Research Article

Argument Properties for a Class of Analytic Functions Involving Libera Transform

Badr S. Alkahtani,¹ Teodor Bulboacă,² Rakesh Kumar,³ and Rubayyi Alqahtani⁴

¹Mathematics Department, College of Science, King Saud University, Riyadh 11989, Saudi Arabia

²Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania

³Department of Mathematics, Amity University Rajasthan, NH-IIIC, Jaipur 302002, India

⁴Department of Mathematics and Statistics, College of Science, Al-Imam Mohammad Ibn Saud Islamic University (IMSIU), P.O. Box 65892, Riyadh 11566, Saudi Arabia

Correspondence should be addressed to Rakesh Kumar; rkyadav11@gmail.com

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The purpose of this paper is to find sufficient argument properties, such that the images of some subclasses of functions by the Libera transform have bounded arguments.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathcal{U}$, if and only if it satisfies the condition

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathcal{U}, \quad (\alpha < 1). \quad (2)$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}^*(\alpha)$ of convex functions of order α in $\mathcal{U}$, if and only if it satisfies the condition

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in \mathcal{U}, \quad (\alpha < 1). \quad (3)$$

It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*$, $\mathcal{K}(\alpha) \subset \mathcal{K}$ whenever $0 \leq \alpha < 1$, where $\mathcal{S}^* \equiv \mathcal{S}^*(0)$ and $\mathcal{K} \equiv \mathcal{K}(0)$ represent, respectively, the class of starlike and convex (normalized) functions.

If $f \in \mathcal{A}$ satisfies the condition

$$\operatorname{Re} \frac{f(z)}{zf'(z)} > \beta, \quad z \in \mathcal{U}, \quad (0 \leq \beta < 1), \quad (4)$$

then f is said to be *star-like of reciprocal order* β , and we denote this class by $\mathcal{NS}^*(\beta)$.

The above definition was recently discussed by Nunokawa et al. [1] and Ravichandran and Sivaprasad Kumar [2].

Motivated by their works, we define a class of convex functions of reciprocal order β as follows.

Definition 1. If $f \in \mathcal{A}$ satisfies the condition

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right)^{-1} > \beta, \quad z \in \mathcal{U}, \quad (0 \leq \beta < 1), \quad (5)$$

then f is said to be *convex function of reciprocal order* β , and we denote this class by $\mathcal{NK}(\beta)$.

Definition 2 (see [3, 4]). Suppose that f and g are two analytic functions in $\mathcal{U}$. We say that the function f is subordinate to g , written $f(z) \prec g(z)$, if there exists a Schwarz function w , that is, a function w analytic in $\mathcal{U}$, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in \mathcal{U}$.

It is well-known that (see, e.g., [3, 4]) if g is univalent in $\mathcal{U}$, then the following *subordination property* holds:

$$\begin{aligned} f(z) < g(z) &\iff \\ f(0) &= g(0), \\ f(\mathcal{U}) &\subset g(\mathcal{U}). \end{aligned} \quad (6)$$

Remarks 1. (1) In view of the fact that

$$\begin{aligned} \operatorname{Re} p(z) > \beta, \quad z \in \mathcal{U}, \quad (0 \leq \beta < 1) \\ \Downarrow \\ \operatorname{Re} \frac{1}{p(z)} = \operatorname{Re} \frac{p(z)}{|p(z)|^2} > 0, \quad z \in \mathcal{U}, \end{aligned} \quad (7)$$

it follows that a star-like (convex) function of reciprocal order β is a star-like (convex) function. Thus,

$$\begin{aligned} \mathcal{NS}^*(\beta) &\subset \mathcal{S}^*(0), \\ \mathcal{NK}(\beta) &\subset \mathcal{K}(0), \\ (0 \leq \beta < 1), \end{aligned} \quad (8)$$

and the equality holds in both cases if and only if $\beta = 0$.

(2) Let $0 < \beta < 1$, and suppose that $f \in \mathcal{A}$. Then

$$\begin{aligned} (i) \quad f \in \mathcal{NS}^*(\beta) &\iff \\ \left| \frac{zf'(z)}{f(z)} - \frac{1}{2\beta} \right| &< \frac{1}{2\beta}, \quad z \in \mathcal{U}, \end{aligned} \quad (9)$$

(ii)

$$\begin{aligned} f \in \mathcal{NK}(\beta) &\iff \\ \left| 1 + \frac{zf''(z)}{f'(z)} - \frac{1}{2\beta} \right| &< \frac{1}{2\beta}, \quad z \in \mathcal{U}. \end{aligned} \quad (10)$$

(3) We note that, for $0 \leq \beta < 1$, we have

$$\begin{aligned} f \in \mathcal{NK}(\beta) &\iff \\ zf'(z) &\in \mathcal{NS}^*(\beta). \end{aligned} \quad (11)$$

In the following, we give some examples of functions belonging to the class of star-like functions of reciprocal order and the class of convex functions of reciprocal order.

Example 3. (1) For $0 \leq \beta < 1$, define the function $f_\beta : \mathcal{U} \rightarrow \mathbb{C}$ by

$$\begin{aligned} f_\beta(z) &= \begin{cases} z[1 + (1 - 2\beta)z]^{-2(1-\beta)/(1-2\beta)}, & \text{if } \beta \in [0, 1) \setminus \left\{\frac{1}{2}\right\}, \\ ze^{-z}, & \text{if } \beta = \frac{1}{2}. \end{cases} \end{aligned} \quad (12)$$

Since $f_\beta \in \mathcal{A}$ and a simple computation shows that

$$\operatorname{Re} \frac{f_\beta(z)}{zf'_\beta(z)} = \operatorname{Re} \frac{1 + (1 - 2\beta)z}{1 - z} > \beta, \quad z \in \mathcal{U}, \quad (13)$$

we conclude that $f_\beta \in \mathcal{NS}^*(\beta)$.

(2) Using the third part of the Remarks 1, we deduce that

$$\begin{aligned} \varphi_\beta \in \mathcal{NK}(\beta) &\iff \\ f_\beta(z) = z\varphi'_\beta(z) &\in \mathcal{NS}^*(\beta). \end{aligned} \quad (14)$$

Since we proved that $f_\beta \in \mathcal{NS}^*(\beta)$, and from the above differential equation we obtain

$$\begin{aligned} \varphi_\beta(z) &= \begin{cases} \int_0^z [1 + (1 - 2\beta)u]^{-2(1-\beta)/(1-2\beta)} du, & \text{if } \beta \in [0, 1) \setminus \left\{\frac{1}{2}\right\}, \\ 1 - e^{-z}, & \text{if } \beta = \frac{1}{2}, \end{cases} \end{aligned} \quad (15)$$

it follows that $\varphi_\beta \in \mathcal{NK}(\beta)$.

Example 4. (1) For $1/2 \leq \beta < 1$, consider the function $\psi_\beta : \mathcal{U} \rightarrow \mathbb{C}$, defined by

$$\psi_\beta(z) = ze^{((1-\beta)/\beta)z}. \quad (16)$$

This gives that

$$\operatorname{Re} \frac{\psi_\beta(z)}{z\psi'_\beta(z)} = \operatorname{Re} \frac{\beta}{\beta + (1 - \beta)z} = \operatorname{Re} h(z), \quad z \in \mathcal{U}, \quad (17)$$

where

$$h(z) = \frac{\beta}{\beta + (1 - \beta)z}. \quad (18)$$

Since $\operatorname{Re} \beta \geq 1/2$, it follows that h is analytic in $\mathcal{U}$. Using the fact that h is a convex (not necessary normalized) function in $\mathcal{U}$ and $h(\mathcal{U})$ is symmetric with respect to the real axis, we deduce that $\operatorname{Re} h(z) > \beta$ for all $z \in \mathcal{U}$; hence $\psi_\beta \in \mathcal{NS}^*(\beta)$.

(2) From the above result, using the same reasons like in the second part of Example 3, we obtain that

$$\begin{aligned} \chi_\beta(z) &= \int_0^z e^{((1-\beta)/\beta)u} du \in \mathcal{NK}(\beta), \\ &\left(\frac{1}{2} \leq \beta < 1\right). \end{aligned} \quad (19)$$

We mention that several authors have investigated the strongly star-like functions and the strongly convex functions (see [5–18]). In the present investigation we give some argument properties of analytic functions belonging to $\mathcal{A}$, such that the images of these functions by the Libera transform have bounded arguments.

2. Main Results

The following lemma will be used to prove our main results.

Lemma 5 (see [16]). *Let α_0 be the solution of*

$$\alpha\pi = \frac{3\pi}{2} - \tan^{-1}\gamma \quad (20)$$

for a suitable fixed $\gamma > 0$, so that $\lambda : \mathcal{U} \rightarrow \mathbb{C}$ satisfies

$$\frac{\alpha \operatorname{Re} \lambda(z)}{1 + \alpha |\operatorname{Im} \lambda(z)|} \geq \gamma, \quad z \in \mathcal{U}, \quad (21)$$

and let

$$\eta := \eta(\alpha, \gamma) = \alpha + \frac{2}{\pi} \tan^{-1} \gamma, \quad 0 < \alpha \leq \alpha_0. \quad (22)$$

If p is analytic in $\mathcal{U}$, $p(0) = 1$ and

$$\left| \arg [p(z) + \lambda(z) z p'(z)] \right| < \frac{\pi}{2} \eta, \quad z \in \mathcal{U}, \quad (23)$$

then

$$\left| \arg p(z) \right| < \frac{\pi}{2} \alpha, \quad z \in \mathcal{U}. \quad (24)$$

Remark that in the article [19] the author considered some special situations improving many results with a lot of applications. Thus, in [19, Lemma 2], the author proved the next result.

Let λ be a function defined on $\mathcal{U}$ satisfying

$$\begin{aligned} \left| \arg (\lambda(z) - \eta) \right| &< \frac{\pi}{3}, \quad z \in \mathcal{U}, \\ \text{such that } \eta &= \inf_{z \in \mathcal{U}} \left(\operatorname{Re} \lambda(z) - \frac{|\operatorname{Im} \lambda(z)|}{\sqrt{3}} \right), \end{aligned} \quad (25)$$

and let

$$\begin{aligned} \beta'(\eta) &= \left(\frac{6 + 5\eta^2 + 2\sqrt{9 + 15\eta^2}}{25\eta^2} \right)^{1/3} \\ &\quad \cdot \frac{9 - 2\sqrt{9 + 15\eta^2}}{10} \end{aligned} \quad (26)$$

be such that $2\beta'(\eta) + \eta \geq 0$. If p is analytic in $\mathcal{U}$, $p(0) = 1$, then

$$\begin{aligned} \operatorname{Re} [p(z) + \lambda(z) z p'(z)] &> \beta'(\eta), \quad z \in \mathcal{U} \\ \Downarrow \\ \left| \arg p(z) \right| &< \frac{\pi}{3}, \quad z \in \mathcal{U}. \end{aligned} \quad (27)$$

We emphasize that a special case of this lemma improves the conclusion of Lemma 1 from [19], and in the same article the author derived a number of interesting consequences of it.

Let $\mathcal{L} : \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$\mathcal{L}f(z) = \frac{2}{z} \int_0^z f(t) dt \quad (28)$$

be the well-known *Libera transform* [20].

We first prove the following theorem, which is essential for proving our other results.

Theorem 6. *For $g \in \mathcal{A}$ suppose that the Libera transform $G = \mathcal{L}g$ satisfies the condition (21), with*

$$\lambda(z) = \left(2 + \frac{zG''(z)}{G'(z)} \right)^{-1}, \quad (29)$$

where $0 < \alpha \leq \alpha_0$ and α_0 is defined by (20). If $f \in \mathcal{A}$ and $F = \mathcal{L}f$, then

$$\left| \arg \frac{f'(z)}{g'(z)} \right| < \frac{\pi}{2} \eta(\alpha, \gamma) = \frac{\pi}{2} \left(\alpha + \frac{2}{\pi} \tan^{-1} \gamma \right), \quad (30)$$

$z \in \mathcal{U}$

implies

$$\left| \arg \frac{F'(z)}{G'(z)} \right| < \frac{\pi}{2} \alpha, \quad z \in \mathcal{U}. \quad (31)$$

Proof. Let us define the function p by

$$p(z) = \frac{F'(z)}{G'(z)}, \quad z \in \mathcal{U}. \quad (32)$$

Suppose that there exists a number $z_0 \in \mathcal{U}$, such that $G'(z_0) = 0$; since $G'(0) = 1$, then $z_0 \in \mathcal{U} \setminus \{0\}$. It follows that there exists a unique number $n \in \mathbb{N}$ and a unique function Q , analytic in $\mathcal{U}$, with $Q(z_0) \neq 0$, such that

$$G'(z) = (z - z_0)^n Q(z), \quad z \in \mathcal{U}. \quad (33)$$

From the above relation we deduce that

$$\frac{zG''(z)}{G'(z)} = \frac{nz}{z - z_0} + \frac{zQ'(z)}{Q(z)}, \quad z \in \mathcal{U} \setminus \{z_0\}, \quad (34)$$

which implies that z_0 is a simple pole for the function $zG''(z)/G'(z)$. Consequently, from (29) we obtain that $\lambda(z_0) = 0$, and using the fact that γ satisfies the inequality (21) for all $z \in \mathcal{U}$ it follows that

$$\gamma \leq \frac{\alpha \operatorname{Re} \lambda(z_0)}{1 + \alpha |\operatorname{Im} \lambda(z_0)|} = 0, \quad (35)$$

which contradicts the assumption $\gamma > 0$.

Thus, $G'(z) \neq 0$ for all $z \in \mathcal{U}$, which implies that the function p is analytic in $\mathcal{U}$, and $p(0) = 1$.

Now, a simple calculus shows that

$$\frac{f'(z)}{g'(z)} = \frac{zF''(z) + 2F'(z)}{zG''(z) + 2G'(z)} = p(z) + \lambda(z) z p'(z), \quad (36)$$

and our result follows immediately from Lemma 5. $\square$

Taking $g(z) = z$ in Theorem 6, for the function λ defined by (29) we have $\lambda(z) = 1/2$, and we obtain the following result.

Corollary 7. Suppose that the parameters α and γ satisfy the conditions (20) and (22), for a given $\gamma \leq \alpha/2$. If $f \in \mathcal{A}$ and $F = \mathcal{L}f$; then

$$|f'(z)| < \frac{\pi}{2} \eta(\alpha, \gamma) = \frac{\pi}{2} \left(\alpha + \frac{2}{\pi} \tan^{-1} \gamma \right), \quad z \in \mathcal{U} \quad (37)$$

implies

$$|\arg F'(z)| < \frac{\pi}{2} \alpha, \quad z \in \mathcal{U}. \quad (38)$$

Example 8. For $f \in \mathcal{A}$ and $F = \mathcal{L}f$, considering the special cases $\gamma = 1/\sqrt{3}$, $\gamma = \sqrt{2} - 1$ and $\gamma = 2 - \sqrt{3}$, from the above corollary we obtain respectively the next implications:

(1) If $2/\sqrt{3} \leq \alpha \leq 4/3$, then

$$|f'(z)| < \frac{\pi}{2} \eta\left(\alpha, \frac{1}{\sqrt{3}}\right) = \frac{\pi}{2} \left(\alpha + \frac{1}{3} \right), \quad z \in \mathcal{U} \quad (39)$$

$$\Downarrow$$

$$|\arg F'(z)| < \frac{\pi}{2} \alpha, \quad z \in \mathcal{U}.$$

(2) If $2(\sqrt{2} - 1) \leq \alpha \leq 11/8$, then

$$|f'(z)| < \frac{\pi}{2} \eta(\alpha, \sqrt{2} - 1) = \frac{\pi}{2} \left(\alpha + \frac{1}{4} \right), \quad z \in \mathcal{U} \quad (40)$$

$$\Downarrow$$

$$|\arg F'(z)| < \frac{\pi}{2} \alpha, \quad z \in \mathcal{U}.$$

(3) If $2(2 - \sqrt{3}) \leq \alpha \leq 17/12$, then

$$|f'(z)| < \frac{\pi}{2} \eta(\alpha, 2 - \sqrt{3}) = \frac{\pi}{2} \left(\alpha + \frac{1}{6} \right), \quad z \in \mathcal{U} \quad (41)$$

$$\Downarrow$$

$$|\arg F'(z)| < \frac{\pi}{2} \alpha, \quad z \in \mathcal{U}.$$

Theorem 9. For $g \in \mathcal{A}$ suppose that the Libera transform $G = \mathcal{L}g \in \mathcal{N}\mathcal{K}(\beta)$, $0 < \beta < 1$. Suppose that α_0 is defined by (20), and for $0 < \alpha \leq \alpha_0$ let γ satisfy the condition

$$\begin{aligned} 0 < \gamma \leq \bar{\gamma} &:= \bar{\gamma}(\alpha, \beta) \\ &= \frac{2\alpha\beta}{1 + 2\beta + \sqrt{(\beta/(1+\beta))\alpha^2 + 1}}. \end{aligned} \quad (42)$$

If $f \in \mathcal{A}$ and $F = \mathcal{L}f$, then

$$\left| \arg \frac{f'(z)}{g'(z)} \right| < \frac{\pi}{2} \eta(\alpha, \gamma) = \frac{\pi}{2} \left(\alpha + \frac{2}{\pi} \tan^{-1} \gamma \right), \quad z \in \mathcal{U} \quad (43)$$

implies

$$\left| \arg \frac{F'(z)}{G'(z)} \right| < \frac{\pi}{2} \alpha, \quad z \in \mathcal{U}. \quad (44)$$

Proof. Let p be the function defined by

$$p(z) = \frac{F'(z)}{G'(z)}, \quad z \in \mathcal{U}. \quad (45)$$

Similarly like in the proof of Theorem 6, we will prove that the function p is analytic in $\mathcal{U}$. Supposing that there exists a number $z_0 \in \mathcal{U}$, such that $G'(z_0) = 0$, since $G'(0) = 1$, then $z_0 \in \mathcal{U} \setminus \{0\}$. Hence, there exists a unique number $n \in \mathbb{N}$ and a unique function Q , analytic in $\mathcal{U}$, with $Q(z_0) \neq 0$, such that

$$G'(z) = (z - z_0)^n Q(z), \quad z \in \mathcal{U}. \quad (46)$$

From here we deduce that

$$\frac{zG''(z)}{G'(z)} = \frac{nz}{z - z_0} + \frac{zQ'(z)}{Q(z)}, \quad z \in \mathcal{U} \setminus \{z_0\}, \quad (47)$$

which implies that z_0 is a simple pole for the function $zG''(z)/G'(z)$. Now, using the fact that $G = \mathcal{L}g \in \mathcal{N}\mathcal{K}(\beta)$, we obtain

$$0 = \operatorname{Re} \left(1 + \frac{zG''(z_0)}{G'(z_0)} \right)^{-1} > \beta, \quad (48)$$

which contradicts the assumption $0 \leq \beta$.

Consequently, $G'(z) \neq 0$ for all $z \in \mathcal{U}$, and this implies that the function p is analytic in $\mathcal{U}$, with $p(0) = 1$.

From (45) we easily get

$$\frac{f'(z)}{g'(z)} = p(z) + \lambda(z)zp'(z), \quad (49)$$

where the function λ is given by (29).

The assumption $G = \mathcal{L}g \in \mathcal{N}\mathcal{K}(\beta)$ is equivalent to

$$\left(1 + \frac{zG''(z)}{G'(z)} \right)^{-1} < \frac{1 + (1 - 2\beta)z}{1 - z}; \quad (50)$$

hence there exists a Schwarz function w such that

$$\left(1 + \frac{zG''(z)}{G'(z)} \right)^{-1} = \frac{1 + (1 - 2\beta)w(z)}{1 - w(z)}, \quad z \in \mathcal{U}. \quad (51)$$

By simple calculations, we get

$$\left(2 + \frac{zG''(z)}{G'(z)} \right)^{-1} = \frac{1 + (1 - 2\beta)w(z)}{2(1 - \beta w(z))}, \quad z \in \mathcal{U}; \quad (52)$$

that is,

$$\lambda(z) = \left(2 + \frac{zG''(z)}{G'(z)} \right)^{-1} < \frac{1 + (1 - 2\beta)z}{2(1 - \beta z)} =: \phi(z). \quad (53)$$

Since the circular transform ϕ maps the unit disk $\mathcal{U}$ onto the disk

$$\Omega = \{w \in \mathbb{C} : |w - \omega| < r\}, \quad (54)$$

with $\omega = \frac{1+2\beta}{2(1+\beta)}, r = \frac{1}{2(1+\beta)},$

from the subordination property (6) we deduce that the subordination relation (53) is equivalent to

$$\lambda(z) = u(z) + iv(z) \in \Omega, \quad z \in \mathcal{U}. \quad (55)$$

With the above notation, the condition (21) becomes

$$\alpha u(z) - \alpha \gamma |v(z)| - \gamma \geq 0, \quad z \in \mathcal{U}. \quad (56)$$

We will prove that if γ satisfies the assumption (42), that is, $0 < \gamma \leq \tilde{\gamma}$, then the subordination (53) implies that the inequality (56) holds, that is,

$$\Omega \subset \Delta, \quad (57)$$

where

$$\Delta = \{w = u + iv \in \mathbb{C} : \alpha u - \alpha \gamma |v| - \gamma \geq 0\}. \quad (58)$$

Since the sets Ω and Δ are symmetric with respect to the real axis, the above inclusion is equivalent to

$$\tilde{\Omega} \subset \tilde{\Delta}, \quad (59)$$

where

$$\begin{aligned} \tilde{\Omega} &= \{w = u + iv \in \Omega : v \geq 0\}, \\ \tilde{\Delta} &= \{w = u + iv \in \Delta : v \geq 0\}. \end{aligned} \quad (60)$$

The line (d) containing the half-line

$$\alpha u - \alpha \gamma v - \gamma = 0, \quad v \geq 0, \quad (61)$$

which is a part of the boundary the set $\tilde{\Delta}$, has the equation

$$\alpha u - \alpha \gamma v - \gamma = 0. \quad (62)$$

Thus, the line (d) intersects the real axis in the point γ/α and contains the point $-i/\alpha$ (see Figure 1). The *smallest* set $\tilde{\Delta}$ that includes the domain $\tilde{\Omega}$ is obtained for the case when the line (d) , containing the point $-i/\alpha$, is tangent to the upper boundary of $\tilde{\Omega}$, which is the half-circle

$$(\mathcal{E}) = \{w = u + iv \in \mathbb{C} : (u - \omega)^2 + v^2 = r^2, v \geq 0\}. \quad (63)$$

In this case, as it is shown in the figure, the line (d) becomes the tangent line (d_τ) to $(\mathcal{E})$.

We will determine now the equation of the tangent line (d_τ) . If we consider the family (d_m) of all the lines containing the point $-i/\alpha$, with nonnegative angular coefficient, that is,

$$(d_m) : v = mu - \frac{1}{\alpha}, \quad m \in (0, +\infty), \quad (64)$$

FIGURE 1: The inclusion $\tilde{\Omega} \subset \tilde{\Delta}$.

we will solve the system

$$\begin{aligned} (u - \omega)^2 + v^2 &= r^2, \quad v \geq 0 \\ v &= mu - \frac{1}{\alpha}. \end{aligned} \quad (65)$$

It follows that

$$\begin{aligned} \alpha^2 (1 + m^2) u^2 - 2\alpha (\alpha \omega + m) u + \alpha^2 (\omega^2 - r^2) + 1 \\ = 0, \end{aligned} \quad (66)$$

and the line is tangent to the half-circle if and only if the discriminant of the above quadratic form is zero, that is,

$$\alpha^2 (\omega^2 - r^2) m^2 - 2\alpha \omega m - \alpha^2 r^2 + 1 = 0. \quad (67)$$

This last equation has the biggest positive root

$$\tilde{m} = \frac{\omega + r \sqrt{(\omega^2 - r^2) \alpha^2 + 1}}{\alpha (\omega^2 - r^2)}; \quad (68)$$

hence the tangent line to the half-circle $(\mathcal{E})$, which contains the point $-i/\alpha$, has the equation

$$(d_\tau) : v = \tilde{m}u - \frac{1}{\alpha}. \quad (69)$$

The tangent (d_τ) intersects the real axis in the point $1/(\alpha \tilde{m})$; hence we deduce that the inclusion (59) holds if and only if

$$0 < \frac{\gamma}{\alpha} \leq \frac{1}{\alpha \tilde{m}}, \quad (70)$$

that is

$$\begin{aligned} 0 < \gamma \leq \frac{1}{\tilde{m}} &= \frac{\alpha (\omega^2 - r^2)}{\omega + r \sqrt{(\omega^2 - r^2) \alpha^2 + 1}} \\ &= \frac{2\alpha \beta}{1 + 2\beta + \sqrt{(\beta/(1+\beta)) \alpha^2 + 1}} = \tilde{\gamma}(\alpha, \beta). \end{aligned} \quad (71)$$

Thus, if the parameter γ satisfies the assumption (42), then the condition (21) of Lemma 5 is fulfilled, and our result follows from Theorem 6. $\square$

Putting $g(z) = z$ in Theorem 9, for $\beta \rightarrow 1^-$, we get the next result.

Corollary 10. Suppose that α_0 is defined by (20), and for $0 < \alpha \leq \alpha_0$ let γ satisfy the condition

$$0 < \gamma \leq \frac{2\alpha}{3 + \sqrt{\alpha^2/2 + 1}}. \quad (72)$$

If $f \in \mathcal{A}$ and $F = \mathcal{L}f$, then

$$|f'(z)| < \frac{\pi}{2}\eta(\alpha, \gamma) = \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\gamma\right), \quad z \in \mathcal{U} \quad (73)$$

implies

$$|\arg F'(z)| < \frac{\pi}{2}\alpha, \quad z \in \mathcal{U}. \quad (74)$$

Example 11. For $f \in \mathcal{A}$ and $F = \mathcal{L}f$, considering the special case $\gamma = 2 - \sqrt{3}$, from Corollary 10 we deduce the next implication.

If

$$\begin{aligned} 0.54552\dots &\approx \frac{-168 + 108\sqrt{3} + 164\sqrt{2} - 92\sqrt{6}}{47} \leq \alpha \\ &\leq \frac{17}{12}, \end{aligned} \quad (75)$$

then

$$\begin{aligned} |f'(z)| &< \frac{\pi}{2}\eta(\alpha, 2 - \sqrt{3}) = \frac{\pi}{2}\left(\alpha + \frac{1}{6}\right), \quad z \in \mathcal{U} \\ &\Downarrow \\ |\arg F'(z)| &< \frac{\pi}{2}\alpha, \quad z \in \mathcal{U}. \end{aligned} \quad (76)$$

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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