Research Article

On the Hyers-Ulam Stability of First-Order Impulsive Delay Differential Equations

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This paper proves the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability of nonlinear first-order ordinary differential equation with single constant delay and finite impulses on a compact interval. Our approach uses abstract Gronwall lemma together with integral inequality of Gronwall type for piecewise continuous functions.

1. Introduction

Ulam, in [1], put a question regarding the stability of functional equation for homomorphism in front of a Mathematical Colloquium. The question was “*when an approximate homomorphism from a group G_1 to a metric group G_2 can be approximated by an exact homomorphism?*”

Within the next two years, Hyers [2] brilliantly gave a partial answer to this question for the case when G_1 and G_2 are assumed to be Banach spaces by using direct method. Aoki [3] and Rassias [4] latter extended the partial answer by Hyers. In fact, the most exciting result was by Rassias [4] which weakens the condition for the bound of the norm of Cauchy difference $f(x+y) - f(x) - f(y)$. For further details and discussions, the reader is referred to the book by Jung [5].

As far we know, among the functional equations, Obloza for the first time investigated the stability of differential equations [6, 7]. After him, Alsina and Ger [8] proved the stability for differential equation $y'(x) = y(x)$, which was then generalized for the Banach space valued linear differential equation of first-order $y'(x) = \lambda y(x)$, by Takahasi et al. [9]. To study Hyers-Ulam stability of differential equations, different researchers presented their works with different approaches; for example, see [10–25].

Many real world phenomenons are represented by smooth differential equations. However, the situation becomes quite different in the case when a physical phenomenon has sudden

changes in its state such as mechanical systems with impact, biological systems like heart beats, blood flows, population dynamics [26, 27], theoretical physics, radio physics, pharmacokinetics, mathematical economy, chemical technology, electric technology, metallurgy, ecology, industrial robotics, biotechnology processes, chemistry [28], engineering [29], control theory, and medicine. Adequate mathematical models of such processes are systems of differential equations with impulses, that is, impulsive differential equations.

An impulsive differential equation is described by three components: a continuous-time differential equation, which governs the state of the system between impulses; an impulse equation, which models an impulsive jump defined by a jump function at the instant an impulse occurs; and a jump criterion, which defines a set of jump events in which the impulse equation is active.

To the best of our knowledge, the first mathematicians who investigated Ulam's type stability of impulsive ordinary differential equations are Wang et al. [30]. They, in 2012, obtained four Ulam's type stability concepts for first-order nonlinear impulsive ordinary differential equation on closed bounded interval with finite impulses. Following their own work, in 2014, they proved the Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for impulsive evolution equations on a compact interval [31] which then they extended for infinite impulses in the same paper. To study more work on impulsive ODEs we recommend [32–35].

However, as far as we know, Ulam's type stability results of impulsive delay ordinary differential equations have not been investigated yet.

In this paper, we obtain Hyers-Ulam stability and Hyers-Ulam-Rassias stability of impulsive delay differential equation of the form

$$\begin{aligned} z'(t) &= F(t, z(t), z(h(t))), \\ t \in I' &= [t_0, t_f] \setminus \{t_1, t_2, \dots, t_m\}, \\ z(t) &= \alpha(t), \quad t \in [t_0 - \lambda, t_0], \\ \Delta z(t_k) &= z(t_k^+) - z(t_k^-) = Y_k(z(t_k^-)), \quad k = 1, \dots, m, \end{aligned} \quad (1)$$

where $\lambda > 0$, $t_f > t_0 \geq 0$, $F : [t_0, t_f] \times R^2 \rightarrow R$, $Y_k : R \rightarrow R$, and $\alpha : [t_0 - \lambda, t_0] \rightarrow R$ are continuous. $z(t_k^+) = \lim_{\tau \rightarrow 0^+} z(t_k + \tau)$ and $z(t_k^-) = \lim_{\tau \rightarrow 0^+} z(t_k - \tau)$ are, respectively, the right and left side limits of $z(t)$ at t_k , where t_k satisfy $t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = t_f < +\infty$. Moreover, $h : [t_0, t_f] \rightarrow [t_0 - \lambda, t_f]$ is such that $h(t) \leq t$.

2. Preliminaries

In this section we list some important notations, definitions, and lemmas that would be used in our main results.

Throughout this paper, the following spaces appear mostly:

- (a) $C[J, R]$ is the Banach space of all continuous real valued functions from J with norm $\|x\|_C = \sup\{|x(t)| : t \in J\}$, where J is a compact interval.
- (b) $PC[t_0 - \lambda, t_f]$ denotes the Banach space of all functions $x : [t_0 - \lambda, t_f] \rightarrow R$ such that $x \in C[[t_0 - \lambda, t_0], R] \cap C[(t_k, t_{k+1}], R]$, $k = 0, 1, 2, \dots, m$, where $x(t_k^+)$, $x(t_k^-)$ exist and are finite for $k = 1, 2, \dots, m$, with norm $\|x\|_{PC} = \sup\{|x(t)| : t \in [t_0 - \lambda, t_f]\}$.
- (c) $PC^1[t_0, t_f] = \{x \in PC[t_0 - \lambda, t_f] : x' \in PC[t_0 - \lambda, t_f]\}$ is Banach space with norm $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$.

For problem (1), for some $\epsilon > 0$, $\varphi \in PC[[t_0 - \lambda, t_f], R^+]$, where φ is increasing, $\varphi(t_i) = \kappa > 0$ for some $t_i \in [t_0 - \lambda, t_f]$, we focus on the following inequalities:

$$\begin{aligned} |y'(t) - F(t, y(t), y(h(t)))| &\leq \epsilon, \quad t \in I', \\ |\Delta y(t_k) - Y_k(y(t_k^-))| &\leq \epsilon, \quad k = 1, \dots, m, \\ |y'(t) - F(t, y(t), y(h(t)))| &\leq \varphi(t), \quad t \in I', \\ |\Delta y(t_k) - Y_k(y(t_k^-))| &\leq \kappa, \quad k = 1, \dots, m. \end{aligned} \quad (2)$$

Definition 1. Equation (1) is Hyers-Ulam stable on $[t_0 - \lambda, t_f]$ if for every $y \in PC[t_0 - \lambda, t_f] \cap PC^1[t_0, t_f]$ satisfying (2), there exists a solution $y_0 \in PC[t_0 - \lambda, t_f] \cap PC^1[t_0, t_f]$ of (1) with $|y_0(t) - y(t)| \leq K\epsilon$, $K > 0$, for all $t \in [t_0 - \lambda, t_f]$.

Definition 2. Equation (1) is Hyers-Ulam-Rassias stable on $[t_0 - \lambda, t_f]$ with respect to (φ, κ) if for every $y \in PC[t_0 - \lambda, t_f] \cap PC^1[t_0, t_f]$ satisfying (3), there exists a solution $y_0 \in PC[t_0 - \lambda, t_f] \cap PC^1[t_0, t_f]$ of (1) with $|y_0(t) - y(t)| < M\varphi(t)$, $M > 0$, for all $t \in [t_0 - \lambda, t_f]$.

Definition 3. Let $(X; d)$ be a metric space. An operator $\Lambda : X \rightarrow X$ is a Picard operator if it has a unique fixed point $x^* \in X$ such that, for every $x \in X$, the sequence $\{\Lambda^{(n)}(x)\}_{n \in \mathbb{N}}$ converges to x^* .

We consume what just follow in deriving our main results.

Lemma 4 (Gronwall lemma [36]). *If for $t \geq t_0 \geq 0$ we have*

$$x(t) \leq a(t) + \int_{t_0}^t b(s)x(s)ds + \sum_{t_0 < t_k < t} \xi_k x(t_k^-), \quad (4)$$

where $x, a, b \in PC[[t_0, \infty), R^+]$, a is nondecreasing and $b(t)$, $\xi_k > 0$. Then for $t \geq t_0$ the following inequality works:

$$x(t) \leq a(t) \prod_{t_0 < t_k < t} (1 + \xi_k) \exp\left(\int_{t_0}^t b(s)ds\right). \quad (5)$$

Lemma 5 (abstract Gronwall lemma [37]). *Let $(X, d, \leq)$ be an ordered metric space and let $\Lambda : X \rightarrow X$ be an increasing Picard operator with fixed point x^* . Then for any $x \in X$, $x \leq \Lambda(x)$ implies $x \leq x^*$ and $x \geq \Lambda(x)$ implies $x \geq x^*$, where x^* is the fixed point of Λ in X .*

Remark 6 (see [30]). A function $y \in PC^1[t_0, t_f]$ satisfies (2) if and only if there is a function $f \in PC[t_0 - \lambda, t_f]$ and a sequence f_k (which depends on y) such that $|f(t)| \leq \epsilon$ for all $t \in [t_0 - \lambda, t_f]$, $|f_k| \leq \epsilon$ for all $k = 1, 2, 3, \dots, m$, and

$$\begin{aligned} y'(t) &= F(t, y(t), y(h(t))) + f(t), \quad t \in I', \\ \Delta y(t_k) &= Y_k(y(t_k^-)) + f_k, \quad k = 1, 2, 3, \dots, m. \end{aligned} \quad (6)$$

We do similar remark for (3).

Lemma 7. *Every $y \in PC^1[t_0, t_f]$ that satisfies (2) also comes out perfect on the following inequality:*

$$\begin{aligned} \left| y(t) - y(t_0) - \sum_{j=1}^k Y(y(t_j^-)) \right. \\ \left. - \int_{t_0}^t F(s, y(s), y(h(s)))ds \right| \leq (m + t - t_0)\epsilon, \end{aligned} \quad (7)$$

for $t \in (t_k, t_{k+1}] \subset [t_0, t_f]$.

Proof. If $y \in PC^1[t_0, t_f]$ satisfies (2), then by Remark 6 we have

$$\begin{aligned} y'(t) &= F(t, y(t), y(h(t))) + f(t), \quad t \in I', \\ \Delta y(t_k) &= Y_k(y(t_k^-)) + f_k, \quad k = 1, 2, 3, \dots, m. \end{aligned} \quad (8)$$

Then

$$\begin{aligned} y(t) &= y(t_0) + \sum_{j=1}^k Y(y(t_j^-)) + \sum_{i=1}^k f_i \\ &\quad + \int_{t_0}^t F(s, y(s), y(h(s))) ds + \int_{t_0}^t f(s) ds, \end{aligned} \quad (9)$$

$$t \in (t_k, t_{k+1}].$$

From this the following follows:

$$\begin{aligned} &\left| y(t) - y(t_0) - \sum_{j=1}^k Y(y(t_j^-)) \right. \\ &\quad \left. - \int_{t_0}^t F(s, y(s), y(h(s))) ds \right| \leq \int_{t_0}^t |f(s)| ds \\ &\quad + \sum_{i=1}^k |f_i| \leq ((t - t_0) + k) \epsilon \leq (t - t_0 + m) \epsilon, \end{aligned} \quad (10)$$

$$t \in (t_k, t_{k+1}].$$

We have similar remarks for (3). $\square$

3. Main Results

Thus far, we were warming up the environment for our main results. Now, we are in position to present our main results.

First we are going to give our result on Hyers-Ulam stability.

Theorem 8. *If*

- (a) $F : [t_0, t_f] \times R^2 \rightarrow R$ is continuous with the Lipschitz condition: $|F(t, x_1, x_2) - F(t, y_1, y_2)| \leq \sum_{i=1}^2 L|x_i - y_i|$, $L > 0$, for all $t \in [t_0, t_f]$ and $x_i, y_i \in R$, $i \in \{1, 2\}$;
- (b) $Y_k : R \rightarrow R$ is such that $|Y_k(x_1) - Y_k(x_2)| \leq M_k|x_1 - x_2|$, $M_k > 0$, for all $k \in \{1, 2, \dots, m\}$ and $x_1, x_2 \in R$, $i \in \{1, 2\}$;
- (c) $(\sum_{i=1}^m M_i + 2L(t_f - t_0)) < 1$,

then (1) has

- (i) a unique solution in $PC[t_0 - \lambda, t_f] \cap PC^1[t_0, t_f]$;
- (ii) Hyers-Ulam's stability on $[t_0 - \lambda, t_f]$.

Proof. (i) Define an operator $\Lambda : PC[t_0 - \lambda, t_f] \rightarrow PC[t_0 - \lambda, t_f]$ by

$$(\Lambda z)(t) = \begin{cases} \alpha(t), & t \in [t_0 - \lambda, t_0], \\ \alpha(t_0) + \int_{t_0}^t F(r, z(r), z(h(r))) dr, & t \in (t_0, t_1], \\ \alpha(t_0) + Y_1(z(t_1^-)) + \int_{t_0}^t F(r, z(r), z(h(r))) dr, & t \in (t_1, t_2], \\ \alpha(t_0) + \sum_{j=1}^2 Y_j(z(t_j^-)) + \int_{t_0}^t F(r, z(r), z(h(r))) dr, & t \in (t_2, t_3], \\ \vdots \\ \alpha(t_0) + \sum_{j=1}^m Y_j(z(t_j^-)) + \int_{t_0}^t F(r, z(r), z(h(r))) dr, & t \in (t_m, t_{m+1}]. \end{cases} \quad (11)$$

We see that for any $z_1, z_2 \in PC[t_0 - \lambda, t_f]$ and for all $t \in [t_0 - \lambda, t_0]$ we have $|(\Lambda z_1)(t) - (\Lambda z_2)(t)| = 0$. For $t \in [t_k, t_{k+1}]$ consider

$$\begin{aligned} &|(\Lambda z_1)(t) - (\Lambda z_2)(t)| \leq \sum_{j=1}^k |Y_j(z_1(t_j^-)) \\ &\quad - Y_j(z_2(t_j^-))| + \int_{t_0}^t |F(r, z_1(r), z_1(h(r))) \\ &\quad - F(r, z_2(r), z_2(h(r)))| dr \\ &\leq \sum_{j=1}^k M_j |(z_1(t_j^-)) - (z_2(t_j^-))| + L \int_{t_0}^t |z_1(r) \\ &\quad - z_2(r)| dr + L \int_{t_0}^t |z_1(h(r)) - z_2(h(r))| dr \end{aligned}$$

$$\begin{aligned} &\leq \left(\sum_{j=1}^k M_j + 2L(t_f - t_0) \right) \sup_{t \in [t_0 - \lambda, t_f]} |z_1(t) - z_2(t)| \\ &\leq \left(\sum_{j=1}^m M_j + 2L(t_f - t_0) \right) \|z_1 - z_2\|. \end{aligned} \quad (12)$$

Following (c), the operator is strictly contractive on $(t_k, t_{k+1}]$, $k = 0, 1, 2, \dots, m$, and hence a Picard operator on $PC[t_0 - \lambda, t_f]$. From (11), it follows that the unique fixed point of this operator is in fact the unique solution of (1) in $PC[t_0 - \lambda, t_f] \cap PC^1[t_0, t_f]$.

Next, let $y \in PC[t_0 - \lambda, t_f] \cap PC^1[t_0, t_f]$ be a solution to (2). The unique solution $z \in PC[t_0 - \lambda, t_f] \cap PC^1[t_0, t_f]$ of the differential equation

$$z'(t) = F(t, z(t), z(h(t))), \quad t \in I'$$

$$z(t) = y(t), \quad t \in [t_0 - \lambda, t_0],$$

is given by

$$z(t) = \begin{cases} y(t), & t \in [t_0 - \lambda, t_0], \\ y(t_0) + \int_{t_0}^t F(r, z(r), z(h(r))) dr, & t \in (t_0, t_1], \\ y(t_0) + Y_1(z(t_1^-)) + \int_{t_0}^t F(r, z(r), z(h(r))) dr, & t \in (t_1, t_2], \\ y(t_0) + \sum_{j=1}^2 Y_j(z(t_j^-)) + \int_{t_0}^t F(r, z(r), z(h(r))) dr, & t \in (t_2, t_3], \\ \vdots \\ y(t_0) + \sum_{j=1}^m Y_j(z(t_j^-)) + \int_{t_0}^t F(r, z(r), z(h(r))) dr, & t \in (t_m, t_{m+1}]. \end{cases} \quad (14)$$

We observe that for all $t \in [t_0 - \lambda, t_0]$ we have $|y(t) - z(t)| = 0$. For $t \in (t_k, t_{k+1}]$, using Lemma 7, we have

$$\begin{aligned} |y(t) - z(t)| &\leq \left| y(t) - y(t_0) - \sum_{j=1}^k Y_j(y(t_j^-)) \right. \\ &\quad \left. - \int_{t_0}^t F(s, y(s), y(h(s))) dr \right| \\ &\quad + \int_{t_0}^t |F(r, y(r), y(h(r))) \\ &\quad - F(r, z(r), z(h(r)))| dr + \sum_{j=1}^k |Y_j(y(t_j^-)) \end{aligned}$$

$$\begin{aligned} &- Y_j(z(t_j^-))| \leq (m + t - t_0) \epsilon + L \int_{t_0}^t |y(r) \\ &- z(r)| dr + L \int_{t_0}^t |y(h(r)) - z(h(r))| dr \\ &+ \sum_{j=1}^k M_j |y(t_j^-) - z(t_j^-)|. \end{aligned}$$

(15)

Next, we show that the operator $T : PC[t_0 - \lambda, t_f] \rightarrow PC[t_0 - \lambda, t_f]$ given below is an increasing Picard operator on $PC[t_0 - \lambda, t_f]$:

$$(Tg)(t) = \begin{cases} 0, & t \in [t_0 - \lambda, t_0], \\ (t - t_0) \epsilon + L \int_{t_0}^t g(r) dr + L \int_{t_0}^t g(h(r)) dr, & t \in (t_0, t_1], \\ (1 + t - t_0) \epsilon + M_1 g(t_1^-) + L \int_{t_0}^t g(r) dr + L \int_{t_0}^t g(h(r)) dr, & t \in (t_1, t_2], \\ (2 + t - t_0) \epsilon + \sum_{j=1}^2 M_j g(t_j^-) + L \int_{t_0}^t g(r) dr + L \int_{t_0}^t g(h(r)) dr, & t \in (t_2, t_3], \\ \vdots \\ (m + t - t_0) \epsilon + \sum_{j=1}^m M_j g(t_j^-) + L \int_{t_0}^t g(r) dr + L \int_{t_0}^t g(h(r)) dr, & t \in (t_m, t_{m+1}]. \end{cases} \quad (16)$$

For any $g_1, g_2 \in PC[t_0 - \lambda, t_f]$, $|(Tg_1)(t) - (Tg_2)(t)| = 0$ for all $t \in [t_0 - \lambda, t_0]$. For $t \in (t_k, t_{k+1}]$ consider

$$\begin{aligned}
 & |(Tg_1)(t) - (Tg_2)(t)| \\
 & \leq \sum_{j=1}^k M_j |g_1(t_j^-) - g_2(t_j^-)| \\
 & \quad + L \int_{t_0}^t |g_1(r) - g_2(r)| dr \\
 & \quad + L \int_{t_0}^t |g_1(h(r)) - g_2(h(r))| dr \\
 & \leq \sum_{j=1}^m M_j \sup_{t \in [t_0 - \lambda, t_f]} |g_1(t) - g_2(t)| \\
 & \quad + 2L(t_f - t_0) \sup_{t \in [t_0 - \lambda, t_f]} |g_1(h(t)) - g_2(h(t))| \\
 & \leq \left(\sum_{j=1}^m M_j + 2L(t_f - t_0) \right) \|g_1 - g_2\|.
 \end{aligned} \tag{17}$$

Since $(\sum_{j=1}^m M_j + 2L(t_f - t_0)) < 1$, the operator is contractive on $PC[t_0 - \lambda, t_f]$ for $t \in (t_k, t_{k+1}]$, where $k = 0, 1, \dots, m$. Applying Banach contraction principle, T is Picard operator with unique fixed $g^* \in PC[t_0 - \lambda, t_f]$; that is,

$$\begin{aligned}
 g^*(t) &= (k + t - t_0)\epsilon + \sum_{j=1}^k M_j g^*(t_j^-) \\
 & \quad + L \int_{t_0}^t g^*(r) dr + L \int_{t_0}^t g^*(h(r)) dr, \\
 & \quad t \in (t_k, t_{k+1}].
 \end{aligned} \tag{18}$$

g^* is increasing, so $g^*(h(t)) \leq g^*(t)$ and hence we can write

$$\begin{aligned}
 g^*(t) &\leq (k + t_f - t_0)\epsilon + 2L \int_{t_0}^t g^*(r) dr \\
 & \quad + \sum_{j=1}^k M_j g^*(t_j^-).
 \end{aligned} \tag{19}$$

Using Lemma 4, we get

$$\begin{aligned}
 & g^*(t) \\
 & \leq (k + t_f - t_0)\epsilon \prod_{t_0 < t_k < t} (1 + M_k) \exp(2L(t_f - t_0)).
 \end{aligned} \tag{20}$$

If we set $g = |y - z|$, then from (16), $g(t) \leq (Tg)(t)$ from which, by using abstract Gronwall Lemma, it follows that $g(t) \leq g^*$; thus

$$\begin{aligned}
 & |y(t) - z(t)| \\
 & \leq (m + t_f - t_0)\epsilon \prod_{t_0 < t_k < t} (1 + M_k) \exp(2L(t_f - t_0)).
 \end{aligned} \tag{21}$$

□

In the following theorem, we state about the Hyers-Ulam-Rassias stability of (1) on $I = [t_0 - \lambda, t_f]$. The proof follows the same steps as that of the above theorem. The remarked Lemma 7 for inequality (3) is consumed in the proof.

Theorem 9. *If*

- (a) $F : [t_0, t_f] \times R^2 \rightarrow R$ is continuous with the Lipschitz condition: $|F(t, x_1, x_2) - F(t, y_1, y_2)| \leq L \sum_{i=1}^2 |x_i - y_i|$, $L > 0$, for all $t \in [t_0, t_f]$ and $x_i, y_i \in R$, $i \in \{1, 2\}$;
- (b) $Y_k : R \rightarrow R$ is such that $|Y_k(x_1) - Y_k(x_2)| \leq M_k |x_1 - x_2|$, $M_k > 0$, for all $k \in \{1, 2, \dots, m\}$ and $x_1, x_2 \in R$, $i \in \{1, 2\}$;
- (c) $(\sum_{i=1}^m M_i + 2L(t_f - t_0)) < 1$;
- (d) $\varphi : [t_0 - \lambda, t_f] \rightarrow R^+$ is increasing such that, for some $\rho > 0$,

$$\int_{t_0}^t \varphi(r) dr \leq \rho \varphi(t), \tag{22}$$

then (1) has

- (i) a unique solution in $PC[t_0 - \lambda, t_f] \cap PC^1[t_0, t_f]$;
- (ii) Hyers-Ulam-Rassias stability on $[t_0 - \lambda, t_f]$.

Remark 10. The implications in the above theorem can also be established by assuming that, in (a), “there exists a function $f : [t_0, t_f] \rightarrow R^+$ such that, for all $t \in [t_0, t_f]$ and $x_i, y_i \in R$, $i \in \{1, 2\}$, we have $|F(t, x_1, x_2) - F(t, y_1, y_2)| \leq \sum_{i=1}^2 f(t)|x_i - y_i|$.”

Remark 11. We notice that our results in this paper have also some practical meanings in the following sense. Consider a phenomenon which has sudden changes in state at some finite instants of time which follows model (1). In general, we can not expect any exact solution to (1). So what is required is to find a function that approximates (1); our results guarantee that there is an exact solution $y(t)$ of (1) close to that approximate solution. In fact, our results are important when finding an exact solution is quit difficult and hence are important in approximation theory and so forth.

Competing Interests

The authors declare that they have no competing interests.

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