Research Article

A Fixed Point Approach to the Stability of an Additive-Quadratic-Cubic-Quartic Type Functional Equation

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By applying a fixed point approach, we investigate the stability problems for an AQCQ functional equation of the form $f(x+2y) + f(x-2y) - 4f(x+y) - 4f(x-y) + 6f(x) - f(2y) - f(-2y) + 4f(y) + 4f(-y) = 0$.

1. Introduction

In 1940, Ulam [1] posed the problem concerning the stability of group homomorphisms. In the following year, Hyers [2] gave an affirmative answer to Ulam's problem for additive mappings between Banach spaces. Thereafter, many mathematicians came to deal with this subject (cf. [3, 4]).

We now consider the following functional equation:

$$\begin{aligned} & f(x+2y) + f(x-2y) - 4f(x+y) - 4f(x-y) \\ & + 6f(x) - f(2y) - f(-2y) + 4f(y) + 4f(-y) \quad (1) \\ & = 0. \end{aligned}$$

The mappings $f(x) = ax^4 + bx^3 + cx^2 + dx$ are solutions of this functional equation, where a, b, c, d are real constants.

The stability problems of (1) were investigated by Lee et al. [5]. Indeed, they proved the stability of the equation by dividing the relevant mapping into the additive, quadratic, cubic, and quartic parts and applying the fixed point method to each part separately. Unfortunately, they could not prove the uniqueness of the exact solution of (1) which approximates the (given) mapping that is a solution of the perturbed equation of (1).

In this paper, we will show that every solution of functional equation (1) is an AQCQ mapping (see below for

the definition of AQCQ mapping) and we introduce a strictly contractive mapping which allows us to use the fixed point theory in the sense of Cădariu and Radu (see [6–8]). In a different way from Lee et al. [5], we do not split the (given) mapping which satisfies the perturbed equation of (1) and we apply the fixed point method to construct the explicit form of the exact solution of (1) near the (given) mapping by

$$\begin{aligned} F(x) = \lim_{n \rightarrow \infty} \sum_{i=0}^n {}^nC_i \left(\frac{(-1)^{n-i} 10^i}{16^n} f_o(2^{2n-i}x) \right. \\ \left. + \frac{(-1)^{n-i} 20^i}{64^n} f_e(2^{2n-i}x) \right) \end{aligned} \quad (2)$$

or

$$\begin{aligned} F(x) = \lim_{n \rightarrow \infty} \sum_{i=0}^n {}^nC_i \left(10^i (-16)^{n-i} f_o\left(\frac{x}{2^{2n-i}}\right) \right. \\ \left. + 20^i (-64)^{n-i} f_e\left(\frac{x}{2^{2n-i}}\right) \right), \end{aligned} \quad (3)$$

where we denote by f_o and f_e the odd and even parts of f , respectively.

Using this method, we can prove the uniqueness of the exact solution of (1) near the (given) mapping.

Throughout this paper, let $\mathbb{N}$ denote the set of all positive integers and let $\mathbb{N}_0$ denote the set of all nonnegative integers.

2. Preliminaries

We recall the fixed point theorem of Diaz and Margolis [9].

Theorem 1. Assume that (X, d) is a complete generalized metric space and $J : X \rightarrow X$ is a strictly contractive mapping with the Lipschitz constant $0 < L < 1$. It then holds that, for each $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all $n \in \mathbb{N}_0$ or there exists $k \in \mathbb{N}_0$ such that

- (i) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq k$;
- (ii) the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J ;
- (iii) y^* is the unique fixed point of J in $Y = \{y \in X : d(J^k x, y) < \infty\}$;
- (iv) $d(y, y^*) \leq (1/(1-L))d(y, Jy)$ for all $y \in Y$.

Throughout this paper, let V and W be real vector spaces, let X be a real normed space, and let Y be a real Banach space.

For a given mapping $f : V \rightarrow W$, we use the following abbreviations:

$$f_o(x) := \frac{f(x) - f(-x)}{2},$$

$$f_e(x) := \frac{f(x) + f(-x)}{2},$$

$$Af(x, y) := f(x+y) - f(x) - f(y),$$

$$Qf(x, y) := f(x+y) + f(x-y) - 2f(x) - 2f(y),$$

$$Cf(x, y) := f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y),$$

$$Q'f(x, y) := f(x+2y) - 4f(x+y) + 6f(x) - 4f(x-y) + f(x-2y) - 24f(y), \quad (4)$$

$$Ef(x, y) := f(x+2y) - 4f(x+y) + 6f(x) - 4f(x-y) + f(x-2y),$$

$$\begin{aligned} Df(x, y) &:= f(x+2y) + f(x-2y) - 4f(x+y) \\ &\quad - 4f(x-y) + 6f(x) - f(2y) \\ &\quad - f(-2y) + 4f(y) + 4f(-y) \\ &= Ef(x, y) - f(2y) - f(-2y) + 4f(y) \\ &\quad + 4f(-y) \\ &= Q'f(x, y) - f(2y) - f(-2y) + 28f(y) \\ &\quad + 4f(-y) \end{aligned}$$

for all $x, y \in V$. Each solution of $Af(x, y) = 0$, $Qf(x, y) = 0$, $Cf(x, y) = 0$, and $Q'f(x, y) = 0$ is called an additive mapping, a quadratic mapping, a cubic mapping, and a quartic

mapping, respectively. Every mapping $f : V \rightarrow W$ is called an AQCQ mapping (additive-quadratic-cubic-quartic mapping) if f can be expressed by the sum of an additive mapping, a quadratic mapping, a cubic mapping, and a quartic mapping.

The following lemmas have been proved in [10].

Lemma 2 ([10], Theorem 2.3). If $f : V \rightarrow W$ is an odd mapping satisfying $Ef(x, y) = 0$ for all $x, y \in V$, then there exists a cubic mapping $C : V \rightarrow W$ and an additive mapping $A : V \rightarrow W$ such that $f(x) = C(x) + A(x)$ for all $x \in V$.

Lemma 3 ([10], Theorem 2.4). If $f : V \rightarrow W$ is an even mapping satisfying the equalities $Ef(x, y) = 0$ for all $x, y \in V$ and $f(2x) = 4f(x)$ for all $x \in V$, then f is a quadratic mapping.

In the following theorem, we investigate the general solution of the functional equation $Df(x, y) = 0$.

Theorem 4. A mapping $f : V \rightarrow W$ satisfies $Df(x, y) = 0$ for all $x, y \in V$ if and only if f is an AQCQ mapping.

Proof. First, we assume that a mapping $f : V \rightarrow W$ satisfies $Df(x, y) = 0$ for all $x, y \in V$. Then, the odd part f_o satisfies the equalities

$$\begin{aligned} Ef_o(x, y) &= Df_o(x, y) = \frac{1}{2}Df(x, y) - \frac{1}{2}Df(-x, -y) \\ &= 0 \end{aligned} \quad (5)$$

for all $x, y \in V$. It follows from Lemma 2 that f_o is a cubic-additive mapping.

Let $f_1(x) := f_e(2x) - 16f_e(x)$ and $f_2(x) := f_e(2x) - 4f_e(x)$ for all $x \in V$. By a direct calculation and using the assumption that $Df(x, y) = 0$ for all $x, y \in V$, we obtain $f(0) = (1/6)Df(0, 0) = 0$ and

$$\begin{aligned} f_1(2x) - 4f_1(x) &= f_e(4x) - 20f_e(2x) + 64f_e(x) \\ &= f_2(2x) - 16f_2(x) \\ &= Df_e(2x, x) + 4Df_e(x, x) = 0; \end{aligned} \quad (6)$$

that is,

$$\begin{aligned} f_1(2x) &= 4f_1(x), \\ f_2(2x) &= 16f_2(x) \end{aligned} \quad (7)$$

for all $x \in V$. Using these equalities, we further get the equalities $Ef_1(x, y) = Df_1(x, y)$ and $Q'f_2(x, y) = Df_2(x, y)$ which imply the equalities $Ef_1(x, y) = 0$ and $Q'f_2(x, y) = 0$ for all $x, y \in V$. Hence, by (7) and the fact that $Ef_1(x, y) = 0$, Lemma 3 implies that f_1 is quadratic. Moreover, f_2 is quartic because of $Q'f_2(x, y) = 0$ for all $x, y \in V$. Since

$$f(x) = f_o(x) + f_e(x) = f_o(x) - \frac{1}{12}f_1(x) + \frac{1}{12}f_2(x) \quad (8)$$

for all $x \in V$, f is an AQCQ mapping.

Conversely, assume that f_1, f_2, f_3, f_4 are mappings such that the equalities $f(x) := f_1(x) + f_2(x) + f_3(x) + f_4(x)$,

$Af_1(x, y) = 0$, $Qf_2(x, y) = 0$, $Cf_3(x, y) = 0$, and $Q'f_4(x, y) = 0$ hold for all $x, y \in V$. Then the equalities $f_1(x) = -f_1(-x)$, $f_2(x) = f_2(-x)$, $f_3(x) = -f_3(-x)$, $f_4(x) = f_4(-x)$, $f_1(2x) = 2f_1(x)$, $f_2(2x) = 4f_2(x)$, $f_3(2x) = 8f_3(x)$, and $f_4(2x) = 16f_4(x)$ hold for all $x \in V$. From the above equalities, we obtain the equalities

$$\begin{aligned} Df_1(x, y) &= 4Af_1(x + y, x - y) \\ &\quad - Af_1(x + 2y, x - 2y), \\ Df_2(x, y) &= Qf_2(x, 2y) - 4Qf_2(x, y), \\ Df_3(x, y) &= Cf_3(x, y) - Cf_3(x - y, y), \\ Df_4(x, y) &= Q'f_4(x, y) \end{aligned} \quad (9)$$

for all $x, y \in V$, which imply that

$$\begin{aligned} Df(x, y) &= Df_1(x, y) + Df_2(x, y) + Df_3(x, y) \\ &\quad + Df_4(x, y) = 0 \end{aligned} \quad (10)$$

for all $x, y \in V$. $\square$

3. Main Results

Throughout this section, let V be a real vector space and let Y be a real Banach space.

In the following theorems, we can prove the generalized Hyers-Ulam stability of the functional equation (1) by using a fixed point approach.

Theorem 5. Let $f : V \rightarrow Y$ be a mapping, for which there exists a mapping $\varphi : V^2 \rightarrow [0, \infty)$ such that the inequality

$$\|Df(x, y)\| \leq \varphi(x, y) \quad (11)$$

holds for all $x, y \in V$, and let $f(0) = 0$. If there exists a constant $0 < L < 1$ such that φ satisfies the condition

$$\varphi(2x, 2y) \leq (\sqrt{41} - 5)L\varphi(x, y) \quad (12)$$

for all $x, y \in V$, then there exists a unique solution $F : V \rightarrow Y$ of (1) satisfying the inequality

$$\|f(x) - F(x)\| \leq \frac{5}{128(1-L)}\Phi(x) \quad (13)$$

for all $x \in V$, where $\Phi(x) := \varphi(2x, x) + 4\varphi(x, x) + \varphi(-2x, -x) + 4\varphi(-x, -x)$. In particular, F is represented by

$$\begin{aligned} F(x) &= \lim_{n \rightarrow \infty} \sum_{i=0}^n {}^nC_i \left(\frac{(-1)^{n-i} 10^i}{16^n} f_o(2^{2n-i}x) \right. \\ &\quad \left. + \frac{(-1)^{n-i} 20^i}{64^n} f_e(2^{2n-i}x) \right) \end{aligned} \quad (14)$$

for all $x \in V$.

Proof. Let S be the set of all functions $g : V \rightarrow Y$ with $g(0) = 0$. We introduce a generalized metric on S by

$$\begin{aligned} d(g, h) &= \inf \{K \in [0, \infty) : \|g(x) - h(x)\| \\ &\leq K\Phi(x) \quad \forall x \in V\}. \end{aligned} \quad (15)$$

It is not difficult to see that (S, d) is a generalized complete metric space (see [11, 12]).

We now consider the mapping $J : S \rightarrow S$, which is defined by

$$\begin{aligned} Jg(x) &:= -\frac{5}{128}g(4x) + \frac{3}{128}g(-4x) + \frac{60}{128}g(2x) \\ &\quad - \frac{20}{128}g(-2x) \end{aligned} \quad (16)$$

for all $x \in V$. We assert that the equality

$$\begin{aligned} J^n g(x) &= \sum_{i=0}^n {}^nC_i \frac{(-1)^{n-i} 10^i}{16^n} g_o(2^{2n-i}x) \\ &\quad + \sum_{i=0}^n {}^nC_i \frac{(-1)^{n-i} 20^i}{64^n} g_e(2^{2n-i}x) \end{aligned} \quad (17)$$

holds for all $n \in \mathbb{N}_0$ and $x \in V$. Equality (17) is true if $n = 0$. We assume that (17) is true for some $n \in \mathbb{N}_0$. Then we have

$$\begin{aligned} J^{n+1} g(x) &= J^n(Jg(x)) \\ &= \sum_{i=0}^n {}^nC_i \frac{(-1)^{n-i} 10^i}{16^n} Jg_o(2^{2n-i}x) \\ &\quad + \sum_{i=0}^n {}^nC_i \frac{(-1)^{n-i} 20^i}{64^n} Jg_e(2^{2n-i}x) \\ &= \sum_{i=0}^n {}^nC_i \frac{(-1)^{n-i} 10^i}{16^n} \frac{-5}{128} g_o(2^{2n+2-i}x) \\ &\quad + \sum_{i=0}^n {}^nC_i \frac{(-1)^{n-i} 10^i}{16^n} \frac{3}{128} g_o(-2^{2n+2-i}x) \\ &\quad + \sum_{i=0}^n {}^nC_i \frac{(-1)^{n-i} 10^i}{16^n} \frac{60}{128} g_o(2^{2n+1-i}x) \\ &\quad + \sum_{i=0}^n {}^nC_i \frac{(-1)^{n-i} 10^i}{16^n} \frac{-20}{128} g_o(-2^{2n+1-i}x) \\ &\quad + \sum_{i=0}^n {}^nC_i \frac{(-1)^{n-i} 20^i}{64^n} \frac{-5}{128} g_e(2^{2n+2-i}x) \\ &\quad + \sum_{i=0}^n {}^nC_i \frac{(-1)^{n-i} 20^i}{64^n} \frac{3}{128} g_e(-2^{2n+2-i}x) \\ &\quad + \sum_{i=0}^n {}^nC_i \frac{(-1)^{n-i} 20^i}{64^n} \frac{60}{128} g_e(2^{2n+1-i}x) \\ &\quad + \sum_{i=0}^n {}^nC_i \frac{(-1)^{n-i} 20^i}{64^n} \frac{-20}{128} g_e(-2^{2n+1-i}x) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^n {}_nC_i \frac{(-1)^{n+1-i} 10^i}{16^{n+1}} g_o(2^{2n+2-i}x) \\
&\quad + \sum_{i=0}^n {}_nC_i \frac{(-1)^{n-i} 10^{i+1}}{16^{n+1}} g_o(2^{2n+1-i}x) \\
&\quad + \sum_{i=0}^n {}_nC_i \frac{(-1)^{n+1-i} 20^i}{64^{n+1}} g_e(2^{2n+2-i}x) \\
&\quad + \sum_{i=0}^n {}_nC_i \frac{(-1)^{n-i} 20^{i+1}}{64^{n+1}} g_e(2^{2n+1-i}x) \\
&= \sum_{i=0}^{n+1} {}_{n+1}C_i \frac{(-1)^{n+1-i} 10^i}{16^{n+1}} g_o(2^{2n+2-i}x) \\
&\quad + \sum_{i=0}^{n+1} {}_{n+1}C_i \frac{(-1)^{n+1-i} 20^i}{64^{n+1}} g_e(2^{2n+2-i}x)
\end{aligned} \tag{18}$$

for all $n \in \mathbb{N}_0$ and $x \in V$, which we can obtain from (17) provided we replace n with $n+1$.

Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of d , we have

$$\begin{aligned}
\|Jg(x) - Jh(x)\| &\leq \frac{5}{128} \|g(4x) - h(4x)\| \\
&\quad + \frac{3}{128} \|g(-4x) - h(-4x)\| + \frac{60}{128} \|g(2x) \\
&\quad - h(2x)\| + \frac{20}{128} \|g(-2x) - h(-2x)\|
\end{aligned}$$

$$\begin{aligned}
&\leq K \left(\frac{5}{128} \Phi(4x) + \frac{3}{128} \Phi(-4x) + \frac{60}{128} \Phi(2x) \right. \\
&\quad \left. + \frac{20}{128} \Phi(-2x) \right) \leq K \left(\frac{8}{128} \Phi(4x) + \frac{80}{128} \Phi(2x) \right) \\
&\leq K \left(\frac{\sqrt{41}-5}{16} L\Phi(2x) + \frac{10}{16} \Phi(2x) \right) \\
&\leq K \left(\frac{(\sqrt{41}-5)^2}{16} L^2\Phi(x) \right. \\
&\quad \left. + \frac{10(\sqrt{41}-5)}{16} L\Phi(x) \right) \leq K \\
&\quad \cdot \frac{(\sqrt{41}-5)^2 + 10(\sqrt{41}-5)}{16} L\Phi(x) \leq LK\Phi(x)
\end{aligned} \tag{19}$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \leq Ld(g, h) \tag{20}$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L .

Moreover, by a long calculation and (11), we see that

$$\begin{aligned}
\|f(x) - Jf(x)\| &= \frac{\|5Df(2x, x) + 20Df(x, x) - 3Df(-2x, -x) - 12Df(-x, -x)\|}{128} \\
&\leq \frac{5\varphi(2x, x) + 20\varphi(x, x) + 3\varphi(-2x, -x) + 12\varphi(-x, -x)}{128} \leq \frac{5}{128} \Phi(x)
\end{aligned} \tag{21}$$

for all $x \in V$. It means that $d(f, Jf) \leq 5/128 < \infty$ by the definition of d . Therefore, according to Theorem 1(ii) and (iii), the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \rightarrow Y$ of J in the set $T = \{g \in S : d(f, g) < \infty\}$, which is represented by (14) for all $x \in V$.

Notice that

$$d(f, F) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{5}{128(1-L)}, \tag{22}$$

which implies (13).

By the definition of F , together with (11) and (12), we have

$$\begin{aligned}
\|DF(x, y)\| &= \lim_{n \rightarrow \infty} \|DJ^n f(x, y)\| \\
&= \lim_{n \rightarrow \infty} \left\| \sum_{i=0}^n {}_nC_i \frac{(-1)^{n-i} 10^i}{16^n} Df_o(2^{2n-i}x, 2^{2n-i}y) \right. \\
&\quad \left. + \sum_{i=0}^n {}_nC_i \frac{(-1)^{n-i} 20^i}{64^n} Df_e(2^{2n-i}x, 2^{2n-i}y) \right\|
\end{aligned}$$

$$\begin{aligned}
&\leq \lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{{}_nC_i}{2} \left(\frac{10^i}{16^n} + \frac{20^i}{64^n} \right) \times (\varphi(2^{2n-i}x, 2^{2n-i}y) \\
&\quad + \varphi(-2^{2n-i}x, -2^{2n-i}y)) \leq \lim_{n \rightarrow \infty} \sum_{i=0}^n {}_nC_i \frac{10^i}{16^n} \times (\sqrt{41} \\
&\quad - 5)^{n-i} L^{n-i} (\varphi(2^n x, 2^n y) + \varphi(-2^n x, -2^n y)) \\
&\leq \lim_{n \rightarrow \infty} \sum_{i=0}^n {}_nC_i \frac{(\sqrt{41}-5)^{n-i} 10^i}{16^n} (\varphi(2^n x, 2^n y) \\
&\quad + \varphi(-2^n x, -2^n y)) \\
&\leq \lim_{n \rightarrow \infty} \frac{(\sqrt{41}+5)^n}{16^n} (\varphi(2^n x, 2^n y) + \varphi(-2^n x, \\
&\quad - 2^n y)) \leq \lim_{n \rightarrow \infty} \frac{(\sqrt{41}+5)^n (\sqrt{41}-5)^n}{16^n}
\end{aligned}$$

$$\begin{aligned} & \cdot L^n(\varphi(x, y) + \varphi(-x, -y)) \leq \lim_{n \rightarrow \infty} L^n(\varphi(x, y) \\ & + \varphi(-x, -y)) = 0 \end{aligned} \quad (23)$$

for all $x, y \in V$; that is, F is a solution of the functional equation (1). Notice that if F is a solution of the functional equation (1), then the equality

$$\begin{aligned} F(x) - JF(x) &= \frac{5}{128}DF(2x, x) + \frac{20}{128}DF(x, x) \\ &\quad - \frac{3}{128}DF(-2x, -x) \\ &\quad - \frac{12}{128}DF(-x, -x) \end{aligned} \quad (24)$$

implies that F is a fixed point of J . $\square$

In the following theorem, we replace condition (12) for φ with another (25) and prove the generalized Hyers-Ulam stability of AQCQ equation (1) by applying a fixed point approach.

Theorem 6. Let $f : V \rightarrow Y$ be a mapping, for which there exists a mapping $\varphi : V^2 \rightarrow [0, \infty)$ such that inequality (11) holds for all $x, y \in V$, and let $f(0) = 0$. If there exists a constant $0 < L < 1$ such that φ satisfies the condition

$$L\varphi(2x, 2y) \geq \frac{32}{\sqrt{41} - 5}\varphi(x, y) \quad (25)$$

for all $x, y \in V$, then there exists a unique solution $F : V \rightarrow Y$ of (1) satisfying the inequality

$$\|f(x) - F(x)\| \leq \frac{(\sqrt{41} - 5)^2 L^2}{32^2(1 - L)}\Phi(x) \quad (26)$$

for all $x \in V$. In particular, F is represented by

$$\begin{aligned} F(x) &= \lim_{n \rightarrow \infty} \sum_{i=0}^n {}_nC_i \left(10^i (-16)^{n-i} f_o \left(\frac{x}{2^{2n-i}} \right) \right. \\ &\quad \left. + 20^i (-64)^{n-i} f_e \left(\frac{x}{2^{2n-i}} \right) \right) \end{aligned} \quad (27)$$

for all $x \in V$.

Proof. Let (S, d) be the set defined in the proof of Theorem 5. We now consider the mapping $J : S \rightarrow S$ defined by

$$\begin{aligned} Jg(x) &:= 15g\left(\frac{x}{2}\right) + 5g\left(\frac{-x}{2}\right) - 40g\left(\frac{x}{4}\right) \\ &\quad - 24g\left(\frac{-x}{4}\right) \end{aligned} \quad (28)$$

for all $x \in V$. We assert that the equality

$$\begin{aligned} J^n g(x) &= \sum_{i=0}^n {}_nC_i 10^i (-16)^{n-i} g_o \left(\frac{x}{2^{2n-i}} \right) \\ &\quad + \sum_{i=0}^n {}_nC_i 20^i (-64)^{n-i} g_e \left(\frac{x}{2^{2n-i}} \right) \end{aligned} \quad (29)$$

holds for all $n \in \mathbb{N}$ and $x \in V$. Equality (29) is obviously true when $n = 0$. We assume that (29) holds for some $n \in \mathbb{N}_0$. Then we have

$$\begin{aligned} J^{n+1} g(x) &= J^n(Jg(x)) \\ &= \sum_{i=0}^n {}_nC_i 10^i (-16)^{n-i} Jg_o \left(\frac{x}{2^{2n-i}} \right) \\ &\quad + \sum_{i=0}^n {}_nC_i 20^i (-64)^{n-i} Jg_e \left(\frac{x}{2^{2n-i}} \right) \\ &= \sum_{i=0}^n {}_nC_i 10^i (-16)^{n-i} 15g_o \left(\frac{x}{2^{2n-1-i}} \right) \\ &\quad + \sum_{i=0}^n {}_nC_i 10^i (-16)^{n-i} 5g_o \left(\frac{-x}{2^{2n-1-i}} \right) \\ &\quad - \sum_{i=0}^n {}_nC_i 10^i (-16)^{n-i} 40g_o \left(\frac{x}{2^{2n-2-i}} \right) \\ &\quad - \sum_{i=0}^n {}_nC_i 10^i (-16)^{n-i} 24g_o \left(\frac{-x}{2^{2n-2-i}} \right) \\ &\quad + \sum_{i=0}^n {}_nC_i 20^i (-64)^{n-i} 15g_e \left(\frac{x}{2^{2n-1-i}} \right) \\ &\quad + \sum_{i=0}^n {}_nC_i 20^i (-64)^{n-i} 5g_e \left(\frac{-x}{2^{2n-1-i}} \right) \\ &\quad - \sum_{i=0}^n {}_nC_i 20^i (-64)^{n-i} 40g_e \left(\frac{x}{2^{2n-2-i}} \right) \\ &\quad - \sum_{i=0}^n {}_nC_i 20^i (-64)^{n-i} 24g_e \left(\frac{-x}{2^{2n-2-i}} \right) \\ &= \sum_{i=0}^n {}_nC_i 10^{i+1} (-16)^{n-i} g_o \left(\frac{x}{2^{2n-1-i}} \right) \\ &\quad + \sum_{i=0}^n {}_nC_i 10^i (-16)^{n+1-i} g_o \left(\frac{x}{2^{2n-1-i}} \right) \\ &\quad + \sum_{i=0}^n {}_nC_i 20^{i+1} (-64)^{n-i} g_e \left(\frac{x}{2^{2n-1-i}} \right) \\ &\quad + \sum_{i=0}^n {}_nC_i 20^i (-64)^{n+1-i} g_e \left(\frac{x}{2^{2n-1-i}} \right) \\ &= \sum_{i=0}^{n+1} {}_{n+1}C_i 10^i (-16)^{n+1-i} g_o \left(\frac{x}{2^{2n+2-i}} \right) \\ &\quad + \sum_{i=0}^{n+1} {}_{n+1}C_i 20^i (-64)^{n+1-i} g_e \left(\frac{x}{2^{2n+2-i}} \right) \end{aligned} \quad (30)$$

for all $n \in \mathbb{N}_0$ and $x \in V$, which we get from (29) when we replace n by $n + 1$.

Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of d , we have

$$\begin{aligned}
 \|Jg(x) - Jh(x)\| &\leq 15 \left\| g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right) \right\| \\
 &\quad + 5 \left\| g\left(\frac{-x}{2}\right) - h\left(\frac{-x}{2}\right) \right\| \\
 &\quad + 40 \left\| g\left(\frac{x}{4}\right) - h\left(\frac{x}{4}\right) \right\| \\
 &\quad + 24 \left\| g\left(\frac{-x}{4}\right) - h\left(\frac{-x}{4}\right) \right\| \\
 &\leq 64K\Phi\left(\frac{x}{4}\right) + 20K\Phi\left(\frac{x}{2}\right) \\
 &\leq L^2 \frac{2(\sqrt{41}-5)^2}{32} K\Phi(x) \\
 &\quad + 20 \frac{\sqrt{41}-5}{32} LK\Phi(x) \\
 &\leq LK\Phi(x)
 \end{aligned} \tag{31}$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \leq Ld(g, h) \tag{32}$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L .

Moreover, by (11), we see that

$$\begin{aligned}
 \|f(x) - Jf(x)\| &= \left\| Df\left(\frac{x}{2}, \frac{x}{4}\right) + 4Df\left(\frac{x}{4}, \frac{x}{4}\right) \right\| \\
 &\leq \varphi\left(\frac{x}{2}, \frac{x}{4}\right) + 4\varphi\left(\frac{x}{4}, \frac{x}{4}\right) \\
 &\leq \frac{(\sqrt{41}-5)^2 L^2}{32^2} (\varphi(2x, x) + 4\varphi(x, x)) \\
 &\leq \frac{(\sqrt{41}-5)^2 L^2}{32^2} \Phi(x)
 \end{aligned} \tag{33}$$

for all $x \in V$, which implies that

$$d(f, Jf) \leq \frac{(\sqrt{41}-5)^2 L^2}{32^2} < \infty \tag{34}$$

by the definition of d .

Therefore, according to Theorem 1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \rightarrow Y$ of J in the set $T = \{g \in S : d(f, g) < \infty\}$, which is represented by (27) for all $x \in V$. Notice that

$$d(f, F) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{(\sqrt{41}-5)^2 L^2}{32^2 (1-L)}, \tag{35}$$

which implies (26).

By definition of F , together with (11) and (25), we have

$$\begin{aligned}
 \|DF(x, y)\| &= \lim_{n \rightarrow \infty} \|DJ^n f(x, y)\| \\
 &= \lim_{n \rightarrow \infty} \left\| \sum_{i=0}^n {}_n C_i 10^i (-16)^{n-i} f_o\left(\frac{x}{2^{2n-i}}, \frac{y}{2^{2n-i}}\right) \right. \\
 &\quad \left. + \sum_{i=0}^n {}_n C_i 20^i (-64)^{n-i} f_e\left(\frac{x}{2^{2n-i}}, \frac{y}{2^{2n-i}}\right) \right\| \\
 &\leq \lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{{}_n C_i}{2} (10^i 16^{n-i} + 20^i 64^{n-i}) \\
 &\quad \times \left(\varphi\left(\frac{x}{2^{2n-i}}, \frac{y}{2^{2n-i}}\right) + \varphi\left(\frac{-x}{2^{2n-i}}, \frac{-y}{2^{2n-i}}\right) \right) \\
 &\leq \lim_{n \rightarrow \infty} \sum_{i=0}^n {}_n C_i 20^i 64^{n-i} \frac{(\sqrt{41}-5)^{n-i} L^{n-i}}{32^{n-i}} \\
 &\quad \times \left(\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + \varphi\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right) \right) \\
 &\leq \lim_{n \rightarrow \infty} \sum_{i=0}^n {}_n C_i 20^i (2(\sqrt{41}-5))^{n-i} \left(\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right. \\
 &\quad \left. + \varphi\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right) \right) \leq \lim_{n \rightarrow \infty} (20 + 2(\sqrt{41}-5))^n \\
 &\quad \cdot \left(\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + \varphi\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right) \right) \leq \lim_{n \rightarrow \infty} 2^n (\sqrt{41} \\
 &\quad + 5)^n \frac{(\sqrt{41}-5)^n L^n}{32^n} (\varphi(x, y) + \varphi(-x, -y)) \\
 &\leq \lim_{n \rightarrow \infty} L^n (\varphi(x, y) + \varphi(-x, -y)) = 0
 \end{aligned} \tag{36}$$

for all $x, y \in V$; that is, F is a solution of the functional equation (1). Notice that if F is a solution of the functional equation (1), then the equality

$$F(x) - JF(x) = DF\left(\frac{x}{2}, \frac{x}{4}\right) + 4DF\left(\frac{x}{4}, \frac{x}{4}\right) \tag{37}$$

implies that F is a fixed point of J . $\square$

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final paper.

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