Research Article
Estimates for Parameter Littlewood-Paley $g^*_\kappa$ Functions on Nonhomogeneous Metric Measure Spaces

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Let $(\mathcal{X}, d, \mu)$ be a metric measure space which satisfies the geometrically doubling measure and the upper doubling measure conditions. In this paper, the authors prove that, under the assumption that the kernel of $M^\ast\varphi$ satisfies a certain Hörmander-type condition, $M^\ast\varphi\varphi$ is bounded from Lebesgue spaces $L^p(\mu)$ to Lebesgue spaces $L^p(\mu)$ for $p \geq 2$ and is bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$. As a corollary, $M^\ast\varphi\varphi$ is bounded on $L^p(\mu)$ for $1 < p < 2$. In addition, the authors also obtain that $M^\ast\varphi\varphi$ is bounded from the atomic Hardy space $H^1(\mu)$ into the Lebesgue space $L^1(\mu)$.

1. Introduction

In 1958, Stein in [1] firstly introduced the Littlewood-Paley operators of the higher-dimensional case; meanwhile, the author also obtained the boundedness of the Marcinkiewicz integrals and area integrals. In 1970, Fefferman in [2] proved that the Littlewood-Paley $g^*_\kappa$ function is weak type $(p, p)$ for $p \in (1, 2)$ and $\kappa = 2/p$. With further research about Littlewood-Paley operators, some authors turn their attentions to study the parameter Littlewood-Paley operators. For example, in 1999, Sakamoto and Yabuta in [3] considered the parameter $g^*_\kappa$ function. Since then, many papers focus on the behaviours of the operators; among them we refer readers to see [4–6].

In the past ten years or so, most authors mainly study the classical theory of harmonic analysis on $\mathbb{R}^n$ under nondoubling measures which only satisfy the polynomial growth condition; see [7–12]. Exactly, we assume that $\mu$ which is a positive Radon measure on $\mathbb{R}^n$ satisfies the following growth conditions; namely, for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, there exist constant $C$ and $0 < d \leq n$ such that

$$\mu(B(x, r)) \leq Cr^d, \quad (1)$$

where $B(x, r) = \{ y \in \mathbb{R}^n : |x - y| < r \}$. The analysis associated with nondoubling measures $\mu$ as in (1) has important applications in solving long-standing open Painlevé’s problem and Vitushkin’s conjecture (see [13, 14]). Besides, Coifman and Weiss have showed that the measure $\mu$ is a key assumption in harmonic analysis on homogeneous-type spaces (see [15, 16]). However, Hytönen in [17] pointed that the measure $\mu$ as in (1) may not contain the doubling measure as special cases. To solve the problem, in 2010, Hytönen in [17] introduced a new class of metric measure spaces satisfying the so-called upper doubling conditions and the geometrically doubling (resp., see Definitions 1 and 2 below), which are now claimed nonhomogeneous metric measure spaces. Therefore, if we replace the underlying spaces with nonhomogeneous metric measure spaces, many known-consequences have been proved still true; for example, see [18–22].

In this paper, we always assume that $(\mathcal{X}, d, \mu)$ is a nonhomogeneous metric measure space. In this setting, we will establish the boundedness of the parameter Littlewood-Paley $g^*_\kappa$ functions on $(\mathcal{X}, d, \mu)$.

In order to state our main results, we firstly recall some necessary notions and notation. Hytönen in [17] gave out the definition of upper doubling metric spaces as follows.
Definition 1 (see [17]). A metric measure space \((\mathcal{X}, d, \mu)\) is said to be upper doubling, if \(\mu\) is Borel measure on \(\mathcal{X}\) and there exist a dominating function \(\lambda: \mathcal{X} \times (0, \infty) \to (0, \infty)\) and a positive constant \(C_\lambda\) such that for each \(x \in \mathcal{X}\), \(r \to \lambda(x, r)\) is nondecreasing and, for all \(x \in \mathcal{X}\) and \(r \in (0, \infty)\),

\[
\mu(B(x, r)) \leq \lambda(x, r) \leq C_\lambda \lambda(x, r/2),
\]

which is very similar to the number \(K_{Q, R}\) introduced in [7] by Tolsa. For any two balls \(B \subset S, \overline{K}_{B_S}\) is defined by

\[
\overline{K}_{B_S} = 1 + \sum_{i=1}^{N_B} \mu(B_i),
\]

where the radii of the balls \(B\) and \(S\) are denoted by \(r_B\) and \(r_S\), respectively, and \(N_B\) is the smallest integer satisfying \(6^{N_B-r_B} \geq r_S\). It is easy to obtain \(\overline{K}_{B_S} \leq CK_{B_S}\). Bui and Duong in [21] also pointed out that it is incorrect that \(K_{B_S} \sim \overline{K}_{B_S}\).

Now we recall the following notion of \((\alpha, \beta)\)-doubling property (see [17]).

Definition 5 (see [17]). Let \(\alpha, \beta \in (1, \infty)\). A ball \(B \subset \mathcal{X}\) is claimed to be \((\alpha, \beta)\)-doubling if \(\mu(\alpha B) \leq \beta \mu(B)\).

It was stated in [17] that, there exist many balls which have the above \((\alpha, \beta)\)-doubling property. In the latter part of the paper, if \(\alpha\) and \(\beta\) are not specified, \((\alpha, \beta)\)-doubling ball always stands for \((6, \beta_6)\)-doubling ball with a fixed number \(\beta_6 > \max\{C_\alpha^{\log \beta_6}, 6^n\}\), where \(n = \log N_B\) is considered as a geometric dimension of the space. Moreover, the smallest \((6, \beta_6)\)-doubling ball of the form \(6^j B\) with \(j \in \mathbb{N}\) is denoted by \(B\), and sometimes \(B^\delta\) can be simply denoted by \(B\).

Now we give the definition of the parameter Littlewood-Paley \(g^\delta_x\) functions on \((\mathcal{X}, d, \mu)\).

Definition 6 (see [22]). Let \(K(x, y)\) be a locally integrable function on \((\mathcal{X} \times \mathcal{X}) \setminus \{(x, y): x = y\}\). Assume that there exists a positive constant \(C\) such that, for all \(x, y \in \mathcal{X}\) with \(x \neq y\),

\[
|K(x, y)| \leq C \frac{d(x, y)}{\lambda(x, d(x, y))}
\]

and, for all \(x, y, y' \in \mathcal{X}\),

\[
\int_{d(x, y') \leq 2d(x, y)} \left[|K(x, y) - K(x, y')| + |K(y, x) - K(y', x)|\right] \frac{1}{d(x, y)} d\mu(x) \leq C.
\]

The parameter Marcinkiewicz integral \(M^\rho\) associated with the above \(K(x, y)\) which satisfies (6) and (7) is defined by

\[
M^\rho(f)(x) = \left( \int_0^\infty \left[ \frac{1}{t^\rho} \int_{d(x, y) \leq t} |K(x, y)| \frac{f(y)}{d(x, y)} d\mu(y) \right]^2 \frac{dt}{t} \right)^{1/2},
\]

\(x \in \mathcal{X}\),
Let $\rho \in (0, \infty)$. The parameter $g^\ast_\rho$ function $M^\ast_\rho$ is defined by
\[
M^\ast_\rho (f)(x) = \left\{ \begin{array}{ll}
\int_{\mathbb{X} \times (0, \infty)} \left( \frac{t}{t + d(x, y)} \right)^{1/\rho} f(y) \, dy \\
\int_{d(y, z) \leq t} \frac{K(y, z)}{d(y, z)^{1/\rho}} f(z) \, dz
\end{array} \right. \quad (9)
\]
where $x \in \mathbb{X}$, $\mathbb{X} \times (0, \infty) = \{(y, t) : y \in \mathbb{X}, \ t > 0\}$, $\rho > 0$, and $\lambda \in (1, \infty)$.

Remark 7. (1) When $\rho = 1$, the operator $M^\rho$ as in (8) is just the Marcinkiewicz integral on $(\mathcal{X}, d, \mu)$ (see [22]).

(2) If we take $(\mathcal{X}, d, \mu) = (\mathbb{R}^n, |\cdot|, \mu)$ and $\lambda(y, t) = r^\rho$, then the parameter $g^\ast_\rho$ function $M^\ast_\rho$ as in (9) is just a parameter Littlewood-Paley operator with non-decreasing measures in [8].

The following definition of the atomic Hardy space was introduced by Hytönen et al. (see [18]).

Definition 8 (see [18]). Let $\zeta \in (1, \infty)$ and $\rho \in (1, \infty)$. A function $b \in L^1_{\text{loc}}(\mu)$ is called a $(\rho, 1)_\zeta$-atomic block if

(a) there exists a ball $B$ such that $supp b \subset B$,

\[
\sup_{r > 0} \left( \sum_{i=1}^{\infty} \int_{d(y, z) \leq r} |K(x, y) - K(x, y')| + \frac{|K(y, x) - K(y', x)|}{d(x, y)} \, dy \right) \leq C.
\]

Notice this condition is slightly stronger than (7).

Now let us state the main theorems which generalize and improve the corresponding results in [8].

**Theorem 9.** Let $K(x, y)$ satisfy (6) and (7), and let $M^\ast_\rho$ be as in (9) with $\rho \in (0, \infty)$ and $\kappa \in (1, \infty)$. Then $M^\ast_\rho$ is bounded on $L^p(\mu)$ for any $p \in [2, \infty]$.

**Theorem 10.** Let $K(x, y)$ satisfy (6) and (11), and let $M^\ast_\rho$ be as in (9) with $\rho \in (1/2, \infty)$ and $\kappa \in (1, \infty)$. Then $M^\ast_\rho$ is bounded from $L^1(\mu)$ into weak $L^1(\mu)$; namely, there exists a positive constant $C$ such that, for any $\tau > 0$ and $f \in L^1(\mu)$,

\[
\mu\left( \{ x \in \mathbb{X} : M^\ast_\rho (f)(x) > \tau \} \right) \leq C \frac{\| f \|_{L^1(\mu)}}{\tau}.
\]

**Theorem 11.** Let $K(x, y)$ satisfy (6) and (11), and let $M^\ast_\rho$ be as in (9) with $\rho > 1/2$ and $\kappa > 1$. Suppose that $M^\ast_\rho$ is bounded on $L^2(\mu)$. Then, $M^\ast_\rho$ is bounded from $H^1(\mu)$ into $L^1(\mu)$.

Applying the Marcinkiewicz interpolation theorem and Theorems 9 and 10, it is easy to get the following result.

**Corollary 12.** Under the assumption of Theorem 10, $M^\ast_\rho$ is bounded on $L^p(\mu)$ for $p \in (1, 2)$.

The organization of this paper is as follows. In Section 2, we will give some preliminary lemmas. The proofs of the main theorems will be given in Section 3. Throughout this paper, $C$ stands for a positive constant which is independent of the main parameters, but it may be different from line to line. For any $E \subset \mathcal{X}$, we use $\chi_E$ to denote its characteristic function.

**2. Preliminary Lemmas**

In this section, we make some preliminary lemmas which are used in the proof of the main results. Firstly, we recall some properties of $K_{B, B}$ as in (4) (see [17]).

**Lemma 13** (see [17]). (1) For all balls $B < R \subset S$, it holds true that $K_{B, R} \leq K_{B, S}$.

(2) For any $\zeta \in [1, \infty)$, there exists a positive constant $C_\zeta$ such that, for all balls $B \subset S$ with $r_B \leq \zeta r_S$, $K_{B, S} \leq C_\zeta$.

(3) For any $\zeta \in (1, \infty)$, there exists a positive constant $C_\zeta$ depending on $\zeta$, such that, for all balls $B, K_{B, B} \leq C_\zeta$.

(4) There exists a positive constant $C$ such that, for all balls $B < R \subset S$, $K_{B, S} \leq K_{B, R} + c K_{B, S}$. In particular, if $B$ and $R$ are concentric, then $c = 1$.

(5) There exists a positive constant $\bar{c}$ such that, for all balls $B < R \subset S$, $K_{B, R} \leq \bar{c} K_{B, S}$; moreover, if $B$ and $R$ are concentric, then $K_{B, S} \leq K_{B, S}$. 

To state the following lemmas, let us give a known-result (see [19]). For \( \eta \in (0, \infty) \), the maximal operator is defined, by setting that, for all \( f \in L^1_{\text{loc}}(\mu) \) and \( x \in \mathcal{X} \),

\[
M_{\eta}(f)(x) := \sup_{Q \ni x, Q \text{ doubling} \mu} \frac{1}{\eta(Q)} \int_Q |f(y)| \, d\mu(y) \tag{13}
\]
is bounded on \( L^p(\mu) \) provided that \( p \in (1, \infty) \) and also bounded from \( L^1(\mu) \) into \( L^{1,\infty}(\mu) \).

The following lemma is slightly changed from [8].

**Lemma 14.** Let \( K(x, y) \) satisfy (6) and (7), and \( \eta \in (0, \infty) \). Assume that \( \mathcal{M}^p \) is as in (8) and \( \mathcal{M}^p_\rho \) is as in (9) with \( \rho \in (0, \infty) \) and \( \kappa \in (1, \infty) \). Then for any nonnegative function \( \phi \), there exists a positive constant \( C \) such that, for all \( f \in L^p(\mu) \) with \( p \in (1, \infty) \),

\[
\int_{\mathcal{X}} \left[ \mathcal{M}^{p}_{\rho}(f)(x) \right]^2 \phi(x) \, d\mu(x) \leq C \int_{\mathcal{X}} \left[ \mathcal{M}^{p}(f)(x) \right]^2 M_\eta(\phi)(x) \, d\mu(x). 
\tag{14}
\]

**Proof.** By the definition of \( \mathcal{M}^{p}_{\rho}(f) \), we have

\[
\int_{\mathcal{X}} \left[ \mathcal{M}^{p}_{\rho}(f)(x) \right]^2 \phi(x) \, d\mu(x) 
= \int_{\mathcal{X}} \int_{x \times (0, \infty]} \left( \frac{t}{t + d(x, y)} \right)^{\beta} \frac{1}{\mu(B(y, t))} \int_B K(y, z) \frac{d\mu(z)}{d(y, z)} \left( \frac{f(y)}{\lambda(y, t)} \right) \, d\mu(x) 
\leq C \int_{\mathcal{X}} \left[ \mathcal{M}^{p}(f)(x) \right]^2 M_\eta(\phi)(x) \, d\mu(x). 
\tag{15}
\]

Thus, to prove Lemma 14, we only need to estimate that

\[
\sup_{t > 0} \int_{\mathcal{X}} \left( \frac{t}{t + d(x, y)} \right)^{\beta} \frac{\phi(x)}{\lambda(y, t)} \, d\mu(x) \leq C M_\eta(\phi)(y). 
\tag{16}
\]

For any \( x \in \mathcal{X} \) and \( t > 0 \), write

\[
\int_{\mathcal{X}} \left( \frac{t}{t + d(x, y)} \right)^{\beta} \frac{\phi(x)}{\lambda(y, t)} \, d\mu(x) 
= \int_{B(y, t)} \left( \frac{t}{t + d(x, y)} \right)^{\beta} \frac{\phi(x)}{\lambda(y, t)} \, d\mu(x) 
+ \int_{\mathcal{X} \setminus B(y, t)} \left( \frac{t}{t + d(x, y)} \right)^{\beta} \frac{\phi(x)}{\lambda(y, t)} \, d\mu(x) 
= D_1 + D_2. 
\tag{17}
\]

For \( D_1 \), it is not difficult to obtain that

\[
D_1 \leq \int_{B(y, t)} \frac{\phi(x)}{\lambda(y, t)} \, d\mu(x) 
= \frac{\mu(\eta B(y, t))}{\mu(\eta B(y, t))} \int_{B(y, t)} \phi(x) \, d\mu(x) 
\leq C M_\eta(\phi)(y). 
\tag{18}
\]

Now we turn to estimate \( D_2 \), by (2) and (13); we have

\[
D_2 \leq \sum_{k=1}^{\infty} \int_{B(y, 6^k t) \setminus B(y, 6^{k-1} t)} \left( \frac{t}{t + d(x, y)} \right)^{\beta} \frac{\phi(x)}{\lambda(y, t)} \, d\mu(x) 
\leq C \sum_{k=1}^{\infty} 6^{-(k-1)\beta} \int_{B(y, 6^k t)} \frac{\phi(x)}{\lambda(y, t)} \, d\mu(x) 
\leq C \sum_{k=1}^{\infty} 6^{-(k-1)\beta} M_\eta(\phi)(y) 
\leq C \sum_{k=1}^{\infty} 6^{-(k-1)\beta} \frac{\lambda(y, 6^k t)}{\lambda(y, t)} M_\eta(\phi)(y) 
\leq C \sum_{k=1}^{\infty} 6^{-(k-1)\beta} M_\eta(\phi)(y) 
\leq C \sum_{k=1}^{\infty} 6^{-(k-1)\beta} M_\eta(\phi)(y) 
\leq C \sum_{k=1}^{\infty} 6^{-(k-1)\beta} \leq C M_\eta(\phi)(y). 
\tag{19}
\]

Combining the estimates for \( D_1 \) and \( D_2 \), we obtain (16) and hence complete the proof of Lemma 14. \( \square \)

Finally, we recall the Calderón-Zygmund decomposition theorem (see [21]). Suppose that \( \gamma_0 \) is a fixed positive constant...
satisfying that \( \gamma_0 > \max\{c_1^3 \log_2 6, 6^3\} \), where \( C_\Lambda \) is as in (2) and \( n \) as in Remark 3.

**Lemma 15** (see [21]). Let \( p \in [1, \infty) \), \( f \in L^p(\mu) \), and \( t \in (0, \infty) \) \((t > \gamma_0 \|f\|_{L^p(\mu)/\mu(\mathcal{X})} < \infty) \). Then

1. there exists a family of finite overlapping balls \( \{6B_i\}_i \) such that \( \{B_i\}_i \) is pairwise disjoint:

\[
\frac{1}{\mu(6^2 B_i)} \int_{6B_i} |f(x)|^p \, d\mu(x) > \frac{t^p}{\gamma_0} \quad \forall i,
\]

(\( i \)th integral for \( \mu \)-almost every \( x \in \mathcal{X} \backslash \bigcup_i 6B_i \));

2. for each \( i \), let \( S_i \) be a \( (3 \times 6^2, c_1^3 \log_2 6^3 + 1) \)-doubling ball of the family \( \{(3 \times 6^2)^2 B_i\}_{i \in \mathbb{N}_0} \) and \( \omega_i = \chi_{6B_i}/(\sum \chi_{6B_i}) \). Then there exists a family \( \{\psi_i\}_i \) of functions that, for each \( i \), \( \supp(\psi_i) \subset S_i \), \( \psi_i \) has a constant sign on \( S_i \), and

\[
\int_{\mathcal{X}} \psi_i(x) \, d\mu(x) = \int_{6B_i} |f(x)| \, \omega_i(x) \, d\mu(x),
\]

(\( \sum_i |\psi_i(x)| \leq \gamma t \) for \( \mu \)-almost every \( x \in \mathcal{X} \), where \( \gamma \) is some positive constant depending only on \( (\mathcal{X}, \mu) \), and there exists a positive constant \( C \), independent of \( f \), \( t \), and \( i \), such that if \( p = 1 \), then

\[
\|\psi_i\|_{L^\infty(\mu)} \leq C \int_{\mathcal{X}} |f(x)| \, \omega_i(x) \, d\mu(x),
\]

(24)

and if \( p \in (1, \infty) \),

\[
\left( \int_{S_i} |\psi_i(x)|^p \, d\mu(x) \right)^{1/p} \left( \mu(S_i) \right)^{1/p'} \leq \frac{C}{t^{p'}} \int_{\mathcal{X}} |f(x)|^p \, \omega_i(x) \, d\mu(x).
\]

(25)

\section{Proofs of Theorems}

**Proof of Theorem 9.** For the case of \( p = 2 \), assume \( \phi(x) = 1 \) in Lemma 14; then it is easy to get that

\[
\int_{\mathcal{X}} g(x)^2 \, d\mu(x) \leq C \int_{\mathcal{X}} \mathcal{M}_\phi^p(f)(x)^2 \, d\mu(x),
\]

(26)

which, along with \( L^2(\mu) \)-boundedness of \( \mathcal{M}_\phi \), easily yields that Theorem 9 holds.

For the case of \( p > 2 \), let \( q \) be the index conjugate to \( p/2 \). By applying Hölder inequality and Lemma 14, we can conclude

\[
\|\mathcal{M}_\phi^p(f)\|_{L^q(\mu)}^2 \leq C \|f\|_{L^p(\mu)} \|\phi\|_{L^q(\mu)} \leq C \|f\|_{L^p(\mu)}^2
\]

(27)

which is desired. Thus, we complete the proof of Theorem 9. \( \square \)

**Proof of Theorem 10.** Without loss of generality, we may assume that \( \|f\|_{L^1(\mu)}^2 = 1 \). It is easy to see that the conclusion of Theorem 10 naturally holds if \( \tau \leq \beta_0(\|f\|_{L^1(\mu)}/\mu(\mathcal{X})) \) when \( \mu(\mathcal{X}) < \infty \). Thus, we only need to discuss the case that \( \tau > \beta_0(\|f\|_{L^1(\mu)}/\mu(\mathcal{X})) \). Applying Lemma 15 to \( f \) at the level \( \tau \) and letting \( \omega, \phi, B_i, \) and \( S_i \) be the same as in Lemma 15, we see that \( f(x) = b(x) + h(x) \), where \( b(x) = f(x) \chi_{\mathcal{X} \cup \bigcup_i S_i} \) and \( h(x) = \sum_i \omega_i(x) f(x) - \phi(x) \). It is easy to obtain that \( \|b\|_{L^1(\mu)} \leq C \mathcal{R} \) and \( \|b\|_{L^1(\mu)} \leq C \). By \( L^2(\mu) \)-boundedness of \( \mathcal{M}_\phi^p \), we have

\[
\mu \left( \{ x \in \mathcal{X} : \mathcal{M}_\phi^p(b)(x) > \tau \} \right) \leq \frac{\|\mathcal{M}_\phi^p(b)\|_{L^1(\mu)}^2}{\tau^2}
\]

(28)

On the other hand, by (20) with \( p = 1 \) and the fact that the sequence of balls, \( \{B_i\}_i \), is pairwise disjoint, we see that

\[
\mu \left( \bigcup_i 6^2 B_i \right) \leq C \tau^{-1} \int_{\mathcal{X}} |f(x)| \, d\mu(x) \leq C \tau^{-1},
\]

(29)

and thus the proof of Theorem 10 can be reduced to prove that

\[
\mu \left( \{ x \in \mathcal{X} \cap \bigcup_i 6^2 B_i : \mathcal{M}_\phi^p(h)(x) > \tau \} \right) \leq C \tau^{-1}.
\]

(30)

For each fixed \( i \), denote the center of \( B_i \) by \( x_i \), and let \( N_1 \) be the positive integer satisfying \( S_i = (3 \times 6^2)^2 N_1 B_i \). We have

\[
\mu \left( \{ x \in \mathcal{X} \cap \bigcup_i 6^2 B_i : \mathcal{M}_\phi^p(h)(x) > \tau \} \right) \leq \tau^{-1} \sum_i \int_{X \cap \bigcup_i 6^2 B_i} \mathcal{M}_\phi^p(h_i)(x) \, d\mu(x)
\]
\[
\begin{align*}
& \leq \tau^{-1} \sum_i \int_{X \times 6S_i} \mathcal{M}^\kappa_{\phi} (h_i) (x) \, d\mu (x) \\
& + \tau^{-1} \sum_i \int_{6S_i \setminus \delta^2 B_i} \mathcal{M}^\kappa_{\phi} (h_i) (x) \, d\mu (x) \\
& = \tau^{-1} \sum_i (E_1 + E_2).
\end{align*}
\]

Firstly, let us estimate \( E_2 \) and write it as
\[
E_2 \leq \int_{6S_i \setminus \delta^2 B_i} \mathcal{M}^\kappa_{\phi} (f \omega_i) (x) \, d\mu (x)
\]
\[
+ \int_{6S_i \setminus \delta^2 B_i} \mathcal{M}^\kappa_{\phi} (\varphi_i) (x) \, d\mu (x) = E_{21} + E_{22},
\]
where \( h_i = \omega_i f - \varphi_i \). By Hölder inequality, (24), and \( L^2(\mu) \)-boundedness of \( \mathcal{M}^\kappa_{\phi} \), we have
\[
E_{22} \leq \left( \int_{6S_i} \left| \mathcal{M}^\kappa_{\phi} (\varphi_i) (x) \right|^2 \, d\mu (x) \right)^{1/2} \mu (6S_i)^{1/2}
\]
\[
\leq C \left( \int_{6S_i} |\varphi_i (x)|^2 \, d\mu (x) \right)^{1/2} \mu (6S_i)^{1/2}
\]
\[
\leq C \int_{6S_i} |f \omega_i (x)| \, d\mu (x).
\]

For \( E_{21} \), by Minkowski inequality and (6), write
\[
E_{21} = \int_{6S_i \setminus \delta^2 B_i} \left[ \int_{X \times (0, \infty)} \left| \frac{t}{t + d (x, y)} \right|^{x/2} \frac{1}{t^\rho} \int_{d(y,z) \geq t} \frac{K (y, z) \int_{\lambda (y, t) \leq t} \omega_i (z) \, d\mu (z) \left[ \frac{\int_{d(y,z) \leq t} \frac{[d (y, z)]^{2p} \, d\mu (y) \, dt \, d\mu (z)}{[\lambda (y, t)^{1+2p}]^{1/2}} \right]}{t^{1+2p}} \right]^x \, d\mu (x)
\]
\[
\leq \int_{6S_i \setminus \delta^2 B_i} \left[ \int_{X \times (0, \infty)} \left| \frac{t}{t + d (x, y)} \right|^{x/2} \frac{1}{t^\rho} \int_{d(y,z) \geq t} \frac{K (y, z) \int_{\lambda (y, t) \leq t} \omega_i (z) \, d\mu (z) \left[ \frac{\int_{d(y,z) \leq t} \frac{[d (y, z)]^{2p} \, d\mu (y) \, dt \, d\mu (z)}{[\lambda (y, t)^{1+2p}]^{1/2}} \right]}{t^{1+2p}} \right]^x \, d\mu (x)
\]
\[
\leq \int_{6S_i \setminus \delta^2 B_i} \left[ \int_{X \times (0, \infty)} \left| \frac{t}{t + d (x, y)} \right|^{x/2} \frac{1}{t^\rho} \int_{d(y,z) \geq t} \frac{K (y, z) \int_{\lambda (y, t) \leq t} \omega_i (z) \, d\mu (z) \left[ \frac{\int_{d(y,z) \leq t} \frac{[d (y, z)]^{2p} \, d\mu (y) \, dt \, d\mu (z)}{[\lambda (y, t)^{1+2p}]^{1/2}} \right]}{t^{1+2p}} \right]^x \, d\mu (x)
\]
\[
\leq \int_{6S_i \setminus \delta^2 B_i} \left[ \int_{X \times (0, \infty)} \left| \frac{t}{t + d (x, y)} \right|^{x/2} \frac{1}{t^\rho} \int_{d(y,z) \geq t} \frac{K (y, z) \int_{\lambda (y, t) \leq t} \omega_i (z) \, d\mu (z) \left[ \frac{\int_{d(y,z) \leq t} \frac{[d (y, z)]^{2p} \, d\mu (y) \, dt \, d\mu (z)}{[\lambda (y, t)^{1+2p}]^{1/2}} \right]}{t^{1+2p}} \right]^x \, d\mu (x)
\]
\[
= F_1 + F_2 + F_3.
\]

To this end, let \( B_i \) be as in Lemma 15 with \( c_{B_i} \) and \( r_{B_i} \) being, respectively, its center and radius. For any \( x \in 6S_i \setminus \delta^2 B_i \) and \( z \in \delta^2 B_i \), by (2) and (3), we have
\[
F_1 \leq \int_{6S_i \setminus \delta^2 B_i} \left| f (z) \right| \int_{6S_i \setminus \delta^2 B_i} \left[ \int_{2d(y,x) \geq d(x,y)} \int_{d(y,z) \leq t} \frac{[d (y, z)]^{2p} \, d\mu (y) \, dt \, d\mu (z)}{[\lambda (y, t)^{1+2p}]^{1/2}} \right]^{1/2} \, d\mu (x) \, d\mu (z)
\]
\[
\leq \int_{6S_i \setminus \delta^2 B_i} \left| f (z) \right| \int_{6S_i \setminus \delta^2 B_i} \left[ \int_{2d(y,x) \geq d(x,y)} \frac{[d (y, z)]^{2p} \, d\mu (y) \, dt \, d\mu (z)}{[\lambda (y, t)^{1+2p}]^{1/2}} \right]^{1/2} \, d\mu (x) \, d\mu (z)
\]
\[
\leq \int_{6S_i \setminus \delta^2 B_i} \left| f (z) \right| \int_{6S_i \setminus \delta^2 B_i} \left[ \int_{2d(y,x) \geq d(x,y)} \frac{1}{[\lambda (y, t)^{1+2p}]} \, d\mu (y) \right]^{1/2} \, d\mu (x) \, d\mu (z)
\]
\[
\begin{align*}
&\leq C \int_{6B} |f(z)| \int_{6S} \left[ \int_{2d(y,z) \leq d(x,z)} \frac{1}{\lambda(y, d(x,z))} \left[ \frac{1}{\lambda(y, d(x,z))} \right] \frac{d\mu(y)}{1} \right] d\mu(x) d\mu(z) \\
&\leq C \int_{6B} |f(z)| \int_{6S} \left[ \frac{1}{\lambda(z, (1/2) d(x,z))} \int_{2d(y,z) > d(x,z)} \frac{d\mu(y)}{1} \right] d\mu(x) d\mu(z) \\
&\leq C \int_{6B} |f(z)| \int_{6S} \left[ \frac{1}{\lambda \left( \frac{1}{2} d(x,z) \right)} \int_{2d(y,z) > d(x,z)} \frac{d\mu(y)}{1} \right] d\mu(x) d\mu(z) \\
&\leq C \int_{6B} |f(z)| \int_{6S} \left[ \frac{1}{\lambda \left( \frac{1}{2} d(x,z) \right)} \right] d\mu(x) d\mu(z) \leq C \int_{6B} |f(z)| d\mu(z),
\end{align*}
\]

where we use the fact that
\[
\int_{6S} \frac{1}{\lambda \left( c_{B_i}, d \left( x, c_{B_i} \right) \right)} d\mu(x) \leq CK_{B_i, S_i}. \tag{36}
\]

Next we estimate \( F_2 \). For any \( x \in 6S \setminus 6B \), \( y \in \mathcal{X} \), and \( z \in 6B \) satisfying \( d(y,x) < t \), \( 2d(y,z) \leq d(x,z) \), and \((1/2)d(x,z) < t\), we have

\[
F_2 \leq C \int_{6B} |f(z)| \int_{6S} \left[ \int_{2d(y,z) \leq d(x,z)} \frac{d\mu(y)}{1} \right] d\mu(x) d\mu(z) \\
\leq C \int_{6B} |f(z)| \int_{6S} \left[ \int_{2d(y,z) \leq d(x,z)} \frac{d\mu(y)}{1} \right] d\mu(x) d\mu(z) \\
\leq C \int_{6B} |f(z)| \int_{6S} \left[ \frac{1}{\lambda \left( \frac{1}{2} d(x,z) \right)} \right] d\mu(x) d\mu(z) \leq C \int_{6B} |f(z)| d\mu(z). \tag{37}
\]

Finally, for any \( x \in 6S \setminus 6^2B \), \( y \in \mathcal{X} \), and \( z \in 6B \) satisfying \( 2d(y,z) \leq d(x,z) \), \( 2d(y,z) \geq d(x,z) \), and \( d(x,y) < (3/2) \cdot d(x,z) \), by applying (2), we have

\[
F_3 \leq C \int_{6B} |f(z)| \int_{6S} \left[ \int_{2d(y,z) \leq d(x,z)} \frac{d\mu(y)}{1} \right] d\mu(x) d\mu(z) \\
\leq C \int_{6B} |f(z)| \int_{6S} \left[ \int_{2d(y,z) \leq d(x,z)} \frac{d\mu(y)}{1} \right] d\mu(x) d\mu(z) \\
\leq C \int_{6B} |f(z)| \int_{6S} \left[ \frac{1}{\lambda \left( \frac{1}{2} d(x,z) \right)} \right] d\mu(x) d\mu(z) \leq C \int_{6B} |f(z)| d\mu(z). \tag{38}
\]
Combining the estimates for $F_1$, $F_2$, and $F_3$, we obtain that
\[ E_2 \leq C \int_{\delta \mathcal{B}} |f(z)| \, d\mu(z), \]
where, together with the fact that
\[ E_2 \leq C \int_{\delta \mathcal{B}} |f(z)| \, d\mu(z), \]
we have
\[ E_2 \leq C \int_{\delta \mathcal{B}} |f(x)| \, d\mu(x). \]  
(39)

Now we turn to estimate $E_1$. Let $Q_t = B(c_z, r_z)$, and write
\[ E_1 \leq \int_{X \setminus 2 \mathcal{B}} \left[ \int_{d(x,y) \leq t} \left( \frac{t}{t+d(x,y)} \right)^{1/2} \frac{d\mu(y) \, d\mu(z)}{\lambda(y,t) t} \right] \, d\mu(x) \]
\[ + \int_{X \setminus 2 \mathcal{B}} \left[ \int_{d(x,y) \leq t} \left( \frac{t}{t+d(x,y)} \right)^{1/2} \frac{d\mu(y) \, d\mu(z)}{\lambda(y,t) t} \right] \, d\mu(x) \]  
(40)

For each fixed $i$, decompose $E_{11}$ as
\[ E_{11} \leq \int_{X \setminus 3 \mathcal{B}} \left[ \int_{d(x,y) \leq t} \left( \frac{t}{t+d(x,y)} \right)^{1/2} \frac{d\mu(y) \, d\mu(z)}{\lambda(y,t) t} \right] \, d\mu(x) \]  
(41)

For any $x \in X \setminus 2 \mathcal{B}$, $y \in 2 \mathcal{B}$, $d(y,x) < t$, and $z \in \mathcal{B}, d(x,c_z) - 2r_z \leq d(x,y) < t$ and $d(x,z) < 3r_z$, together with Minkowski inequality and (6), we can conclude
\[ I_1 \leq C \int_{X \setminus 3 \mathcal{B}} \left[ \int_{d(x,y) \leq t} \frac{d(y,z)}{\lambda(y,d(x,y))} \, d\mu(z) \right] \, d\mu(x) \]  
(42)
For $I_2$, write
\[
I_2 \leq \int_{\mathcal{X} \times \mathcal{S}_1} \left[ \int_{d(x,y) < t, y \in \mathcal{X}} \frac{t}{t + d(x,y)} K(y, z) \frac{1}{[d(y, z)]^{1-p}} \frac{d \mu(y)}{\lambda(y, t)} \right]^{1/2} \frac{d \mu(y)}{\lambda(y, t)} \left[ \int_{d(y, z) \leq t} h_2(z) \frac{d \mu(z)}{\lambda(y, t)} \right]^{1/2} \frac{d \mu(z)}{\lambda(y, t)}
\]
\[
+ \int_{\mathcal{X} \times \mathcal{S}_1} \left[ \int_{d(x,y) < t, y \in \mathcal{X}} \frac{t}{t + d(x,y)} K(y, z) \frac{1}{[d(y, z)]^{1-p}} \frac{d \mu(y)}{\lambda(y, t)} \right]^{1/2} \frac{d \mu(y)}{\lambda(y, t)} \left[ \int_{d(y, z) \leq t} h_1(z) \frac{d \mu(z)}{\lambda(y, t)} \right]^{1/2} \frac{d \mu(z)}{\lambda(y, t)}
\]
\[
= I_{21} + I_{22}.
\]

For $I_{21}$, by Minkowski inequality and (6), we deduce
\[
I_{21} \leq C \int_{\mathcal{X} \times \mathcal{S}_1} \left[ \int_{d(x,y) < t, y \in \mathcal{X}} \frac{d(y,z)^2 \mu(y)}{\lambda(y, d(y,z))^{1/2}} \left( \int_{d(x,y)} \frac{1}{t^{1+2p}} \frac{d \mu(y)}{\lambda(y, t)} \right)^{1/2} \frac{d \mu(y)}{\lambda(y, t)} \ \right]
\]
\[
\leq C \int_{\mathcal{X} \times \mathcal{S}_1} \left[ \int_{d(x,y) < t, y \in \mathcal{X}} \frac{1}{\lambda(y, d(y,z))^{1/2}} \left( \int_{d(x,y)} \frac{1}{t^{1+2p}} \frac{d \mu(y)}{\lambda(y, t)} \right)^{1/2} \frac{d \mu(y)}{\lambda(y, t)} \ \right]
\]
\[
\leq C \int_{\mathcal{X} \times \mathcal{S}_1} \left[ \int_{d(x,y) < t, y \in \mathcal{X}} \frac{1}{\lambda (c_{B_1}, d(x,c_{B_1}))^{1/2}} \sum_{k=1}^{\infty} \left[ \int_{2^{k+1} \mathcal{S}_1 \times \mathcal{X}} \frac{1}{\lambda(y, d(y,c_{B_1}))^{1/2}} \frac{d \mu(y)}{\lambda(y, t)} \right] \right]^{1/2} \frac{d \mu(y)}{\lambda(y, t)} \ \right]
\]
\[
\leq C \left\| h \right\|_{L^p(\mathcal{X})}.
\]

Now we estimate $I_{22}$. Applying Minkowski inequality and the vanishing moment, we have
\[
I_{22} \leq C \int_{\mathcal{X} \times \mathcal{S}_1} \left[ \int_{d(x,y) < t, y \in \mathcal{X}} \frac{K(y, z) - K(y, c_{B_1})}{[d(y,c_{B_1})]^{1-p}} \frac{1}{[d(y,z)]^{1-p}} \frac{d \mu(y)}{\lambda(y, t)} \right]^{1/2} \frac{d \mu(y)}{\lambda(y, t)} \ \right]
\]
\[
\leq C \int_{\mathcal{X} \times \mathcal{S}_1} \left[ \int_{d(x,y) < t, y \in \mathcal{X}} \frac{K(y, z) - K(y, c_{B_1})}{[d(y,c_{B_1})]^{1-p}} \frac{1}{[d(y,z)]^{1-p}} \frac{d \mu(y)}{\lambda(y, t)} \right]^{1/2} \frac{d \mu(y)}{\lambda(y, t)} \ \right]
\]
\[
= I_1 + I_2.
\]

With a way similar to that used in the proof of $I_1$, we have
\[
I_1 \leq C \left\| h \right\|_{L^p(\mathcal{X})}.
\]

Thus, we only need to estimate $I_2$ by Minkowski inequality and (11), it follows that
\[
I_2 \leq C \left\| h \right\|_{L^p(\mathcal{X})} \left[ \int_{\mathcal{X} \times \mathcal{S}_1} \left[ K(y, z) - K(y, c_{B_1}) \right]^2 \frac{1}{[d(y,c_{B_1})]^{2-2p}} \frac{1}{\lambda(y, d(y,c_{B_1}) + r_{B_1})} \left( \int_{d(y,c_{B_1}) + r_{B_1}}^{\infty} \frac{d \mu(y)}{\lambda(y, t)^{1+2p}} \right) \frac{d \mu(z)}{\lambda(y, t)} \right]^{1/2} \frac{d \mu(y)}{\lambda(y, t)} \ \right]
Combining the estimates for $J_1, J_2, I_{21}$, and $I_1$, we obtain that

$$E_{11} \leq C \|h_i\|_{L^1(\mu)}.$$  \hfill (47)

Next we estimate $E_{12}$. For any $y \in B_i, x \in \mathcal{X} \setminus 6S_i$, and $z \in S_i$, we have $d(x, y) \geq (1/2)d(x, c_{B_i})$, $d(y, z) \leq 2r_i$, and $d(x, y) < d(x, c_{B_i})$, and together with this fact, Minkowski inequality, and (6), we get

$$E_{12} \leq C \int_{\mathcal{X}\setminus 6S_i} \int_{S_i} |h_i(z)| \left[ \int_{d(y,z) \leq t} \frac{[d(y,z)]^{2p}}{\lambda(t)} \, d\mu(y) \, dt \right] \, d\mu(z) \, d\mu(x).$$

It remains to estimate $E_{13}$. Applying Minkowski inequality and (6), we have

$$E_{13} \leq C \int_{\mathcal{X}\setminus 6S_i} \int_{S_i} |h_i(x)| \left[ \int_{d(y,z) \leq t} \frac{[d(y,z)]^{2p}}{\lambda(t)} \, d\mu(y) \, dt \right] \, d\mu(z) \, d\mu(x).$$

\[= U_1 + U_2.\]
Now we estimate $U_1$. For any $y \in \mathcal{X} \setminus Q_i$, $z \in S_i$, and $d(y, z) \leq t \leq d(x, z)$, it is easy to see $d(y, z) \sim d(y, c_{B_i})$. So we have

\[ U_1 \leq C \int_{\mathcal{X} \setminus S_i} |h_i(z)| \left[ \int_{x \in \mathcal{X}} \left( \frac{[d(x, y)]^{2\rho}}{\lambda(y, d(x, y))} \right)^{1/2} \frac{1}{\lambda(y, d(x, y))} \right] d\mu(y) \]  

\[ \leq C \int_{S_i} |h_i(z)| \left[ \int_{x \in \mathcal{X}} \frac{1}{\lambda(c_{B_i}, d(x, c_{B_i}))} \right] \left[ \int_{x \in \mathcal{X}} \frac{d\mu(y)}{\lambda(y, d(x, c_{B_i}))} \right]^{1/2} d\mu(x) \]

\[ \leq C \int_{S_i} |h_i(z)| \left[ \int_{x \in \mathcal{X}} \frac{1}{\lambda(c_{B_i}, d(x, c_{B_i}))} \right] \left[ \int_{x \in \mathcal{X}} \frac{d\mu(y)}{\lambda(y, d(x, c_{B_i}))} \right]^{1/2} d\mu(x) \]

\[ \leq C \int_{S_i} |h_i(z)| \left[ \int_{x \in \mathcal{X}} \frac{d\mu(x)}{\lambda(c_{B_i}, d(x, c_{B_i}))} \right] d\mu(x) \leq C \|h_i\|_{L^1(\mu)} . \]

On the other hand, by a method similar to that used in the proof of $U_1$, we have

\[ U_2 \leq C \|h_i\|_{L^1(\mu)} . \]

Combining the estimates $U_1, U_2, E_{11}, E_{12}$, and the fact that $\|h_i\|_{L^1(\mu)} \leq C \int_{\mathcal{X}} |f(x)| d\mu(x)$, we conclude that

\[ E_1 \leq C \int_{\mathcal{X}} |f(x)| d\mu(x) . \]

which, together with $E_2$, implies (30) and the proof of Theorem 10 is finished.

**Proof of Theorem 11.** Without loss of generality, we assume $\zeta = 2$. By a standard argument, it suffices to show that, for any $(\alpha, 1)$-atomomic block $b$,

\[ \|\mathcal{M}^\alpha_{\mu}(b)\|_{L^1(\mu)} \leq C |b|_{H^{\iota \alpha}(\mu)} . \]

Assume that $supp b \subset R$ and $b = \sum_{i=1}^{2} v_i a_i$, where, for $i \in \{1, 2\}$, $a_i$ is a function supported in $B_i \subset R$ such that $\|a_i\|_{L^\infty(\mu)} \leq C \|\mu(4B_i)\|^{-1} K_{B_i, R}$ and $|v_1| + |v_2| \sim |b|_{H^{\iota \alpha}(\mu)}$. Write

\[ \int_{\mathcal{X}} \mathcal{M}^\alpha_{\mu}(b)(x) d\mu(x) \]

\[ = \int_{2R} \mathcal{M}^\alpha_{\mu}(b)(x) d\mu(x) + \int_{\mathcal{X} \setminus 2R} \mathcal{M}^\alpha_{\mu}(b)(x) d\mu(x) \]

\[ \leq C \sum_{i=1}^{2} |v_i| \int_{2R} \mathcal{M}^\alpha_{\mu}(a_i)(x) d\mu(x) \]

\[ \leq C \sum_{i=1}^{2} |v_i| \|a_i\|_{L^\infty(\mu)} \|\mu(2B_i)\|^{1/2} \]

\[ \leq C \sum_{i=1}^{2} |v_i| K_{B_i, R} \|a_i\|_{L^\infty(\mu)} \mu(B_i) \leq C |b|_{H^{\iota \alpha}(\mu)} . \]

For $V_1$, we see that

\[ V_1 \leq \sum_{i=1}^{2} |v_i| \int_{B_i} \mathcal{M}^\alpha_{\mu}(a_i)(x) d\mu(x) \]

\[ + \sum_{i=1}^{2} |v_i| \int_{2R \setminus 2B_i} \mathcal{M}^\alpha_{\mu}(a_i)(x) d\mu(x) \]

\[ = V_{11} + V_{12} . \]

Applying the Hölder inequality, $L^2(\mu)$-boundedness of $\mathcal{M}^\alpha_{\mu}$, and the fact that $\|a_i\|_{L^\infty(\mu)} \leq C \|\mu(4B_i)\|^{-1} K_{B_i, R}$ for $i \in \{1, 2\}$, we have

\[ V_{11} \leq \sum_{i=1}^{2} |v_i| \left( \int_{B_i} |\mathcal{M}^\alpha_{\mu}(a_i)(x)|^2 d\mu(x) \right)^{1/2} \mu(2B_i)^{1/2} \]

\[ \leq C \sum_{i=1}^{2} |v_i| \|a_i\|_{L^\infty(\mu)} \mu(2B_i)^{1/2} \leq C |b|_{H^{\iota \alpha}(\mu)} . \]

Now we estimate $V_{12}$, with a method similar to that used in the proof of $F_1$ and $\|a_i\|_{L^\infty(\mu)} \leq C \|\mu(4B_i)\|^{-1} K_{B_i, R}$, and we see that

\[ V_{12} \leq \sum_{i=1}^{2} |v_i| \int_{2R \setminus 2B_i} \mathcal{M}^\alpha_{\mu}(a_i)(x) d\mu(x) \]

\[ \leq C \sum_{i=1}^{2} |v_i| K_{B_i, R} \|a_i\|_{L^\infty(\mu)} \mu(B_i) \leq C |b|_{H^{\iota \alpha}(\mu)} . \]

Therefore, $V_1 \leq C |b|_{H^{\iota \alpha}(\mu)}$.
On the other hand, based on the proof of $E_1$ and Definition 8, it is easy to obtain that

$$V_2 \leq C \|b\|_{L^1(\mu)} \leq C |b|_{H^{1,\infty}(\mu)}.$$  (58)

Combining the estimates for $V_1$ and $V_2$, (53) holds. Thus, Theorem 11 is completed.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

Authors’ Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final paper.

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