Research Article

On a Numerical Radius Preserving Onto Isometry on $\mathcal{L}(X)$

Sun Kwang Kim

Department of Mathematics, Kyonggi University, Suwon 443-760, Republic of Korea

Correspondence should be addressed to Sun Kwang Kim; sunkwang@kgu.ac.kr

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We study a numerical radius preserving onto isometry on $\mathcal{L}(X)$. As a main result, when X is a complex Banach space having both uniform smoothness and uniform convexity, we show that an onto isometry T on $\mathcal{L}(X)$ is numerical radius preserving if and only if there exists a scalar c_T of modulus 1 such that $c_T T$ is numerical range preserving. The examples of such spaces are Hilbert space and L_p spaces for $1 < p < \infty$.

1. Introduction

In this paper, we study a numerical radius preserving onto isometry on the set of bounded linear operators. We begin with some notation to present its definition. Let X be a Banach space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We use X^* and $\mathcal{L}(X)$ for the dual of X and the space of bounded linear operators from X to X , respectively. We denote by B_X (resp., S_X) the closed unit ball of X (resp., the unit sphere of X).

The numerical radius and numerical range of an operator $A \in \mathcal{L}(X)$ are given by

$$\begin{aligned} v(A) &= \sup \{ |x^*(Ax)| : (x, x^*) \in \Pi(X) \}, \\ V(A) &= \overline{\text{conv} \{ x^*(Ax) : (x, x^*) \in \Pi(X) \}}, \end{aligned} \quad (1)$$

where $\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} : x^*(x) = 1\}$.

Note that there are many different definitions of numerical range. The above definition of $V(A)$ is sometimes called Banach algebra numerical range to distinguish from others. It is worth mentioning that it is also usual to study spatial numerical range $\{x^*(Ax) : (x, x^*) \in \Pi(X)\}$. From the definition, we see that $v(A) \leq \|A\|$ for every $A \in \mathcal{L}(X)$.

The concept of numerical range was introduced by G. Lumer and F. Bauer in the sixties. Later this was extended to arbitrary continuous functions on a unit sphere of Banach spaces. On the other hand, the study of numerical radius was initiated by Harris for polynomials and holomorphic functions. We can find more details in [1–3].

There has been many different types of research on these concepts. Among them, we can find a lot of works on linear maps which preserve numerical radius or numerical range [4].

We say that an operator T on $\mathcal{L}(X)$ preserves numerical radius (numerical range) when

$$\begin{aligned} v(T(A)) &= v(A) \\ V(T(A)) &= V(A) \end{aligned} \quad (2)$$

for every $A \in \mathcal{L}(X)$. It is clear that every numerical range preserving map is numerical radius preserving.

In 1975, Pellegrini [5] studied numerical range preserving operators on a Banach algebra. Particularly, when $\mathcal{H}$ is a complex Hilbert space, it was shown that an isomorphism T on $\mathcal{L}(\mathcal{H})$ is C^* -isomorphism if and only if it is numerical range preserving. Later, Chan [6] showed that an isomorphism T on $\mathcal{L}(\mathcal{H})$ is numerical radius preserving if and only if $c_T T$ is a C^* -isomorphism for some scalar c_T of modulus 1. These results say that for each numerical radius preserving isomorphism T on $\mathcal{L}(\mathcal{H})$ there exists a scalar c_T of modulus 1 such that $c_T T$ is a numerical range preserving mapping.

In Section 2, we deduce a similar result for an onto isometry when a complex Banach space X has both uniform convexity and uniform smoothness. Indeed, we show that if T is a numerical radius preserving onto isometry on $\mathcal{L}(X)$ then there exists a scalar c_T of modulus 1 such that $c_T T$ is numerical range preserving. It is known that Hilbert space and L_p for $1 < p < \infty$ have both uniform convexity and

uniform smoothness. However, we see that this does not hold when $X = \ell_\infty^2$.

2. Results

We first recall the definition of uniform convexity.

For every $\epsilon \in (0, 2]$, the *modulus of convexity* of a Banach space $(X, \|\cdot\|)$ is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in B_X, \|x-y\| \geq \epsilon \right\}. \quad (3)$$

A Banach space $(X, \|\cdot\|)$ is said to be uniformly convex if $\delta(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

It is well known that every uniformly convex space is strictly convex and L_p is uniformly convex when $1 < p < \infty$.

Very recently, the following characterization of uniform convexity was shown [7].

Theorem 1. *A Banach space X is uniformly convex if and only if for each $\epsilon > 0$ there is $\eta(\epsilon) > 0$ such that for any $(x, x^*) \in S_X \times S_{X^*}$ with $|x^*(x)| > 1 - \eta(\epsilon)$ there exists $y \in S_Y$ satisfying*

$$\begin{aligned} x^*(y) &= 1, \\ \|y - x\| &< \epsilon. \end{aligned} \quad (4)$$

From this theorem, we show the following lemma.

Lemma 2. *Let X be a uniformly convex space. For any $\epsilon > 0$ there exists $0 < \beta(\epsilon) < \epsilon$ such that if two pairs $(x, x^*), (y, y^*) \in \Pi(X)$ satisfy $\|x^* - y^*\| < \beta(\epsilon)$ then $\|x - y\| < \epsilon$.*

Proof. Let $\eta(\cdot)$ be the function in Theorem 1 and assume that $(x, x^*), (y, y^*) \in \Pi(X)$ satisfy $\|x^* - y^*\| < \eta(\epsilon)$.

Note that x is the unique element in B_X such that $x^*(x) = 1$ by the strict convexity of X . Since $|x^*(y)| \geq |y^*(y)| - |(y^* - x^*)(x)| > 1 - \eta(\epsilon)$, we see that $\|x - y\| < \epsilon$ from Theorem 1. Hence we may take $\beta(\epsilon) = \min\{\eta(\epsilon), \epsilon/2\}$. $\square$

It is worth remarking that the converse of Lemma 2 is true when the space X is finite dimensional. Indeed, if X is not uniformly convex, then there exist $\epsilon > 0$ and a sequence of pairs $(x_i, y_i) \in B_X \times B_X$ such that $\|x_i - y_i\| > \epsilon$ and $\|x_i + y_i\|$ converge to 2. Since B_X is compact, we may assume that (x_i) and (y_i) converge to x and y , respectively. It is clear that $\|x + y\| = 2$ and so there exists a functional $f \in S_{X^*}$ such that $f(x + y) = 2$. Since $(x, f), (y, f) \in \Pi(X)$ and $\|x - y\| > \epsilon$, we get the desired contradiction.

In paper [6] of Chan, it was shown that if T is a numerical radius preserving isomorphism on $\mathcal{L}(\mathcal{H})$, then $T(I) = c_T I$ for some constant c_T of modulus 1, where I is the identity map on $\mathcal{H}$. We show that the same result holds for an onto isometry on $\mathcal{L}(X)$ when X has uniform convexity and uniform smoothness. Before that, we first see the following.

Lemma 3. *If T is a numerical radius preserving map on $\mathcal{L}(X)$, then for each $A \in \mathcal{L}(X)$ there exists a scalar α_A of modulus 1 such that $v(T(I)) + v(T(A)) = v(T(I) + \alpha_A T(A))$.*

Proof. We first see that for any $A \in \mathcal{L}(X)$ there is a scalar α_A of modulus 1 satisfying $v(I + \alpha_A A) = 1 + v(A)$. Indeed,

for a sequence $(x_i, x_i^*) \in \Pi(X)$ such that $|x_i^*(Ax_i)|$ converges to $v(A)$, we may assume that $x_i^*(Ax_i)$ converges to some constant β . Let $\alpha_A = \bar{\beta}/|\beta|$. Then, we have that $x_i^*((I + \alpha_A A)x_i)$ converges to $1 + |\beta| = 1 + v(A)$. Since it is clear that $v(I + \alpha_A A) \leq 1 + v(A)$, we deduce $v(I + \alpha_A A) = 1 + v(A)$.

The fact that T preserves numerical radius gives $v(T(I)) + v(T(A)) = v(I) + v(A) = v(I + \alpha_A A) = v(T(I + \alpha_A A)) = v(T(I) + \alpha_A T(A))$. $\square$

Lemma 4. *Assume X is a strictly convex space and an operator $A \in \mathcal{L}(X)$ satisfies $\|A\| = 1$. If absolute value of every element in $V(A)$ is 1, then $A = c_A I$ for some constant c_A of modulus 1.*

Proof. From the assumption, we see that $\|Ax\| = 1 = |x^*(Ax)|$ for arbitrary $x \in S_X$ and x^* satisfying $x^*(x) = 1$. Since X is strictly convex, x^* attains its norm only at $\{\lambda x : |\lambda| = 1\}$. This implies that $Ax = \lambda_x x$ for some constant λ_x of modulus 1.

Now assume that there exist two elements $x_1, x_2 \in S_X$ such that $\lambda_{x_1} \neq \lambda_{x_2}$. Note that x_1 and x_2 are linearly independent. Consider $z = (x_1 + x_2)/\|x_1 + x_2\|$ and let $Az = \lambda_z z$. This means that

$$\frac{\lambda_z x_1 + \lambda_z x_2}{\|x_1 + x_2\|} = \lambda_z z = A_z = \frac{\lambda_{x_1} x_1 + \lambda_{x_2} x_2}{\|x_1 + x_2\|}. \quad (5)$$

Hence, we have $(\lambda_z - \lambda_{x_1})x_1 = (\lambda_{x_2} - \lambda_z)x_2$ which is a contradiction. $\square$

Theorem 5. *Assume X has both uniform convexity and uniform smoothness. If T is a numerical radius preserving onto isometry on $\mathcal{L}(X)$, then $T(I) = c_T I$ for some constant c_T of modulus 1.*

Proof. From Lemma 3, for each $A \in \mathcal{L}(X)$, there is α_A so that $v(T(I)) + v(T(A)) = v(T(I) + \alpha_A T(A))$.

We now show that $|x^*(T(I)(x))| = v(T(I)) = v(I) = 1$ for any $(x, x^*) \in \Pi(X)$. If this is not true, there exists $(z_0, z_0^*) \in \Pi(X)$ such that $|z_0^*(T(I)(z_0))| < 1$. Take an operator $B \in \mathcal{L}(X)$ so that $T(B)(x) = z_0^*(x)z_0$ for each $x \in X$ and take $0 < \gamma < 1$ such that $|z_0^*(T(I)(z_0))| < \gamma < 1$.

Let $\delta > 0$ be a number satisfying that every $(z, z^*) \in \Pi(X)$ with $\|z - z_0\| < \delta$ and $\|z^* - z_0^*\| < \delta$ implies $|z^*(T(I)(z))| < \gamma$. Since X is uniformly convex, we may take $\beta(\cdot)$ in Lemma 2.

For a pair $(y, y^*) \in \Pi(X)$, if $\|y^* - z_0^*\| < \beta(\delta) < \delta$, then $\|y - z_0\| < \delta$. This gives that

$$\begin{aligned} &|y^*(T(I + \alpha_B B)(y))| \\ &\leq |y^*(T(I)(y))| + |y^*(T(B)(y))| < \gamma + 1. \end{aligned} \quad (6)$$

On the other hand, there exists $\epsilon > 0$ such that if $(y, y^*) \in \Pi(X)$ satisfies $\|y^* - z_0^*\| \geq \beta(\delta)$, then $|y^*(z_0)| \leq 1 - \epsilon$. To see that this is true, we note that dual of uniformly smooth space is uniformly convex. Hence X^* is uniformly convex. We may use again Lemma 2, and then for some small enough $\epsilon > 0$ if $(y, y^*) \in \Pi(X)$ satisfies $|y^*(z_0)| > 1 - \epsilon$ then $\|y^* - z_0^*\| < \beta(\delta)$.

Therefore we see that if $(y, y^*) \in \Pi(X)$ satisfies that $\|y^* - z_0^*\| \geq \beta(\delta)$, then $|y^*(T(B)(y))| = |y^*(z_0^*)z_0^*(y)| \leq 1 - \varepsilon$. We have

$$\begin{aligned} |y^*(T(I + \alpha_B B)(y))| &\leq |y^*(T(I)(y))| + |y^*(T(B)(y))| \\ &\leq 1 + 1 - \varepsilon. \end{aligned} \quad (7)$$

This shows that $v(T(I + \alpha_B B)) \leq \max\{2 - \varepsilon, \gamma + 1\} < 2 = v(T(I)) + v(T(B))$ and so we have desired contradiction.

From Lemma 4, we see that $T(I) = c_T I$ for some modulus 1 constant c_T . $\square$

Remark 6. From Theorem 5, we can easily construct onto isometries which are not numerical radius preserving. For example, consider 2-dimensional Hilbert space $\mathbb{R}^2$; we define an operator $B : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $B(x_1, x_2) = (x_1, -x_2)$ for every $(x_1, x_2) \in \mathbb{R}^2$. It is clear that the operator T on $\mathcal{L}(\mathbb{R}^2)$ given by $T(A) = B \circ A$ for every $A \in \mathcal{L}(\mathbb{R}^2)$ is an onto isometry. However, this does not preserve numerical radius since $T(I) \neq c_T I$ for any c_T .

We denote a state space $\mathcal{S}$ of a Banach algebra $\mathcal{L}(X)$ by

$$\mathcal{S} = \{f \in \mathcal{L}(X)^* : \|f\| = 1 = f(I)\}. \quad (8)$$

When X is a complex Banach space, according to [5, Theorem 2.2], if the adjoint operator S^* of operator $S : \mathcal{L}(X) \rightarrow \mathcal{L}(X)$ satisfies that $S^*(\mathcal{S}) \subset \mathcal{S}$, then $V(S(A)) \subset V(A)$. For T in Theorem 5, since $T_0 = c_T^{-1}T$ is an onto isometry and so satisfies $T_0^*(\mathcal{S}) = \mathcal{S}$, we see that $V(T_0(A)) = V(A)$. Now, we deduce the following main result.

Theorem 7. Assume a complex Banach space X has both uniform convexity and uniform smoothness. An onto isometry T on $\mathcal{L}(X)$ is numerical radius preserving if and only if there exists a scalar c_T of modulus 1 such that $c_T T$ is a numerical range preserving mapping.

Remark 8. Theorem 7 does not hold for some Banach spaces. In order to see this, let us recall numerical index $n(X)$ of a Banach space.

$$n(X) = \inf \{v(T) : T \in \mathcal{S}_{\mathcal{L}(X)}\}. \quad (9)$$

We see that $n(X)$ is the greatest constant $k \geq 0$ such that $k\|T\| \leq v(T)$ for every $T \in \mathcal{L}(X)$, and so v is equivalent to the operator norm if and only if $n(X) > 0$. For more information, we give [8–11].

From the definition of numerical index, if $n(X) = 1$, then every isometry on $\mathcal{L}(X)$ preserves numerical radius. There are many classes of Banach spaces having numerical index 1, like c_0 , ℓ_1 , and ℓ_∞ [12]. Among them, one of the simplest examples having numerical index 1 is $\mathbb{R}^2$ as the typical subspace of ℓ_∞ . Since the same operator T defined in Remark 6 is also an onto isometry on this space, T preserves numerical radius. However, this does not preserve numerical range. The reason is that we have $V(I) = \{1\}$ and $-1 = (0, 1)T(I)(0, 1) \in V(T(I))$.

Remark 9. It is not possible to say that Theorem 7 is true for every isometry. Consider shift operators B_1 and B_2 on ℓ_p ($1 < p < \infty$) defined by $B_1((x_i)_{i=1}^\infty) = (x_i)_{i=2}^\infty$ and $B_2((x_i)_{i=1}^\infty) = (x_i)_{i=0}^\infty$ for $(x_i)_{i=1}^\infty \in \ell_p$, where $x_0 = 0$. Then, it is easy to see that an operator T given by $T(A) = B_2 \circ A \circ B_1$ for each $A \in L(\ell_p)$ is a numerical radius preserving isometry. However, T does not preserve numerical range since $V(I) = \{1\}$ and $0 = (1, 0, 0, 0, \dots)T(I)(1, 0, 0, 0, \dots) \in V(T(I))$.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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