infty)$. We assume, throughout the paper, that $a(s)$ is a positive $C^2(\mathbb{R}_+)$ function, $g(s)$ is a $C^2(\mathbb{R}_+)$ function, $g'(s) > 0$ for any $s \in \mathbb{R}_+$, $f(s)$ is a nonnegative $C^1(\mathbb{R}_+)$ function, $b(x,s,t)$ is a nonnegative $C^1(\partial\Omega \times \mathbb{R}_+ \times \mathbb{R}_+)$ function, and $u_0(x)$ is a positive $C^2(\overline{\mathbb{R}_+})$ function and satisfies the compatibility conditions. Under the above assumptions, the local existence and uniqueness of classical solution of problem (1) were established by Amann [9]. Furthermore, it follows from maximum principle [10] and regularity theorem [11] that the solution $u(x,t)$ is positive and $u(x,t) \in C^3(\Omega \times (0,T)) \cap C^2(\overline{\Omega} \times (0,T))$.

Many authors have investigated blow-up and global solutions of nonlinear reaction-diffusion equations under nonlinear boundary conditions and have obtained a lot of interesting results (see, e.g., [12–20]). To my knowledge, some special cases of (1) have been studied. Zhang [21] considered the following problem:

$$\begin{aligned} (g(u))_t &= \Delta u + f(u) \quad \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial n} &= b(u) \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) > 0 \quad \text{in } \overline{\Omega}, \end{aligned} \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$. By constructing auxiliary functions and using maximum principles, the existence of blow-up and global solutions were obtained under appropriate assumptions on the functions b, f, g, and u_0. Zhang et al. [22] dealt with the following problem:

$$\begin{aligned} (g(u))_t &= \nabla \cdot (a(u)\nabla u) + f(u) \quad \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial n} &= b(u) \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) > 0 \quad \text{in } \overline{\Omega}, \end{aligned} \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$. Some conditions on nonlinearities and the initial data were given to ensure that $u(x,t)$ exists globally or blows up at some finite time T. In addition, the upper estimates of the global solution, the "blow-up time," and the "blow-up rate" were also established.

In this paper, we study reaction-diffusion problem (1). It is well known that $f(u)$, $g(u)$, $a(u)$, and $b(x,u,t)$ are nonlinear reaction, nonlinear diffusion, nonlinear convection, and nonlinear boundary flux, respectively. What interactions among the four nonlinear mechanisms result in the blow-up and global solutions of (1) is investigated in this work. We note that the boundary flux function $b(x,u,t)$ depends not only on the concentration variable u but also on the space variable x and the time variable t. Hence, it seems that the methods of [21, 22] are not applicable for problem (1). In this paper, by constructing completely different auxiliary functions from those in [21, 22] and technically using maximum principles, we obtain the existence theorems of the blow-up and global solution. Moreover, the upper estimates of "blow-up time," "blow-up rate," and global solution are also given. Our results can be seen as the extension and supplement of those obtained in [21, 22].

The present work is organized as follows. In Section 2, we deal with the blow-up solution of (1). Section 3 is devoted to the global solution of (1). As applications of the obtained results, some examples are presented in Section 4.

2. Blow-Up Solution

In this section, we discuss what interactions among the four nonlinear mechanisms of (1) result in the blow-up solution. Our main result in this section is the following theorem.

Theorem 1. *Let $u(x,t)$ be a solution of problem (1). Assume that the following conditions (i)–(iv) are satisfied.*

(i) *For $s \in \mathbb{R}_+$,*

$$\begin{gathered} a(s) - a'(s) \geq 0, \\ f'(s) - f(s) \geq 0, \\ \left(\frac{a(s)}{g'(s)}\right)' \geq 0. \end{gathered} \tag{4}$$

(ii) *For $(x,s,t) \in \partial\Omega \times \mathbb{R}_+ \times \mathbb{R}_+$,*

$$\begin{gathered} b_s(x,s,t) - b(x,s,t) \leq 0, \\ a(s)\, b(x,s,t) - (a(s)\, b(x,s,t))_s \leq 0, \\ b_t(x,s,t) \geq 0. \end{gathered} \tag{5}$$

(iii) *Consider the following:*

$$\alpha := \min_{\overline{D}} \left\{ \frac{a(u_0)\left[\nabla \cdot (a(u_0)\nabla u_0) + f(u_0)\right]}{e^{u_0} g'(u_0)} \right\} > 0. \tag{6}$$

(iv) *Consider the following:*

$$\int_{M_0}^{+\infty} \frac{a(s)}{e^s} ds < +\infty, \quad M_0 := \max_{\overline{D}} u_0(x). \tag{7}$$

Then $u(x,t)$ blows up in a finite time T and

$$\begin{gathered} T \leq \frac{1}{\alpha} \int_{M_0}^{+\infty} \frac{a(s)}{e^s} ds, \\ u(x,t) \leq H^{-1}(\alpha(T-t)), \end{gathered} \tag{8}$$

where

$$H(y) := \int_y^{+\infty} \frac{a(s)}{e^s} ds, \quad y > 0, \tag{9}$$

and H^{-1} is the inverse function of H.

Proof. Introduce an auxiliary function

$$P(x,t) := -e^{-u} u_t + \alpha \frac{1}{a(u)}, \tag{10}$$

and then we have

$$\nabla P = e^{-u} u_t \nabla u - e^{-u} \nabla u_t - \alpha \frac{a'}{a^2} \nabla u, \tag{11}$$

$$\begin{aligned} \Delta P = {} & -e^{-u} u_t |\nabla u|^2 + 2e^{-u} \nabla u \cdot \nabla u_t + e^{-u} u_t \Delta u \\ & - e^{-u} \Delta u_t + \left(2\alpha \frac{(a')^2}{a^3} - \alpha \frac{a''}{a^2}\right) |\nabla u|^2 \\ & - \alpha \frac{a'}{a^2} \Delta u, \end{aligned} \tag{12}$$

$$\begin{aligned} P_t &= e^{-u} (u_t)^2 - e^{-u} (u_t)_t - \alpha \frac{a'}{a^2} u_t \\ &= e^{-u} (u_t)^2 - e^{-u} \left(\frac{a}{g'} \Delta u + \frac{a'}{g'} |\nabla u|^2 + \frac{f}{g'}\right)_t \\ &\quad - \alpha \frac{a'}{a^2} u_t \\ &= e^{-u} (u_t)^2 + \left(\frac{a g''}{(g')^2} - \frac{a'}{g'}\right) e^{-u} u_t \Delta u \\ &\quad - \frac{a}{g'} e^{-u} \Delta u_t + \left(\frac{a' g''}{(g')^2} - \frac{a''}{g'}\right) e^{-u} u_t |\nabla u|^2 \\ &\quad - 2\frac{a'}{g'} e^{-u} (\nabla u \cdot \nabla u_t) + \left(\frac{f g''}{(g')^2} - \frac{f'}{g'}\right) e^{-u} u_t \\ &\quad - \alpha \frac{a'}{a^2} u_t. \end{aligned} \tag{13}$$

It follows from (12) and (13) that

$$\begin{aligned}\frac{a}{g'}\Delta P - P_t = &\left(\frac{a''}{g'} - \frac{a}{g'} - \frac{a'g''}{(g')^2}\right)e^{-u}u_t|\nabla u|^2 \\ &+ \left(2\frac{a}{g'} + 2\frac{a'}{g'}\right)e^{-u}(\nabla u\cdot\nabla u_t) \\ &+ \left(\frac{a'}{g'} + \frac{a}{g'} - \frac{ag''}{(g')^2}\right)e^{-u}u_t\Delta u \\ &+ \left(2\alpha\frac{(a')^2}{a^2g'} - \alpha\frac{a''}{ag'}\right)|\nabla u|^2 \\ &- \alpha\frac{a'}{ag'}\Delta u - e^{-u}(u_t)^2 \\ &+ \left(\frac{f'}{g'} - \frac{fg''}{(g')^2}\right)e^{-u}u_t + \alpha\frac{a'}{a^2}u_t.\end{aligned} \tag{14}$$

The first equation of (1) implies

$$\Delta u = \frac{g'}{a}u_t - \frac{a'}{a}|\nabla u|^2 - \frac{f}{a}. \tag{15}$$

Inserting (15) into (14), we obtain

$$\begin{aligned}\frac{a}{g'}\Delta P - P_t = &\left(\frac{a''}{g'} - \frac{a}{g'} - \frac{(a')^2}{ag'} - \frac{a'}{g'}\right)e^{-u}u_t|\nabla u|^2 \\ &+ \left(2\frac{a}{g'} + 2\frac{a'}{g'}\right)e^{-u}(\nabla u\cdot\nabla u_t) \\ &+ \frac{g'}{a}\left(\frac{a}{g'}\right)'e^{-u}(u_t)^2 \\ &+ \left(\frac{f'}{g'} - \frac{a'f}{ag'} - \frac{f}{g'}\right)e^{-u}u_t \\ &+ \left(3\alpha\frac{(a')^2}{a^2g'} - \alpha\frac{a''}{ag'}\right)|\nabla u|^2 + \alpha\frac{a'f}{a^2g'}.\end{aligned} \tag{16}$$

It follows from (11) that

$$\nabla u_t = -e^u\nabla P + u_t\nabla u - \alpha\frac{a'}{a^2}e^u\nabla u. \tag{17}$$

Substituting (17) into (16), we get

$$\begin{aligned}&\frac{a}{g'}\Delta P + 2\frac{a+a'}{g'}\nabla u\cdot\nabla P - P_t \\ &\quad= \left(\frac{a''}{g'} + \frac{a}{g'} - \frac{(a')^2}{ag'} + \frac{a'}{g'}\right)e^{-u}u_t|\nabla u|^2 \\ &\qquad+ \left(\alpha\frac{(a')^2}{a^2g'} - 2\alpha\frac{a'}{ag'} - \alpha\frac{a''}{ag'}\right)|\nabla u|^2 \\ &\qquad+ \frac{g'}{a}\left(\frac{a}{g'}\right)'e^{-u}(u_t)^2 \\ &\qquad+ \left(\frac{f'}{g'} - \frac{a'f}{ag'} - \frac{f}{g'}\right)e^{-u}u_t + \alpha\frac{a'f}{a^2g'}.\end{aligned} \tag{18}$$

By (10), we have

$$u_t = -e^uP + \alpha e^u\frac{1}{a}. \tag{19}$$

Now, we insert (19) into (18) to deduce

$$\begin{aligned}&\frac{a}{g'}\Delta P + 2\frac{a'+a}{g'}\nabla u\cdot\nabla P \\ &\quad+ \frac{a}{g'}\left\{\left[1 + \frac{a'}{a} + \left(\frac{a'}{a}\right)'\right]|\nabla u|^2 + \left(\frac{f}{a}\right)' - \frac{f}{a}\right\}P \\ &\quad- P_t \\ &= \alpha\frac{1}{ag'}(a-a')|\nabla u|^2 + \frac{g'}{a}\left(\frac{a}{g'}\right)'e^{-u}(u_t)^2 \\ &\quad+ \alpha\frac{1}{ag'}(f'-f).\end{aligned} \tag{20}$$

Assumptions (4) ensure that the right side in equality (20) is nonnegative; that is,

$$\begin{aligned}&\frac{a}{g'}\Delta P + 2\frac{a'+a}{g'}\nabla u\cdot\nabla P \\ &\quad+ \frac{a}{g'}\left\{\left[1 + \frac{a'}{a} + \left(\frac{a'}{a}\right)'\right]|\nabla u|^2 + \left(\frac{f}{a}\right)' - \frac{f}{a}\right\}P \\ &\quad- P_t \geq 0 \quad \text{in } \Omega\times(0,T).\end{aligned} \tag{21}$$

Next, it follows from (1) and (10) that

$$\begin{aligned}\frac{\partial P}{\partial n} &= e^{-u}u_t\frac{\partial u}{\partial n} - e^{-u}\frac{\partial u_t}{\partial n} - \alpha\frac{a'}{a^2}\frac{\partial u}{\partial n}\\ &= e^{-u}u_t b - e^{-u}\left(\frac{\partial u}{\partial n}\right)_t - \alpha\frac{a'}{a^2}b\\ &= e^{-u}u_t b - e^{-u}\left(b(x,u,t)\right)_t - \alpha\frac{a'}{a^2}b\\ &= (b-b_u)e^{-u}u_t - e^{-u}b_t - \alpha\frac{a'}{a^2}b\\ &= (b-b_u)\left(-P+\alpha\frac{1}{a}\right) - e^{-u}b_t - \alpha\frac{a'}{a^2}b\\ &= (b_u-b)P + \alpha\frac{ab-(ab)_u}{a^2} - e^{-u}b_t\\ &\qquad\qquad \text{on } \partial\Omega\times(0,T).\end{aligned} \tag{22}$$

We note that (6) implies

$$\begin{aligned}\max_{\overline{\Omega}} P(x,0) &= \max_{\overline{\Omega}}\left\{-e^{-u_0}(u_0)_t + \alpha\frac{1}{a(u_0)}\right\}\\ &= \max_{\overline{\Omega}}\left\{-\frac{\nabla\cdot(a(u_0)\nabla u_0) + f(u_0)}{e^{u_0}g'(u_0)} + \alpha\frac{1}{a(u_0)}\right\}\\ &= \max_{\overline{\Omega}}\left\{\frac{1}{a(u_0)}\left(\alpha\right.\right.\\ &\quad\left.\left. - \frac{a(u_0)\left[\nabla\cdot(a(u_0)\nabla u_0) + f(u_0)\right]}{e^{u_0}g'(u_0)}\right)\right\} = 0.\end{aligned} \tag{23}$$

There, (21)–(23), assumption (5) and the maximum principle [10] imply that the maximum of the function P in $\overline{\Omega}\times[0,\mathrm{T})$ is zero. In fact, if the function P takes a positive maximum at point $(x_0,t_0)\in\partial\Omega\times(0,T)$, then we have

$$P(x_0,t_0) > 0, \qquad \left.\frac{\partial P}{\partial n}\right|_{(x_0,t_0)} > 0. \tag{24}$$

Using assumption (5) and the fact that $P(x_0,t_0) > 0$, it follows from (22) that

$$\left.\frac{\partial P}{\partial n}\right|_{(x_0,t_0)} \le 0, \tag{25}$$

which contradicts the second inequality in (24). Hence, the maximum of the function P in $\overline{\Omega}\times[0,T)$ is zero. Now, we have

$$P \le 0 \quad \text{in } \overline{\Omega}\times[0,T); \tag{26}$$

that is,

$$\frac{a(u)}{e^u}u_t \ge \alpha. \tag{27}$$

At the point $x_0\in\overline{\Omega}$, where $u_0(x_0) = M_0$, integrating inequality (27) from 0 to t, we arrive at

$$\int_0^t \frac{a(u)}{e^u}u_t\,dt = \int_{M_0}^{u(x_0,t)}\frac{a(s)}{e^s}ds \ge \alpha t. \tag{28}$$

Inequality (28) and assumption (7) imply that $u(x,t)$ blows up in finite time $t = T$. Now, we let $t\to T$ in (28) to deduce

$$T \le \frac{1}{\alpha}\int_{M_0}^{+\infty}\frac{a(s)}{e^s}ds. \tag{29}$$

For each fixed $x\in\overline{\Omega}$, integrating inequality (27) from t to s $(0<t<s<T)$, we get

$$\begin{aligned}H(u(x,t)) &\ge H(u(x,t)) - H(u(x,s))\\ &= \int_{u(x,t)}^{+\infty}\frac{a(s)}{e^s}ds - \int_{u(x,s)}^{+\infty}\frac{a(s)}{e^s}ds\\ &= \int_{u(x,t)}^{u(x,s)}\frac{a(s)}{e^s}ds = \int_t^s\frac{a(u)}{e^u}u_t\,dt\\ &\ge \alpha(s-t).\end{aligned} \tag{30}$$

In the above inequality, letting $s\to T$, we obtain

$$H(u(x,t)) \ge \alpha(T-t). \tag{31}$$

We note that H is a strictly decreasing function. Hence,

$$u(x,t) \le H^{-1}(\alpha(T-t)). \tag{32}$$

The proof is complete. □

3. Global Solution

In this section, we study what interactions among the four nonlinear mechanisms of (1) result in the global solution of (1). The main results of this section are formulated in the following theorem.

Theorem 2. *Let $u(x,t)$ be a solution of problem (1). Assume that the following conditions (i)–(iv) are fulfilled.*

(i) *For $s\in\mathbb{R}_+$,*

$$\begin{aligned}a(s) + a'(s) &\le 0,\\ f(s) + f'(s) &\le 0,\\ \left(\frac{a(s)}{g'(s)}\right)' &\le 0.\end{aligned} \tag{33}$$

(ii) *For $(x,s,t)\in\partial\Omega\times\mathbb{R}_+\times\mathbb{R}_+$,*

$$\begin{aligned}b(x,s,t) + b_s(x,s,t) &\le 0,\\ a(s)b(x,s,t) + (a(s)b(x,s,t))_s &\le 0,\\ b_t(x,s,t) &\le 0.\end{aligned} \tag{34}$$

(iii) *Consider the following:*

$$\beta := \max_{\overline{\Omega}} \left\{ \frac{a(u_0)\left[\nabla \cdot (a(u_0)\nabla u_0) + f(u_0)\right]}{e^{-u_0} g'(u_0)} \right\} > 0. \quad (35)$$

(iv) *Consider the following:*

$$\int_{m_0}^{+\infty} \frac{a(s)}{e^{-s}} ds = +\infty, \quad m_0 := \min_{\overline{\Omega}} u_0(x). \quad (36)$$

Then $u(x,t)$ must be a global solution and

$$u(x,t) \le K^{-1}(\beta t + K(u_0(x))), \quad (37)$$

where

$$K(y) := \int_{m_0}^{y} \frac{a(s)}{e^{-s}} ds, \quad y \ge m_0, \quad (38)$$

and K is the inverse function of K.

Proof. We consider an auxiliary function

$$Q(x,t) := -e^u u_t + \beta \frac{1}{a(u)}. \quad (39)$$

Using the same reasoning process as that of (11)–(20), we obtain

$$\begin{aligned} &\frac{a}{g'}\Delta Q + 2\frac{a'-a}{g'}\nabla u \cdot \nabla Q \\ &\quad + \frac{a}{g'}\left\{\left[1 - \frac{a'}{a} + \left(\frac{a'}{a}\right)'\right]|\nabla u|^2 + \left(\frac{f}{a}\right)' + \frac{f}{a}\right\} Q \\ &\quad - Q_t \\ &= \beta \frac{1}{ag'}(a + a')|\nabla u|^2 + \frac{g'}{a}\left(\frac{a}{g'}\right)' e^u (u_t)^2 \\ &\quad + \beta \frac{1}{ag'}(f' + f). \end{aligned} \quad (40)$$

Assumptions (33) imply that the right side of (40) is nonpositive; that is,

$$\begin{aligned} &\frac{a}{g'}\Delta Q + 2\frac{a'-a}{g'}\nabla u \cdot \nabla Q \\ &\quad + \frac{a}{g'}\left\{\left[1 - \frac{a'}{a} + \left(\frac{a'}{a}\right)'\right]|\nabla u|^2 + \left(\frac{f}{a}\right)' + \frac{f}{a}\right\} Q \\ &\quad - Q_t \le 0 \quad \text{in } \Omega \times (0,T). \end{aligned} \quad (41)$$

By (1) and (39), we get

$$\begin{aligned} \frac{\partial Q}{\partial n} &= -e^u u_t \frac{\partial u}{\partial n} - e^u \frac{\partial u_t}{\partial n} - \beta \frac{a'}{a^2}\frac{\partial u}{\partial n} \\ &= -e^u u_t b - e^u \left(\frac{\partial u}{\partial n}\right)_t - \beta \frac{a'}{a^2} b \\ &= -e^u u_t b - e^u (b(x,u,t))_t - \beta \frac{a'}{a^2} b \\ &= -(b + b_u) e^u u_t - e^u b_t - \beta \frac{a'}{a^2} b \\ &= (b + b_u)\left(Q - \beta \frac{1}{a}\right) - e^u b_t - \beta \frac{a'}{a^2} b \\ &= (b + b_u) Q - \beta \frac{ab + (ab)_u}{a^2} - e^u b_t \\ &\qquad \text{on } \partial\Omega \times (0,T). \end{aligned} \quad (42)$$

Assumption (35) implies

$$\begin{aligned} \min_{\overline{\Omega}} Q(x,0) &= \min_{\overline{\Omega}} \left\{ -e^{u_0}(u_0)_t + \beta \frac{1}{a(u_0)} \right\} \\ &= \min_{\overline{\Omega}} \left\{ -\frac{\nabla \cdot (a(u_0)\nabla u_0) + f(u_0)}{e^{-u_0} g'(u_0)} + \beta \frac{1}{a(u_0)} \right\} \\ &= \min_{\overline{\Omega}} \left\{ \frac{1}{a(u_0)} \left(\beta \right.\right. \\ &\quad \left.\left. - \frac{a(u_0)\left[\nabla \cdot (a(u_0)\nabla u_0) + f(u_0)\right]}{e^{-u_0} g'(u_0)} \right) \right\} = 0. \end{aligned} \quad (43)$$

It follows from (41)–(43), (34), and the maximum principle that the minimum of Q in $\overline{\Omega} \times [0,T)$ is zero. Hence, we have the following inequality:

$$Q \ge 0 \quad \text{in } \overline{\Omega} \times [0,T); \quad (44)$$

that is

$$\frac{a(u)}{e^{-u}} u_t \le \beta. \quad (45)$$

For each fixed $x \in \overline{\Omega}$, we integrate (45) from 0 to t to deduce

$$\frac{1}{\beta}\int_0^t \frac{a(u)}{e^{-u}} u_t dt = \frac{1}{\beta}\int_{u_0(x)}^{u(x,t)} \frac{a(s)}{e^{-s}} ds \le t. \quad (46)$$

Inequality (46) and assumption (36) imply that u must be a global solution. Furthermore, it follows from (45) that

$$\begin{aligned} &K(u(x,t)) - K(u_0(x)) \\ &\quad = \int_{m_0}^{u(x,t)} \frac{a(s)}{e^{-s}} ds - \int_{m_0}^{u_0(x)} \frac{a(s)}{e^{-s}} ds = \int_{u_0(x)}^{u(x,t)} \frac{a(s)}{e^{-s}} ds \\ &\quad = \int_0^t \frac{a(u)}{e^{-u}} u_t dt \le \beta t. \end{aligned} \quad (47)$$

Since K is a strictly increasing function, we obtain

$$u(x,t) \le K^{-1}(\beta t + K(u_0(x))). \tag{48}$$

The proof is complete. □

4. Applications

When $a(u) = 1$ and $b(x,u,t) = b(u)$, the results of Theorems 1–2 still hold. In this sense, our results extend and supplement those obtained in [21, 22].

In the following, we give a few examples to demonstrate the applications of Theorems 1–2.

Example 3. Let u be a solution of the following problem:

$$\begin{aligned} \left(\frac{u}{2} + e^{u/2}\right)_t &= \nabla \cdot \left(e^{u/2}\nabla u\right) + e^u \quad \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial n} &= e^{(3/4)(u-2)} + e^{u-2+t|x|^2} \\ &\qquad \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= 1 + |x|^2 \quad \text{in } \overline{\Omega}, \end{aligned} \tag{49}$$

where $\Omega = \{x = (x_1, x_2, x_3) \mid |x|^2 = x_1^2 + x_2^2 + x_3^2 < 1\}$. We note

$$\begin{aligned} a(u) &= e^{2/u}, \\ g(u) &= \frac{u}{2} + e^{u/2}, \\ f(u) &= e^u, \\ b(x,u,t) &= e^{(3/4)(u-2)} + e^{u-2+t|x|^2}. \end{aligned} \tag{50}$$

In order to calculate the constant α, we set

$$\omega = |x|^2, \tag{51}$$

and then $0 \le \omega \le 1$ and

$$\begin{aligned} \alpha &= \min_{\overline{\Omega}} \left\{ \frac{a(u_0)\left[\nabla \cdot (a(u_0)\nabla u_0) + f(u_0)\right]}{e^{u_0} g'(u_0)} \right\} \\ &= \min_{\overline{\Omega}} \left\{ \frac{2\left[6 + 2|x|^2 + e^{(1/2)(1+|x|^2)}\right]}{e^{(1/2)(1+|x|^2)} + 1} \right\} \\ &= \min_{0\le\omega\le1} \left\{ \frac{2\left[6 + 2\omega + e^{(1/2)(1+\omega)}\right]}{e^{(1/2)(1+\omega)} + 1} \right\} = \frac{2(8+e)}{1+e}. \end{aligned} \tag{52}$$

We can check that (4), (5), and (7) hold. It follows from Theorem 1 that $u(x,t)$ blows up in a finite time T and

$$\begin{aligned} T &\le \frac{1}{\alpha}\int_{M_0}^{+\infty} \frac{a(s)}{e^s} ds = \frac{1+e}{2(8+e)} \int_2^{+\infty} e^{-s/2} ds \\ &= \frac{1+e}{e(8+e)}, \\ u(x,t) &\le H^{-1}(\alpha(T-t)) = 2\ln\frac{1+e}{(8+e)(T-t)}. \end{aligned} \tag{53}$$

Example 4. Let u be a solution of the following problem:

$$\begin{aligned} (ue^u)_t &= \nabla \cdot \left(\frac{e^{-u}}{1+u}\nabla u\right) + e^{-u} \quad \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial n} &= e^{-2u} + e^{-u-t|x|^2} \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) &= 1 + |x|^2 \quad \text{in } \overline{\Omega}, \end{aligned} \tag{54}$$

where $\Omega = \{x = (x_1, x_2, x_3) \mid |x|^2 = x_1^2 + x_2^2 + x_3^2 < 1\}$. Now we have

$$\begin{aligned} a(u) &= \frac{e^{-u}}{1+u}, \\ g(u) &= ue^u, \\ f(u) &= e^{-u}, \\ b(x,u,t) &= e^{-2u} + e^{-u-t|x|^2}. \end{aligned} \tag{55}$$

Setting

$$\omega = |x|^2, \tag{56}$$

we get $0 \le \omega \le 1$ and

$$\begin{aligned} \beta &= \max_{\overline{\Omega}} \left\{ \frac{a(u_0)\left[\nabla \cdot (a(u_0)\nabla u_0) + f(u_0)\right]}{e^{-u_0} g'(u_0)} \right\} \\ &= \max_{\overline{\Omega}} \left\{ \frac{16 - 2|x|^2 - 3|x|^4}{e^{2(1+|x|^2)}\left(2+|x|^2\right)^4} \right\} \\ &= \max_{0\le\omega\le1} \left\{ \frac{16 - 2\omega - 3\omega^2}{e^{2(1+\omega)}(2+\omega)^4} \right\} = e^{-2}. \end{aligned} \tag{57}$$

We can also check that (33), (34), and (36) hold. Hence, Theorem 2 implies u must be a global solution and

$$u(x,t) \le K^{-1}(\beta t + K(u_0(x))) = \left(2 + |x|^2\right) e^{te^{-2}} - 1. \tag{58}$$

Remark 5. When the functions a, b, f, and g are all exponential functions, Theorems 1 and 2 can be used. When not all of them are exponential functions, Theorems 1 and 2 can be used in some special cases.

Competing Interests

The author declares that there are no competing interests regarding the publication of this paper.

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