Research Article

Toeplitz Type Operators Associated with Generalized Calderón-Zygmund Operator on Weighted Morrey Spaces

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Let T_1 be a generalized Calderón-Zygmund operator or $\pm I$ (the identity operator), let T_2 and T_4 be the linear operators, and let $T_3 = \pm I$. Denote the Toeplitz type operator by $T^b = T_1 M^b I_\alpha T_2 + T_3 I_\alpha M^b T_4$, where $M^b f = bf$ and I_α is the fractional integral operator. In this paper, we investigate the boundedness of the operator T^b on weighted Morrey space when b belongs to the weighted BMO spaces.

1. Introduction and Results

The classical Morrey spaces, introduced by Morrey [1] in 1938, have been studied intensively by various authors and together with weighted Lebesgue spaces play an important role in the theory of partial differential equations (see [2, 3]). Komori and Shirai [4] introduced a version of the weighted Morrey space $L^{p,\kappa}(\omega)$, which is a natural generalization of the weighted Lebesgue space $L^p(\omega)$.

Definition 1. Suppose that $T: \mathcal{S} \rightarrow \mathcal{S}'$ is a linear operator with kernel $K(\cdot, \cdot)$ defined initially by

$$T(f)(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy, \quad (1)$$

$$f \in C_c^\infty(\mathbb{R}^n), \quad x \notin \text{supp } f.$$

The operator T is called a generalized Calderón-Zygmund operator provided that the following three conditions are satisfied:

- (1) T can be extended into a continuous operator on $L^2(\mathbb{R}^n)$.

- (2) K is smooth away from the diagonal $\{(x, y) : x = y\}$ with

$$\int_{|x-y| > 2|z-y|} (|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)|) dx \leq C, \quad (2)$$

where $C > 0$ is a constant independent of y and z .

- (3) There is a sequence of positive constant numbers $\{C_j\}$ such that, for each $j \in \mathbb{N}$,

$$\left(\int_{2^j|z-y| \leq |x-y| < 2^{j+1}|z-y|} |K(x, y) - K(x, z)|^q dx \right)^{1/q} \leq C_j (2^j |z-y|)^{-n/q'}, \quad (3)$$

$$\left(\int_{2^j|z-y| \leq |x-y| < 2^{j+1}|z-y|} |K(y, x) - K(z, x)|^q dx \right)^{1/q} \leq C_j (2^j |z-y|)^{-n/q'},$$

where (q, q') is a fixed pair of positive numbers with $1/q + 1/q' = 1$ and $1 < q' < 2$.

If we compare the generalized Calderón-Zygmund operator with the classical Calderón-Zygmund operator, whose kernel K is smooth away from the diagonal $\{(x, y) : x = y\}$ with

$$\begin{aligned} |K(x, y)| &\leq C|x - y|^{-n}, \\ |K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \\ &\leq C|x - y|^{-n} \left(\frac{|z - y|}{|x - y|} \right)^\delta, \end{aligned} \quad (4)$$

where $|x - y| > 2|z - y|$ for some $\delta > 0$, we can find out that the classical Calderón-Zygmund operator is a generalized Calderón-Zygmund operator defined above with $C_j = 2^{-j\delta}$, $j \in \mathbb{N}$, and any $1 < q < \infty$.

Let b be a locally integrable function on $\mathbb{R}^n$. The Toeplitz type operator associated with generalized Calderón-Zygmund operator and fractional integral operator I_α is defined by

$$T^b = T_1 M^b I_\alpha T_2 + T_3 I_\alpha M^b T_4, \quad (5)$$

where T_1 is the generalized Calderón-Zygmund operator or $\pm I$ (the identity operator), T_2 and T_4 are the linear operators, $T_3 = \pm I$, and $M^b f = bf$.

Note that the commutators $[b, I_\alpha](f) = bI_\alpha(f) - I_\alpha(bf)$ are the particular cases of the Toeplitz type operators T^b . The Toeplitz type operators T^b are the nontrivial generalization of these commutators.

The boundedness of the singular integral commutators generated by BMO function was obtained in [5–8]. Motivated by these, in this paper, we investigate the boundedness of T^b on the weighted Morrey space when b belongs to weighted BMO space and we have the following result.

Theorem 2. Suppose that T^b is a Toeplitz type operator associated with generalized Calderón-Zygmund operator and fractional integral operator I_α , and $b \in BMO(\omega)$. Let $0 < \alpha < n$, $q' < p < n/\alpha$, $1/s = 1/p - \alpha/n$, $0 < \kappa < p/s$, $\{jC_j\} \in l^1$, $\omega^{s/p} \in A_1$ and the critical index of ω for the reverse Hölder condition $r_\omega > \max\{(s-1)q'/(s-q'), (1-\kappa)/(p/s-\kappa)\}$. If $T^1(f) = 0$ for any $f \in L^{p,\kappa}(\omega)$, T_2 and T_4 are the bounded operators on $L^{p,\kappa}(\omega)$, and then there exists a constant $C > 0$ such that

$$\|T^b(f)\|_{L^{s,\kappa s/p}(\omega^{1-(1-\alpha/n)s}, \omega)} \leq C \|b\|_{BMO(\omega)} \|f\|_{L^{p,\kappa}(\omega)}. \quad (6)$$

The following results are immediately obtained from Theorem 2.

Corollary 3 (see [5]). Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$, and $\omega^{q/p} \in A_1$. Suppose that $b \in BMO(\omega)$ and the critical index of ω for the reverse Hölder condition $r_\omega > (1-\kappa)/(p/q-\kappa)$; then, $[b, I_\alpha]$ is bounded from $L^{p,\kappa}(\omega)$ to $L^{q,\kappa q/p}(\omega^{1-(1-\alpha/n)q}, \omega)$.

Corollary 4. Suppose that T^b is a Toeplitz type operator associated with generalized Calderón-Zygmund operator and

fractional integral operator I_α and $b \in BMO(\mathbb{R}^n)$. Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/s = 1/p - \alpha/n$, $0 < \kappa < p/s$, and $\{jC_j\} \in l^1$. If $T^1(f) = 0$ for any $f \in L^{p,\kappa}(\mathbb{R}^n)$, $T^{k,2}$ and $T^{k,4}$ are the bounded operators on $L^{p,\kappa}(\mathbb{R}^n)$; then, there exists a constant $C > 0$ such that

$$\|T^b(f)\|_{L^{s,\kappa s/p}(\mathbb{R}^n)} \leq C \|b\|_{BMO} \|f\|_{L^{p,\kappa}(\mathbb{R}^n)}. \quad (7)$$

The paper is organized as follows. In Section 2, we will introduce some notation and definitions and recall some preliminary results. In Section 3, we give the sharp estimates for T^b . In Section 4, we will give the proof of Theorem 2.

2. Some Preliminaries

First, let us recall some notation and the definition of weight classes.

A weight ω is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B = B_r(x_0)$ denote the ball with the center x_0 and radius r , and let $\lambda B = B_{\lambda r}(x_0)$ for any $\lambda > 0$. For a given weight function ω and a measurable set E , we also denote the Lebesgue measure of E by $|E|$ and set weighted measure $\omega(E) = \int_E \omega(x) dx$. For any given weight function ω on $\mathbb{R}^n$, $0 < p < \infty$, denote by $L^p(\omega)$ the space of all function f satisfying

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \right)^{1/p} < \infty. \quad (8)$$

Definition 5 (see [9]). A weight ω is said to belong to the Muckenhoupt class A_p for $1 < p < \infty$, if there exists a constant C such that

$$\left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{-1/(p-1)} dx \right)^{p-1} \leq C \quad (9)$$

for every ball B . The class A_1 is defined by replacing the above inequality with

$$\frac{1}{|B|} \int_B \omega(y) dy \leq C \cdot \operatorname{ess\,inf}_{x \in B} \omega(x). \quad (10)$$

When $p = \infty$, we defined $A_\infty = \cup_{1 \leq p < \infty} A_p$.

Definition 6 (see [10]). A weight function ω belongs to $A_{p,q}$ for $1 < p < q < \infty$, if for every ball B in $\mathbb{R}^n$, there exists a positive constant C which is independent of B such that

$$\left(\frac{1}{|B|} \int_B \omega(y)^{-p'} dy \right)^{1/p'} \left(\frac{1}{|B|} \int_B \omega(y)^q dy \right)^{1/q} \leq C, \quad (11)$$

where p' denotes the conjugate exponent of $p > 1$; that is, $1/p + 1/p' = 1$.

From the definition of $A_{p,q}$, we can get that

$$\omega \in A_{p,q}, \quad \text{iff } \omega^q \in A_{1+q/p'}. \quad (12)$$

Definition 7 (see [11]). A weight function ω belongs to the reverse Hölder class RH_u if there exist two constants $u > 1$ and $C > 0$ such that the following reverse Hölder inequality

$$\left(\frac{1}{|B|} \int_B \omega(x)^u dx \right)^{1/u} \leq C \left(\frac{1}{|B|} \int_B \omega(x) dx \right) \quad (13)$$

holds for every ball $B \subset \mathbb{R}^n$.

It is well known that if $\omega \in A_p$ with $1 < p < \infty$, then $\omega \in A_u$ for all $u > p$ and $\omega \in A_q$ for some $1 < q < p$. If $\omega \in A_p$ with $1 \leq p < \infty$, then there exists $u > 1$ such that $\omega \in \text{RH}_u$. It follows directly from Hölder's inequality that $\omega \in \text{RH}_u$ implies $\omega \in \text{RH}_s$ for all $1 < s < u$. Moreover, if $\omega \in \text{RH}_u$, $u > 1$, then we have $\omega \in \text{RH}_{u+\varepsilon}$ for some $\varepsilon > 0$. We write $r_\omega = \sup\{u > 1 : \omega \in \text{RH}_u\}$ to denote the critical index of ω for the reverse Hölder condition.

Lemma 8 (see [11]). Suppose $\omega \in A_1$. Then, there exist two constants C_1 and C_2 , such that

$$C_1 \omega(B) \leq |B| \inf_{x \in B} \omega(x) \leq C_2 \omega(B). \quad (14)$$

Lemma 9 (see [11]). Let $\omega \in A_p$, $p \geq 1$. Then, for any ball B and any $\lambda > 1$, there exists an absolute constant $C > 0$ such that

$$\omega(\lambda B) \leq C \lambda^{pn} \omega(B), \quad (15)$$

where C does not depend on B or λ .

Next, we will recall the definition of the Hardy-Littlewood maximal operator and several variants, the fractional integral operator, and some function spaces.

Definition 10. The Hardy-Littlewood maximal operator Mf is defined by

$$M(f)(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| dy. \quad (16)$$

For $0 < \delta < 1$, the sharp maximal operator $M_\delta^\sharp f$ is defined by

$$M_\delta^\sharp(f)(x) = \sup_{x \in B} \inf_{c \in \mathbb{C}} \left(\frac{1}{|B|} \int_B \left| |f(y)|^\delta - |c|^\delta \right| dy \right)^{1/\delta}. \quad (17)$$

For $0 \leq \alpha < n$, $t \geq 1$, we define the fractional maximal operator $M_{\alpha,t} f$ by

$$M_{\alpha,t}(f)(x) = \sup_{x \in B} \left(\frac{1}{|B|^{1-\alpha t/n}} \int_B |f(y)|^t dy \right)^{1/t} \quad (18)$$

and define the fractional weighted maximal operator $M_{\alpha,t,\omega} f$ by

$$\begin{aligned} M_{\alpha,t,\omega} f(x) \\ = \sup_{x \in B} \left(\frac{1}{\omega(B)^{1-\alpha t/n}} \int_B |f(y)|^t \omega(y) dy \right)^{1/t}, \end{aligned} \quad (19)$$

where the above supremum is taken over all balls B containing x . In order to simplify the notation, we set $M_\alpha = M_{\alpha,1}$ and $M_{t,\omega} = M_{0,t,\omega}$.

Definition 11. For $0 < \alpha < n$, the fractional integral operator I_α is defined by

$$I_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy. \quad (20)$$

Lemma 12. Let I_α be fractional integral operator, and let E be a measurable set in $\mathbb{R}^n$. Then, for any $f \in L^1(\mathbb{R}^n)$, there exists a constant C such that

$$\int_E |I_\alpha f(x)| dx \leq C \|f\|_{L^1} |E|^{\alpha/n}. \quad (21)$$

Proof. Since

$$\begin{aligned} |\{x \in E : |I_\alpha f(x)| > \lambda\}| &\leq |\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}| \\ &\leq C \left(\frac{\|f\|_{L^1}}{\lambda} \right)^{n/(n-\alpha)}, \end{aligned} \quad (22)$$

we have

$$\begin{aligned} \int_E |I_\alpha f(x)| dx &= \int_0^\infty |\{x \in E : |I_\alpha f(x)| > \lambda\}| d\lambda \\ &\leq \int_0^\infty \min \left\{ C \left(\frac{\|f\|_{L^1}}{\lambda} \right)^{n/(n-\alpha)}, |E| \right\} d\lambda \\ &\leq \int_0^{C\|f\|_{L^1}|E|^{\alpha/n-1}} |E| d\lambda \\ &\quad + \int_{C\|f\|_{L^1}|E|^{\alpha/n-1}}^\infty \left(\frac{\|f\|_{L^1}}{\lambda} \right)^{n/(n-\alpha)} d\lambda \\ &\leq C \|f\|_{L^1} |E|^{\alpha/n}. \end{aligned} \quad (23)$$

□

Definition 13. Let $1 \leq p < \infty$ and ω be a weighted function. A locally integrable function b is said to be in $\text{BMO}_p(\omega)$ if

$$\begin{aligned} \|b\|_{\text{BMO}_p(\omega)} \\ = \sup_B \left[\frac{1}{\omega(B)} \int_B |b(x) - b_B|^p \omega(x)^{1-p} dx \right]^{1/p} \\ < \infty, \end{aligned} \quad (24)$$

where $b_B = (1/|B|) \int_B b(y) dy$ and the supremum is taken over all balls $B \subset \mathbb{R}^n$.

Lemma 14 (see [12]). Let $\omega \in A_1$. Then, for any $1 \leq p < \infty$, there exists an absolute constant $C > 0$ such that $\|b\|_{\text{BMO}_p(\omega)} \leq C \|b\|_{\text{BMO}(\omega)}$.

Definition 15 (see [4]). Let $1 \leq p < \infty$, $0 < \kappa < 1$, and let ω be a weight function. Then, the weighted Morrey space is defined by

$$L^{p,\kappa}(\omega) = \{f \in L_{\text{loc}}^p(\omega) : \|f\|_{L^{p,\kappa}(\omega)} < \infty\}, \quad (25)$$

where

$$\|f\|_{L^{p,\kappa}(\omega)} = \sup_B \left(\frac{1}{\omega(B)^\kappa} \int_B |f(x)|^p \omega(x) dx \right)^{1/p}, \quad (26)$$

and the supremum is taken over all balls $B \subset \mathbb{R}^n$.

In order to deal with the fractional order case, we need to consider the weighted Morrey space with two weights.

Definition 16 (see [4]). Let $1 \leq p < \infty$ and $0 < \kappa < 1$. Then, for two weights μ and ν , the weighted Morrey space is defined by

$$L^{p,\kappa}(\mu, \nu) = \{f \in L^p_{\text{loc}}(\mu) : \|f\|_{L^{p,\kappa}(\mu, \nu)} < \infty\}, \quad (27)$$

where

$$\|f\|_{L^{p,\kappa}(\mu, \nu)} = \sup_B \left(\frac{1}{\nu(B)^\kappa} \int_B |f(x)|^p \mu(x) dx \right)^{1/p}, \quad (28)$$

and the supremum is taken over all balls $B \subset \mathbb{R}^n$.

We list a series of lemmas which will be used in the proof of our theorem.

Lemma 17 (see [5]). Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$, and $\omega \in A_\infty$. Then, for any $1 \leq r < p$, we have

$$\|M_{\alpha, r, \omega}(f)\|_{L^{q, \kappa q/p}(\omega)} \leq C \|f\|_{L^{p, \kappa}(\omega)}. \quad (29)$$

Lemma 18 (see [5]). Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $\omega^{p/q} \in A_1$, and $r_\omega > (1 - \kappa)/(p/q - \kappa)$. Then, for every $0 < \kappa < p/q$ and $1 \leq r < p$, we have

$$\|M_{r, \omega}(f)\|_{L^{q, \kappa q/p}(\omega^{q/p}, \omega)} \leq C \|f\|_{L^{p, \kappa q/p}(\omega^{q/p}, \omega)}. \quad (30)$$

Lemma 19 (see [5]). Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $\omega^{p/q} \in A_1$. Then, if $0 < \kappa < p/q$ and $r_\omega > (1 - \kappa)/(p/q - \kappa)$, we have

$$\|M_\alpha(f)\|_{L^{q, \kappa q/p}(\omega^{q/p}, \omega)} \leq C \|f\|_{L^{p, \kappa}(\omega)}. \quad (31)$$

Lemma 20 (see [4]). Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $\omega^{p/q} \in A_1$. Then, if $0 < \kappa < p/q$, we have

$$\|I_\alpha(f)\|_{L^{q, \kappa q/p}(\omega^{q/p}, \omega)} \leq C \|f\|_{L^{p, \kappa}(\omega)}. \quad (32)$$

3. The Sharp Estimates for T^b

In this section, we will prove the sharp estimates for T^b as follows.

Theorem 21. Suppose that T^b is a Toeplitz type operator associated with generalized Calderón-Zygmund operator and fractional integral operator I_α , and $b \in \text{BMO}(\omega)$. Let $0 < \delta < 1$, $0 < \alpha < n$, $q' < p < n/\alpha$, $\{jC_j\} \in l^1$, $\omega \in A_1$, $r_\omega > q'$, and

$t > (r_\omega - 1)q'/(r_\omega - q')$. If $T^1(f) = 0$ for any $f \in L^{p, \kappa}(\omega)$, then there exists a constant $C > 0$ such that

$$\begin{aligned} M_\delta^\#(T^b(f))(x) &\leq C \|b\|_{\text{BMO}(\omega)} \omega(x) \\ &\cdot (M_{t, \omega}(I_\alpha T_2 f)(x) + M_\alpha(T_4 f)(x)) \\ &+ C \|b\|_{\text{BMO}(\omega)} \omega(x)^{1-\alpha/n} M_{\alpha, t, \omega}(T_4 f)(x). \end{aligned} \quad (33)$$

Proof. For any ball $B = B(x_0, r_B)$ which contains x , without loss generality, we may assume that T_1 is a generalized Calderón-Zygmund operator. We write, by $T^1(f) = 0$,

$$\begin{aligned} T^b(f)(y) &= T_1 M^b I_\alpha T_2(f)(y) + T_3 I_\alpha M^b T_4(f)(y) \\ &= \mathcal{U}^b(y) + \mathcal{V}^b(y) \\ &= \mathcal{U}^{b-b_{2B}}(y) + \mathcal{V}^{b-b_{2B}}(y), \end{aligned} \quad (34)$$

where

$$\begin{aligned} \mathcal{U}^{b-b_{2B}}(y) &= T_1 M^{(b-b_{2B})\chi_{2B}} I_\alpha T_2(f)(y) \\ &\quad + T_1 M^{(b-b_{2B})\chi_{(2B)^c}} I_\alpha T_2(f)(y) \\ &= U_1(y) + U_2(y), \\ \mathcal{V}^{b-b_{2B}}(y) &= T_3 I_\alpha M^{(b-b_{2B})\chi_{2B}} T_4(f)(y) \\ &\quad + T_3 I_\alpha M^{(b-b_{2B})\chi_{(2B)^c}} T_4(f)(y) \\ &= V_1(y) + V_2(y). \end{aligned} \quad (35)$$

Since $0 < \delta < 1$, then

$$\begin{aligned} &\left(\frac{1}{|B|} \int_B \left| |T^b(f)(y)|^\delta - |U_2(x_0) + V_2(x_0)|^\delta \right| dy \right)^{1/\delta} \\ &\leq \left(\frac{1}{|B|} \int_B |T^b(f)(y) - U_2(x_0) - V_2(x_0)|^\delta dy \right)^{1/\delta} \\ &\leq \left(\frac{1}{|B|} \int_B |U_1(y)|^\delta dy \right)^{1/\delta} \\ &\quad + \left(\frac{1}{|B|} \int_B |V_1(y)|^\delta dy \right)^{1/\delta} \\ &\quad + \left(\frac{1}{|B|} \int_B |U_2(y) - U_2(x_0)|^\delta dy \right)^{1/\delta} \\ &\quad + \left(\frac{1}{|B|} \int_B |V_2(y) - V_2(x_0)|^\delta dy \right)^{1/\delta} \\ &= M_1 + M_2 + M_3 + M_4. \end{aligned} \quad (36)$$

We are going to estimate each term, respectively. Since T_1 is bounded from L^1 to WL^1 and $\omega \in A_1$, then, by Kolmogorov's inequality and Lemmas 8 and 9, we get

$$\begin{aligned} M_1 &= \left(\frac{1}{|B|} \int_B |U_1(y)|^\delta dy \right)^{1/\delta} \leq \frac{1}{|B|} \\ &\cdot \int_{2B} |(b(y) - b_{2B}) I_\alpha T_2(f)(y)| dy \\ &\leq \frac{C}{|B|} \left(\int_{2B} |b(y) - b_{2B}|^{t'} \omega(y)^{1-t'} dy \right)^{1/t'} \\ &\cdot \left(\int_{2B} |I_\alpha T_2 f(y)|^t \omega(y) dy \right)^{1/t} \leq C \|b\|_{\text{BMO}(\omega)} \\ &\cdot \frac{\omega(2B)}{|B|} M_{t,\omega}(I_\alpha T_2 f)(x) \leq C \|b\|_{\text{BMO}(\omega)} \omega(x) \\ &\cdot M_{t,\omega}(I_\alpha T_2 f)(x). \end{aligned} \quad (37)$$

Since $T_3 = \pm I$, by Hölder's inequality and Lemma 12, we have

$$\begin{aligned} M_2 &= \left(\frac{1}{|B|} \int_B |T_3 I_\alpha M^{(b-b_{2B})\chi_{2B}} T_4(f)(y)|^\delta dy \right)^{1/\delta} \\ &\leq \frac{C}{|B|} \int_B |T_3 I_\alpha M^{(b-b_{2B})\chi_{2B}} T_4(f)(y)| dy = \frac{C}{|B|} \\ &\cdot \int_B |I_\alpha M^{(b-b_{2B})\chi_{2B}} T_4(f)(y)| dy \leq C \frac{1}{|B|^{1-\alpha/n}} \\ &\cdot \int_{\mathbb{R}^n} |M^{(b-b_{2B})\chi_{2B}} T_4(f)(y)| dy \leq C \\ &\cdot \frac{1}{|B|^{1-\alpha/n}} \left(\int_{2B} |b(y) - b_{2B}|^{t'} \omega(y)^{1-t'} dy \right)^{1/t'} \\ &\cdot \left(\int_{2B} |T_4(f)(y)|^t \omega(y) dy \right)^{1/t} \leq C \|b\|_{\text{BMO}(\omega)} \\ &\cdot \left(\frac{\omega(B)}{|B|} \right)^{1-\alpha/n} \\ &\cdot \left(\frac{1}{\omega(B)^{(1-\alpha t/n)}} \int_{2B} |T_4(f)(y)|^t \omega(y) dy \right)^{1/t} \\ &\leq C \|b\|_{\text{BMO}(\omega)} \omega(x)^{1-\alpha/n} M_{\alpha,t,\omega}(T_4 f)(x). \end{aligned} \quad (38)$$

By the definition of generalized Calderón-Zygmund operator, we have

$$\begin{aligned} |U_2(y) - U_2(x_0)| &= |T_1 M^{(b-b_{2B})\chi_{(2B)^c}} I_\alpha T_2(f)(y) \\ &\quad - T_1 M^{(b-b_{2B})\chi_{(2B)^c}} I_\alpha T_2(f)(x_0)| \\ &\leq C \int_{(2B)^c} |b(z) - b_{2B}| |K(y, z) - K(x_0, z)| \\ &\quad \cdot |I_\alpha T_2(f)(z)| dz. \end{aligned} \quad (39)$$

Then, by Hölder's inequality, we get

$$\begin{aligned} M_3 &= \left(\frac{1}{|B|} \int_B |U_2(y) - U_2(x_0)|^\delta dy \right)^{1/\delta} \leq \frac{1}{|B|} \\ &\cdot \int_B |U_2(y) - U_2(x_0)| dy \leq \frac{1}{|B|} \\ &\cdot \int_B |T_1 M^{(b-b_{2B})\chi_{(2B)^c}} I_\alpha T_2(f)(y) \\ &\quad - T_1 M^{(b-b_{2B})\chi_{(2B)^c}} I_\alpha T_2(f)(x_0)| dy \leq \frac{C}{|B|} \\ &\cdot \int_B \int_{(2B)^c} |b(z) - b_{2B}| |K(y, z) - K(x_0, z)| \\ &\quad \cdot |I_\alpha T_2(f)(z)| dz dy \leq \frac{C}{|B|} \\ &\cdot \sum_{j=1}^{\infty} \int_B \int_{2^j|y-x_0| \leq |z-x_0| < 2^{j+1}|y-x_0|} |b(z) - b_{2^{j+1}B}| \\ &\quad \cdot |K(y, z) - K(x_0, z)| |I_\alpha T_2(f)(z)| dz dy \\ &\quad + \frac{C}{|B|} \sum_{j=1}^{\infty} |b_{2^{j+1}B} - b_{2B}| \\ &\quad \cdot \int_B \int_{2^j|y-x_0| \leq |z-x_0| < 2^{j+1}|y-x_0|} |K(y, z) - K(x_0, z)| \\ &\quad \cdot |I_\alpha T_2(f)(z)| dz dy = M_{31} + M_{32}. \end{aligned} \quad (40)$$

Since $r_\omega > q' > 1$ and $t > (r_\omega - 1)q'/(r_\omega - q') > q'$, then there exists $1 < l < \infty$ such that $1/q + 1/t + 1/l = 1$. Applying Hölder's inequality for q, l , and t and by (3) of Definition 1, we get

$$\begin{aligned} M_{31} &\leq \frac{C}{|B|} \\ &\cdot \sum_{j=1}^{\infty} \int_B \left(\int_{2^j|y-x_0| \leq |z-x_0| < 2^{j+1}|y-x_0|} |K(y, z) - K(x_0, z)|^q dz \right)^{1/q} \\ &\cdot \left(\int_{2^j|y-x_0| \leq |z-x_0| < 2^{j+1}|y-x_0|} |b(z) - b_{2^{j+1}B}|^l \omega(z)^{(1/t'-1)l} dz \right)^{1/l} \\ &\cdot \left(\int_{2^j|y-x_0| \leq |z-x_0| < 2^{j+1}|y-x_0|} |I_\alpha T_2(f)(z)|^t \omega(z) dz \right)^{1/t} dy \\ &\leq \frac{C}{|B|} \sum_{j=1}^{\infty} C_j \int_B (2^j|y-x_0|)^{-n/q'} dy \\ &\cdot \left(\int_{2^{j+1}B} |b(z) - b_{2^{j+1}B}|^l \omega(z)^{(1/t'-1)l} dz \right)^{1/l} \\ &\cdot \left(\int_{2^{j+1}B} |I_\alpha T_2(f)(z)|^t \omega(z) dz \right)^{1/t}. \end{aligned} \quad (41)$$

Note that

$$\int_B (2^j|y-x_0|)^{-n/q'} dy \leq C |2^jB|^{-1/q'} r^n; \quad (42)$$

then,

$$M_{31} \leq CM_{t,\omega} (I_\alpha T_2 f)(x) \sum_{j=1}^{\infty} C_j \left| 2^j B \right|^{-1/q'} \omega \left(2^j B \right)^{1/t} \cdot \left(\int_{2^{j+1}B} |b(z) - b_{2^{j+1}B}|^l \omega(z)^{(1/t'-1)l} dz \right)^{1/l}. \quad (43)$$

Since $t > (r_\omega - 1)q'/(r_\omega - q')$, we have $r_\omega > (t-1)q'/(t-q')$. By $r_\omega = \sup\{u > 1 : \omega \in \text{RH}_u\}$, there is u such that $u > (t-1)q'/(t-q') > q' > 1$. Let $p_0 = (u-1)/((t-1)q'/(t-q') - 1)$; then, $1 < p_0 < \infty$. By $1/q + 1/l + 1/t = 1$, we have $l/t' = (t-1)q'/(t-q')$. Then, $p_0 = (u-1)/(l/t' - 1)$; that is, $u = lp_0/t' - p_0/p_0'$. Then, we have

$$\frac{(1-u)}{(p_0 l)} - \frac{1}{q'} = -1, \quad (44)$$

$$\frac{1}{t} + \frac{1}{(p_0' l)} + \frac{u}{(p_0 l)}.$$

Applying Hölder's inequality for p_0 and p_0' , the definition of weighted BMO, and $\omega \in \text{RH}_u$, we have

$$\left(\int_{2^{j+1}B} |b(z) - b_{2^{j+1}B}|^l \omega(z)^{(1/t'-1)l} dz \right)^{1/l} \leq C \left(\int_{2^{j+1}B} |b(z) - b_{2^{j+1}B}|^{lp_0'} \omega(z)^{1-lp_0'} dz \right)^{1/(p_0' l)} \cdot \left(\int_{2^{j+1}B} \omega(z)^u dz \right)^{1/(p_0' l)} \leq C \|b\|_{\text{BMO}(\omega)} \cdot \left| 2^j B \right|^{(1-u)/(p_0 l)} \omega \left(2^j B \right)^{1/(p_0' l) + u/(p_0 l)}.$$

Hence,

$$M_{31} \leq C \|b\|_{\text{BMO}(\omega)} M_{t,\omega} (I_\alpha T_2 f)(x) \cdot \sum_{j=1}^{\infty} C_j \left| 2^j B \right|^{(1-u)/(p_0 l) - 1/q'} \omega \left(2^j B \right)^{1/t + 1/(p_0' l) + u/(p_0 l)} \leq C \|b\|_{\text{BMO}(\omega)} M_{t,\omega} (I_\alpha T_2 f)(x) \sum_{j=1}^{\infty} C_j \frac{\omega \left(2^j B \right)}{\left| 2^j B \right|} \leq C \omega(x) \|b\|_{\text{BMO}(\omega)} M_{t,\omega} (I_\alpha T_2 f)(x). \quad (46)$$

Note that

$$|b_{2^{j+1}B} - b_{2B}| \leq \sum_{k=1}^j \frac{1}{|2^k B|} \int_{2^{k+1}B} |b(z) - b_{2^{k+1}B}| dz \leq C_j \omega(x) \|b\|_{\text{BMO}(\omega)}. \quad (47)$$

Thus, by Hölder's inequality, we get

$$M_{32} \leq \frac{C}{|B|} \cdot \sum_{j=1}^{\infty} \int_B \left(\int_{2^j |y-x_0| \leq |z-x_0| < 2^{j+1} |y-x_0|} |K(y, z) - K(x_0, z)|^q dz \right)^{1/q} \cdot |b_{2^{j+1}B} - b_{2B}| \left| 2^{j+1} B \right|^{1/l} \left(\int_{2^{j+1}B} |I_\alpha T_2(f)(z)|^t dz \right)^{1/t} dy \leq \frac{C}{|B|} \omega(x) \|b\|_{\text{BMO}(\omega)} \sum_{j=1}^{\infty} j C_j \int_B \left(2^j |y-x_0| \right)^{-n/q'} dy \cdot \left(\frac{1}{\omega(2^{j+1}B)} \int_{2^{j+1}B} |I_\alpha T_2(f)(z)|^t \omega(z) dz \right)^{1/t} \left| 2^{j+1} B \right|^{1/l + 1/t} \leq C \omega(x) \|b\|_{\text{BMO}(\omega)} M_{t,\omega} (I_\alpha T_2 f)(x) \sum_{j=1}^{\infty} j C_j \left| 2^{j+1} B \right|^{1/l + 1/t - 1/q'} \leq C \omega(x) \|b\|_{\text{BMO}(\omega)} M_{t,\omega} (I_\alpha T_2 f)(x) \sum_{j=1}^{\infty} j C_j \leq C \omega(x) \cdot \|b\|_{\text{BMO}(\omega)} M_{t,\omega} (I_\alpha T_2 f)(x).$$

Then,

$$M_3 \leq C \omega(x) \|b\|_{\text{BMO}(\omega)} M_{t,\omega} (I_\alpha T_2 f)(x). \quad (49)$$

For any $y \in B$ and $z \in (2B)^c$, we have $|y-z| \sim |x_0-z|$. Then,

$$M_4 = \left(\frac{1}{|B|} \int_B |V_2(y) - V_2(x_0)|^\delta dy \right)^{1/\delta} \leq \frac{1}{|B|} \cdot \int_B |V_2(y) - V_2(x_0)| dy \leq \frac{1}{|B|} \cdot \int_B |T_3 I_\alpha M^{(b-b_{2B})\chi_{(2B)^c}} T_4(f)(y) - T_3 I_\alpha M^{(b-b_{2B})\chi_{(2B)^c}} T_4(f)(x_0)| dy \leq C \frac{1}{|B|} \cdot \int_B \int_{(2B)^c} |b(z) - b_{2B}| \left| \frac{1}{|y-z|^{n-\alpha}} - \frac{1}{|x_0-z|^{n-\alpha}} \right| \cdot |T_4(f)(z)| dz dy \leq C \frac{1}{|B|} \cdot \int_B \int_{(2B)^c} \left| b(z) - b_{2B} \frac{|x_0-y|}{|x_0-z|^{n-\alpha+1}} \right| \cdot |T_4(f)(z)| dz dy \leq C \sum_{j=1}^{\infty} \frac{r_B}{(2^j r_B)^{n-\alpha+1}} \int_{2^{j+1}B} |b(z) - b_{2B}| \cdot |T_4(f)(z)| dz \leq C \sum_{j=1}^{\infty} 2^{-j} |b_{2^{j+1}B} - b_{2B}|$$

$$\begin{aligned}
& \cdot \frac{1}{|2^{j+1}B|^{1-\alpha/n}} \int_{2^{j+1}B} |T_4(f)(z)| dz + C \sum_{j=1}^{\infty} 2^{-j} \\
& \cdot \frac{1}{|2^{j+1}B|^{1-\alpha/n}} \int_{2^{j+1}B} |b(z) - b_{2^{j+1}B}| \\
& \cdot |T_4(f)(z)| dz = M_{41} + N_{42}.
\end{aligned} \tag{50}$$

Note that

$$|b_{2^{j+1}B} - b_{2B}| \leq C_j \|b\|_{\text{BMO}(\omega)}; \tag{51}$$

then,

$$\begin{aligned}
M_{41} &= C \sum_{j=1}^{\infty} 2^{-j} |b_{2^{j+1}B} - b_{2B}| \frac{1}{|2^{j+1}B|^{1-\alpha/n}} \\
& \cdot \int_{2^{j+1}B} |T_4(f)(z)| dz \leq C \|b\|_{\text{BMO}(\omega)} \omega(x) \\
& \cdot M_{\alpha}(T_4 f)(x) \sum_{j=1}^{\infty} j 2^{-j} \leq C \|b\|_{\text{BMO}(\omega)} \omega(x) \\
& \cdot M_{\alpha}(T_4 f)(x).
\end{aligned} \tag{52}$$

By Hölder's inequality,

$$\begin{aligned}
M_{42} &= C \sum_{j=1}^{\infty} 2^{-j} \frac{1}{|2^{j+1}B|^{1-\alpha/n}} \int_{2^{j+1}B} |b(z) - b_{2^{j+1}B}| \\
& \cdot |T_4(f)(z)| dz \leq C \sum_{j=1}^{\infty} 2^{-j} \\
& \cdot \frac{1}{|2^{j+1}B|^{1-\alpha/n}} \left(\int_{2^{j+1}B} |b(z) - b_{2^{j+1}B}|^{t'} \right. \\
& \cdot \omega(z)^{1-t'} dz \Big)^{1/t'} \left(\int_{2^{j+1}B} |T_4(f)(z)|^t \right. \\
& \cdot \omega(z) dz \Big)^{1/t} \leq C \|b\|_{\text{BMO}(\omega)} \\
& \cdot \sum_{j=1}^{\infty} 2^{-j} \left(\frac{\omega(2^{j+1}B)}{|2^{j+1}B|} \right)^{1-\alpha/n} \left(\frac{1}{\omega(2^{j+1}B)^{1-\alpha t/n}} \right. \\
& \cdot \left. \int_{2^{j+1}B} |T_4(f)(z)|^t \omega(z) dz \right)^{1/t} \leq C \|b\|_{\text{BMO}(\omega)} \\
& \cdot \omega(x)^{1-\alpha/n} M_{\alpha,t,\omega}(T_4 f)(x).
\end{aligned} \tag{53}$$

Then,

$$\begin{aligned}
M_4 &\leq C \|b\|_{\text{BMO}(\omega)} \left(\omega(x) M_{\alpha}(T_4 f)(x) \right. \\
& \left. + \omega(x)^{1-\alpha/n} M_{\alpha,t,\omega}(T_4 f)(x) \right).
\end{aligned} \tag{54}$$

Combining the estimates for M_1 , M_2 , M_3 , and M_4 , the proof of Theorem 21 is completed. $\square$

4. Proof of Theorem 2

To prove Theorem 2, we need the following analogy of the classical Fefferman-Stein inequality for the sharp maximal function $M_{\delta}^{\sharp} f$; its proof can be found in [13].

Lemma 22. *Let $0 < \delta < 1$, $0 < \kappa < 1$, and $1 < p < \infty$. If $\mu, \nu \in A_{\infty}$, then we have*

$$\|M_{\delta}(f)\|_{L^{p,\kappa}(\mu,\nu)} \leq C \|M_{\delta}^{\sharp} f\|_{L^{p,\kappa}(\mu,\nu)} \tag{55}$$

for all functions f such that the left hand side is finite.

Proof. It follows from $r_{\omega} > (s-1)q'/(s-q')$ that $s > (r_{\omega}-1)q'/(r_{\omega}-q')$; then, there exists t such that $s > t > (r_{\omega}-1)q'/(r_{\omega}-q')$. Note that

$$s+1 - \left(1 - \frac{\alpha}{n}\right)s = \frac{s}{p}; \tag{56}$$

then, by Lemma 22 and Theorem 21, we have

$$\begin{aligned}
& \|T^b(f)\|_{L^{s,\kappa s/p}(\omega^{1-(1-\alpha/n)s},\omega)} \\
& \leq \|M_{\delta}^{\sharp}(T^b(f))\|_{L^{s,\kappa s/p}(\omega^{1-(1-\alpha/n)s},\omega)} \leq C \|b\|_{\text{BMO}(\omega)} \\
& \cdot \left(\|M_{t,\omega}(I_{\alpha} T_2 f)\|_{L^{s,\kappa s/p}(\omega^{s/p},\omega)} \right. \\
& \left. + \|M_{\alpha}(T_4 f)\|_{L^{s,\kappa s/p}(\omega^{s/p},\omega)} \right) + C \|b\|_{\text{BMO}(\omega)} \\
& \cdot \|M_{\alpha,t,\omega}(T_4 f)\|_{L^{s,\kappa s/p}(\omega)}.
\end{aligned} \tag{57}$$

Since $r_{\omega} > (1-\kappa)/(p/s-\kappa)$, by Lemmas 17–20, we get

$$\begin{aligned}
& \|T^b(f)\|_{L^{s,\kappa s/p}(\omega^{1-(1-\alpha/n)s},\omega)} \\
& \leq C \|b\|_{\text{BMO}(\omega)} \left(\|I_{\alpha} T_2 f\|_{L^{s,\kappa s/p}(\omega^{s/p},\omega)} + \|T_4 f\|_{L^{p,\kappa}(\omega)} \right) \\
& \leq C \|b\|_{\text{BMO}(\omega)} \|f\|_{L^{p,\kappa}(\omega)}.
\end{aligned} \tag{58}$$

This finishes the proof of Theorem 2. $\square$

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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