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Research Article

A New Extension of Hardy-Hilbert's Inequality in the Whole Plane

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By the use of weight coefficients and Hermite-Hadamard's inequality, a new extension of Hardy-Hilbert's inequality in the whole plane with multiparameters and a best possible constant factor is given. The equivalent forms, the operator expressions, and a few particular inequalities are considered.

1. Introduction

Suppose that $p > 1$, $1/p + 1/q = 1$, $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$, $\|a\|_p = (\sum_{m=1}^{\infty} a_m^p)^{1/p} > 0$, and $\|b\|_q > 0$. We have the following well known Hardy-Hilbert's inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q, \quad (1)$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible one (cf. [1]). The more accurate form of (1) was given as follows (cf. [2], Theorem 323):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n-2\alpha} < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q \quad \left(0 \leq \alpha \leq \frac{1}{2}\right), \quad (2)$$

where the constant factor $\pi/\sin(\pi/p)$ is still the best possible one. For $\alpha = 0$, inequality (2) reduces to (1).

In 2011, Yang gave an extension of (2) as follows (cf. [3]): If $0 < \lambda_1, \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = \lambda$, $a_m, b_n \geq 0$, $\|a\|_{p,\varphi} = \{\sum_{m=1}^{\infty}(m-\alpha)^{p(1-\lambda_1)-1}a_m^p\}^{1/p} \in (0,\infty)$, and $\|b\|_{q,\psi} = \{\sum_{n=1}^{\infty}(n-\alpha)^{q(1-\lambda_2)-1}b_n^q\}^{1/q} \in (0,\infty)$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n-2\alpha)^{\lambda}} < B(\lambda_1,\lambda_2)\|a\|_{p,\varphi}\|b\|_{q,\psi} \quad \left(0 \leq \alpha \leq \frac{1}{2}\right), \quad (3)$$

where the constant factor $B(\lambda_1, \lambda_2)$ is the best possible one and $B(u, v)$ is the beta function defined by (cf. [4])

$$B(u,v) := \int_0^{\infty}\frac{1}{(1+t)^{u+v}}t^{u-1}dt \quad (u,v>0). \quad (4)$$

For $\lambda = 1$, $\lambda_1 = 1/q$, and $\lambda_2 = 1/p$, (3) reduces to (2). Some other results related to (1)–(3) are provided by [5–22].

In this paper, by the use of weight coefficients and Hermite-Hadamard's inequality, an extension of (3) in the whole plane is given as follows: For $\xi, \eta \in [0, 1/2]$, $a_m, b_n \geq 0$, $\sum_{|m|=1}^{\infty} |m-\xi|^{p(1-\lambda_1)-1} a_m^p \in (0, \infty)$, and $\sum_{|n|=1}^{\infty} |n-\eta|^{q(1-\lambda_2)-1} b_n^q \in (0, \infty)$, we have

$$\sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \frac{1}{(|m-\xi| + |n-\eta|)^{\lambda}} a_m b_n < 2B(\lambda_1, \lambda_2) \cdot \left[\sum_{|m|=1}^{\infty} |m-\xi|^{p(1-\lambda_1)-1} a_m^p\right]^{1/p} \cdot \left[\sum_{|n|=1}^{\infty} |n-\eta|^{q(1-\lambda_2)-1} b_n^q\right]^{1/q}. \quad (5)$$

Moreover, a generation of (5) with multiparameters and a best possible constant factor is proved. The equivalent forms, the operator expressions, and a few particular inequalities are also considered.

2. Some Lemmas

First, we make appointment that $\mathbf{N} = \{1, 2, \ldots\}$, $p > 1$, $1/p + 1/q = 1$, $\alpha, \beta \in (0, \pi)$, $\xi, \eta \in [0, 1/2]$, $\lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = \lambda$, and, for $|x|, |y| > 1/2$,

$$k(x, y) := \frac{1}{[|x-\xi| + (x-\xi)\cos\alpha + |y-\eta| + (y-\eta)\cos\beta]^{\lambda}}. \quad (6)$$

In particular, for $\alpha = \beta = \pi/2$, we indicate

$$h(x, y) := \frac{1}{(|x-\xi| + |y-\eta|)^{\lambda}}, \quad |x|, |y| > \frac{1}{2}. \quad (7)$$

Definition 1. Define the following weight coefficients:

$$\omega(\lambda_2, m) := \sum_{|n|=1}^{\infty} k(m, n) \frac{[|m-\xi| + (m-\xi)\cos\alpha]^{\lambda_1}}{[|n-\eta| + (n-\eta)\cos\beta]^{1-\lambda_2}}, \quad |m| \in \mathbf{N}, \quad (8)$$

$$\varpi(\lambda_1, n) := \sum_{|m|=1}^{\infty} k(m, n) \frac{[|n-\eta| + (n-\eta)\cos\beta]^{\lambda_2}}{[|m-\xi| + (m-\xi)\cos\alpha]^{1-\lambda_1}}, \quad |n| \in \mathbf{N}, \quad (9)$$

where $\sum_{|j|=1}^{\infty} \cdots = \sum_{j=-1}^{-\infty} \cdots + \sum_{j=1}^{\infty} \cdots$ $(j = m, n)$.

Lemma 2. *If $\lambda_2 \leq 1$, then, for $k_\beta(\lambda_1) := 2B(\lambda_1, \lambda_2)\csc^2\beta$, one has*

$$k_\beta(\lambda_1)(1 - \theta(\lambda_2, m)) < \omega(\lambda_2, m) < k_\beta(\lambda_1), \quad |m| \in \mathbf{N}, \quad (10)$$

where

$$\theta(\lambda_2, m) := \frac{1}{B(\lambda_1, \lambda_2)} \cdot \int_0^{(1+\eta)(1+\cos\beta)/(|m-\xi|+(m-\xi)\cos\alpha)} \frac{u^{\lambda_2-1}}{(1+u)^{\lambda}} du = O\left(\frac{1}{[|m-\xi| + (m-\xi)\cos\alpha]^{\lambda_2}}\right) \in (0, 1), \quad |m| \in \mathbf{N}. \quad (11)$$

Proof. For $|x| > 1/2$, we set

$$k^{(1)}(x, y) := \frac{1}{[|x-\xi| + (x-\xi)\cos\alpha + (y-\eta)(\cos\beta - 1)]^{\lambda}}, \quad y < -\frac{1}{2},$$
$$k^{(2)}(x, y) := \frac{1}{[|x-\xi| + (x-\xi)\cos\alpha + (y-\eta)(1 + \cos\beta)]^{\lambda}}, \quad y > \frac{1}{2}, \quad (12)$$

and then, for $y > 1/2$,

$$k^{(1)}(x, -y) = \frac{1}{[|x-\xi| + (x-\xi)\cos\alpha + (y+\eta)(1 - \cos\beta)]^{\lambda}}. \quad (13)$$

We find

$$\begin{aligned} \omega(\lambda_2, m) &= \sum_{n=-1}^{-\infty} k^{(1)}(m, n) \frac{[|m-\xi| + (m-\xi)\cos\alpha]^{\lambda_1}}{[(n-\eta)(\cos\beta - 1)]^{1-\lambda_2}} + \sum_{n=1}^{\infty} k^{(2)}(m, n) \frac{[|m-\xi| + (m-\xi)\cos\alpha]^{\lambda_1}}{[(n-\eta)(1 + \cos\beta)]^{1-\lambda_2}} \\ &= \frac{[|m-\xi| + (m-\xi)\cos\alpha]^{\lambda_1}}{(1 - \cos\beta)^{1-\lambda_2}} \sum_{n=1}^{\infty} \frac{k^{(1)}(m, -n)}{(n+\eta)^{1-\lambda_2}} + \frac{[|m-\xi| + (m-\xi)\cos\alpha]^{\lambda_1}}{(1 + \cos\beta)^{1-\lambda_2}} \sum_{n=1}^{\infty} \frac{k^{(2)}(m, n)}{(n-\eta)^{1-\lambda_2}}. \end{aligned} \quad (14)$$

It is evident that, for fixed $m \in \mathbf{N}$, $\lambda_2 \leq 1$ $(\lambda > 0)$, both $k^{(1)}(m, -y)/(y+\eta)^{1-\lambda_2}$ and $k^{(2)}(m, y)/(y-\eta)^{1-\lambda_2}$ are strictly decreasing and strictly convex with respect to $y \in (1/2, \infty)$, satisfying

$$\frac{d}{dy}\frac{k^{(i)}\left(m,(-1)^i y\right)}{\left[y+(-1)^i \eta\right]^{1-\lambda_2}} < 0,$$
$$\frac{d^2}{dy^2}\frac{k^{(i)}\left(m,(-1)^i y\right)}{\left[y+(-1)^i \eta\right]^{1-\lambda_2}} > 0, \quad (i = 1, 2). \tag{15}$$

By Hermite-Hadamard's inequality (cf. [23]), we find

$$\begin{aligned}
\omega(\lambda_2, m) &< \frac{\left[|m-\xi| + (m-\xi)\cos\alpha\right]^{\lambda_1}}{(1-\cos\beta)^{1-\lambda_2}} \int_{1/2}^{\infty} \frac{k^{(1)}(m,-y)}{(y+\eta)^{1-\lambda_2}} dy \\
&+ \frac{\left[|m-\xi| + (m-\xi)\cos\alpha\right]^{\lambda_1}}{(1+\cos\beta)^{1-\lambda_2}} \int_{1/2}^{\infty} \frac{k^{(2)}(m,y)}{(y-\eta)^{1-\lambda_2}} dy \\
&\le \frac{\left[|m-\xi| + (m-\xi)\cos\alpha\right]^{\lambda_1}}{(1-\cos\beta)^{1-\lambda_2}} \int_{-\eta}^{\infty} \frac{k^{(1)}(m,-y)}{(y+\eta)^{1-\lambda_2}} dy \\
&+ \frac{\left[|m-\xi| + (m-\xi)\cos\alpha\right]^{\lambda_1}}{(1+\cos\beta)^{1-\lambda_2}} \int_{\eta}^{\infty} \frac{k^{(2)}(m,y)}{(y-\eta)^{1-\lambda_2}} dy.
\end{aligned} \tag{16}$$

Setting $u = ((y+\eta)(1-\cos\beta)/(|m-\xi|+(m-\xi)\cos\alpha))((y-\eta)(1+\cos\beta)/(|m-\xi|+(m-\xi)\cos\alpha))$ in the above first (second) integral, by simplification, we find

$$\begin{aligned}
\omega(\lambda_2, m) &< \left(\frac{1}{1-\cos\beta} + \frac{1}{1+\cos\beta}\right) \int_0^{\infty} \frac{u^{\lambda_2 - 1}}{(1+u)^{\lambda}} du \\
&= 2B(\lambda_1, \lambda_2)\csc^2\beta = k_\beta(\lambda_1).
\end{aligned} \tag{17}$$

By (14), since both $k^{(1)}(m,-y)/(y+\eta)^{1-\lambda_2}$ and $k^{(2)}(m,y)/(y-\eta)^{1-\lambda_2}$ are strictly decreasing, we still have

$$\begin{aligned}
\omega(\lambda_2, m) &> \frac{\left[|m-\xi| + (m-\xi)\cos\alpha\right]^{\lambda_1}}{(1-\cos\beta)^{1-\lambda_2}} \cdot \int_1^{\infty} \frac{k^{(1)}(m,-y)}{(y+\eta)^{1-\lambda_2}} dy \\
&+ \frac{\left[|m-\xi| + (m-\xi)\cos\alpha\right]^{\lambda_1}}{(1+\cos\beta)^{1-\lambda_2}} \int_1^{\infty} \frac{k^{(2)}(m,y)}{(y-\eta)^{1-\lambda_2}} dy \\
&\ge \frac{1}{1-\cos\beta} \cdot \int_{(1+\eta)(1+\cos\beta)/(|m-\xi|+(m-\xi)\cos\alpha)}^{\infty} \frac{u^{\lambda_2-1}}{(1+u)^{\lambda}} du \\
&+ \frac{1}{1+\cos\beta} \cdot \int_{(1+\eta)(1+\cos\beta)/(|m-\xi|+(m-\xi)\cos\alpha)}^{\infty} \frac{u^{\lambda_2-1}}{(1+u)^{\lambda}} du \\
&= k_\beta(\lambda_2)\left(1 - \theta(\lambda_2, m)\right) > 0.
\end{aligned} \tag{18}$$

We obtain

$$\begin{aligned}
0 < \theta(\lambda_2, m) &= \frac{1}{B(\lambda_1,\lambda_2)} \cdot \int_0^{(1+\eta)(1+\cos\beta)/(|m-\xi|+(m-\xi)\cos\alpha)} \frac{u^{\lambda_2-1}}{(1+u)^{\lambda}} du \\
&< \frac{1}{B(\lambda_1,\lambda_2)} \int_0^{(1+\eta)(1+\cos\beta)/(|m-\xi|+(m-\xi)\cos\alpha)} u^{\lambda_2-1} du \\
&= \frac{1}{\lambda_2 B(\lambda_1,\lambda_2)} \left[\frac{(1+\eta)(1+\cos\beta)}{|m-\xi| + (m-\xi)\cos\alpha}\right]^{\lambda_2},
\end{aligned} \tag{19}$$

and then we have (10) and (11). □

In the same way, we still have the following.

Lemma 3. *If $\lambda_1 \le 1$, then, for $k_\alpha(\lambda_1) = 2B(\lambda_1, \lambda_2)\csc^2\alpha$, one has*

$$k_\alpha(\lambda_1)\left(1 - \vartheta(\lambda_1, n)\right) < \varpi(\lambda_1, n) < k_\alpha(\lambda_1), \quad |n| \in \mathbf{N}, \tag{20}$$

where

$$\begin{aligned}
\vartheta(\lambda_1, n) &:= \frac{1}{B(\lambda_1,\lambda_2)} \cdot \int_0^{(1+\xi)(1+\cos\alpha)/(|n-\eta|+(n-\eta)\cos\beta)} \frac{u^{\lambda_1-1}}{(1+u)^{\lambda}} du \\
&= O\left(\frac{1}{\left[|n-\eta| + (n-\eta)\cos\beta\right]^{\lambda_1}}\right) \in (0,1), \quad |n| \in \mathbf{N}.
\end{aligned} \tag{21}$$

Lemma 4. *If $\zeta \in [0, 1/2]$ and $\theta \in (0, \pi)$, then, for $\rho > 0$,*

$$\begin{aligned}
H_\rho(\zeta, \theta) &:= \sum_{|n|=1}^{\infty} \frac{1}{\left[|n-\zeta| + (n-\zeta)\cos\theta\right]^{1+\rho}} \\
&= \frac{1+o(1)}{\rho}\left[\frac{1}{(1+\cos\theta)^{1+\rho}} + \frac{1}{(1-\cos\theta)^{1+\rho}}\right] \quad (\rho \longrightarrow 0^+).
\end{aligned} \tag{22}$$

Proof. We have

$$\begin{aligned} H_\rho(\zeta,\theta) &= \sum_{n=-1}^{-\infty} \frac{1}{[(n-\zeta)(\cos\theta-1)]^{1+\rho}} + \sum_{n=1}^{\infty} \frac{1}{[(n-\zeta)(\cos\theta+1)]^{1+\rho}} \\ &= \frac{1}{(1-\cos\theta)^{1+\rho}} \sum_{n=1}^{\infty} \frac{1}{(n+\zeta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}} \sum_{n=1}^{\infty} \frac{1}{(n-\zeta)^{1+\rho}}. \end{aligned} \tag{23}$$

For $a = 1/(1-\zeta)^{1+\rho}$, we find

$$\begin{aligned} H_\rho(\zeta,\theta) &\le \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right]\left[a + \sum_{n=2}^{\infty} \frac{1}{(n-\zeta)^{1+\rho}}\right] < \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right]\left[a + \int_1^\infty \frac{dy}{(y-\zeta)^{1+\rho}}\right] \\ &= \frac{1}{\rho}\left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right]\left\{1 + \left[a\rho + \frac{1}{(1-\zeta)^{\rho}} - 1\right]\right\}, \\ H_\rho(\zeta,\theta) &\ge \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right] \cdot \sum_{n=1}^{\infty} \frac{1}{(n+\zeta)^{1+\rho}} > \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right] \int_1^\infty \frac{dy}{(y+\zeta)^{1+\rho}} \\ &= \frac{1 + \left[(1+\zeta)^{-\rho} - 1\right]}{\rho}\left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right]. \end{aligned} \tag{24}$$

Hence, we have (22). □

3. Main Results and Operation Expressions

Theorem 5. *If $\lambda_1, \lambda_2 \le 1$, $a_m, b_n \ge 0$ ($|m|, |n| \in \mathbf{N}$),*

$$\begin{aligned} 0 &< \sum_{|m|=1}^{\infty} \left[|m-\xi| + (m-\xi)\cos\alpha\right]^{p(1-\lambda_1)-1} a_m^p < \infty, \\ 0 &< \sum_{|n|=1}^{\infty} \left[|n-\eta| + (n-\eta)\cos\beta\right]^{q(1-\lambda_2)-1} b_n^q < \infty, \\ k(\lambda_1) &:= k_\beta^{1/p}(\lambda_1)\, k_\alpha^{1/q}(\lambda_1) = 2B(\lambda_1,\lambda_2)\csc^{2/p}\beta \csc^{2/q}\alpha, \end{aligned} \tag{25}$$

then one has the following equivalent inequalities:

$$\begin{aligned} I &:= \sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty} k(m,n)\, a_m b_n < k(\lambda_1) \cdot \left\{\sum_{|m|=1}^{\infty} \left[|m-\xi| + (m-\xi)\cos\alpha\right]^{p(1-\lambda_1)-1} a_m^p\right\}^{1/p} \\ &\quad \cdot \left\{\sum_{|n|=1}^{\infty} \left[|n-\eta| + (n-\eta)\cos\beta\right]^{q(1-\lambda_2)-1} b_n^q\right\}^{1/q}, \end{aligned} \tag{26}$$

$$\begin{aligned} J &:= \left\{\sum_{|n|=1}^{\infty} \left[|n-\eta| + (n-\eta)\cos\beta\right]^{p\lambda_2-1} \cdot \left(\sum_{|m|=1}^{\infty} k(m,n)\, a_m\right)^p\right\}^{1/p} < k(\lambda_1) \\ &\quad \cdot \left\{\sum_{|m|=1}^{\infty} \left[|m-\xi| + (m-\xi)\cos\alpha\right]^{p(1-\lambda_1)-1} a_m^p\right\}^{1/p}. \end{aligned} \tag{27}$$

Proof. By Hölder's inequality (cf. [23]) and (9), we have

$$\begin{aligned} \left(\sum_{|m|=1}^{\infty} k(m,n)\, a_m\right)^p &= \left\{\sum_{|m|=1}^{\infty} k(m,n) \cdot \frac{\left[|m-\xi| + (m-\xi)\cos\alpha\right]^{(1-\lambda_1)/q}}{\left[|n-\eta| + (n-\eta)\cos\beta\right]^{(1-\lambda_2)/p}} a_m \cdot \frac{\left[|n-\eta| + (n-\eta)\cos\beta\right]^{(1-\lambda_2)/p}}{\left[|m-\xi| + (m-\xi)\cos\alpha\right]^{(1-\lambda_1)/q}}\right\}^p \\ &\le \sum_{|m|=1}^{\infty} k(m,n) \cdot \frac{\left[|m-\xi| + (m-\xi)\cos\alpha\right]^{(1-\lambda_1)p/q}}{\left[|n-\eta| + (n-\eta)\cos\beta\right]^{1-\lambda_2}} a_m^p \\ &\quad \cdot \left\{\sum_{|m|=1}^{\infty} k(m,n) \frac{\left[|n-\eta| + (n-\eta)\cos\beta\right]^{(1-\lambda_2)q/p}}{\left[|m-\xi| + (m-\xi)\cos\alpha\right]^{1-\lambda_1}}\right\}^{p-1} \\ &= \frac{(\varpi(\lambda_1,n))^{p-1}}{\left[|n-\eta| + (n-\eta)\cos\beta\right]^{p\lambda_2-1}} \sum_{|m|=1}^{\infty} k(m,n) \cdot \frac{\left[|m-\xi| + (m-\xi)\cos\alpha\right]^{(1-\lambda_1)p/q}}{\left[|n-\eta| + (n-\eta)\cos\beta\right]^{1-\lambda_2}} a_m^p. \end{aligned} \tag{28}$$

By (20), we have

$$J < k_\alpha^{1/q}(\lambda_1)\left\{\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}k(m,n)\cdot\frac{[|m-\xi|+(m-\xi)\cos\alpha]^{(1-\lambda_1)p/q}}{[|n-\eta|+(n-\eta)\cos\beta]^{1-\lambda_2}}a_m^p\right\}^{1/p}$$
$$= k_\alpha^{1/q}(\lambda_1)\left\{\sum_{|m|=1}^{\infty}\sum_{|n|=1}^{\infty}k(m,n)\cdot\frac{[|m-\xi|+(m-\xi)\cos\alpha]^{(1-\lambda_1)p/q}}{[|n-\eta|+(n-\eta)\cos\beta]^{1-\lambda_2}}a_m^p\right\}^{1/p}$$
$$= k_\alpha^{1/q}(\lambda_1)\left\{\sum_{|m|=1}^{\infty}\omega(\lambda_2,m)[|m-\xi|+(m-\xi)\cdot\cos\alpha]^{p(1-\lambda_1)-1}a_m^p\right\}^{1/p}. \quad (29)$$

By (10), we have (27).

By Hölder's inequality (cf. [23]), we have

$$I = \sum_{|n|=1}^{\infty}\left\{[|n-\eta|+(n-\eta)\cos\beta]^{\lambda_2-1/p}\cdot\sum_{|m|=1}^{\infty}k(m,n)a_m\right\}[|n-\eta|+(n-\eta)\cos\beta]^{1/p-\lambda_2}b_n$$
$$\leq J\left\{\sum_{|n|=1}^{\infty}[|n-\eta|+(n-\eta)\cos\beta]^{q(1-\lambda_2)-1}b_n^q\right\}^{1/q}. \quad (30)$$

Then by (27), we have (26).

On the other hand, assuming that (26) is valid, we set

$$b_n := [|n-\eta|+(n-\eta)\cos\beta]^{p\lambda_2-1}\cdot\left(\sum_{|m|=1}^{\infty}k(m,n)a_m\right)^{p-1}, \quad |n|\in\mathbf{N}. \quad (31)$$

Then it follows that

$$J = \left\{\sum_{|n|=1}^{\infty}[|n-\eta|+(n-\eta)\cos\beta]^{q(1-\lambda_2)-1}b_n^q\right\}^{1/p}. \quad (32)$$

By (29), we find $J<\infty$. If $J=0$, then (27) is evidently valid; if $J>0$, then, by (26), we have

$$0 < \sum_{|n|=1}^{\infty}[|n-\eta|+(n-\eta)\cos\beta]^{q(1-\lambda_2)-1}b_n^q = J^p = I$$
$$< k(\lambda_1)\cdot\left\{\sum_{|m|=1}^{\infty}[|m-\xi|+(m-\xi)\cos\alpha]^{p(1-\lambda_1)-1}a_m^p\right\}^{1/p}\cdot\left\{\sum_{|n|=1}^{\infty}[|n-\eta|+(n-\eta)\cos\beta]^{q(1-\lambda_2)-1}b_n^q\right\}^{1/q},$$
$$J = \left\{\sum_{|n|=1}^{\infty}[|n-\eta|+(n-\eta)\cos\beta]^{q(1-\lambda_2)-1}b_n^q\right\}^{1/p}$$
$$< k(\lambda_1)\cdot\left\{\sum_{|m|=1}^{\infty}[|m-\xi|+(m-\xi)\cos\alpha]^{p(1-\lambda_1)-1}a_m^p\right\}^{1/p}; \quad (33)$$

namely, (27) follows, which is equivalent to (26). □

Theorem 6. *As regards the assumptions of Theorem 5, the constant factor $k(\lambda_1)$ in (26) and (27) is the best possible one.*

Proof. For any $\varepsilon\in(0,q\lambda_2)$, we set $\widetilde{\lambda}_1=\lambda_1+\varepsilon/q$, $\widetilde{\lambda}_2=\lambda_2-\varepsilon/q$ $(\in(0,1))$, and

$$\widetilde{a}_m := [|m-\xi|+(m-\xi)\cos\alpha]^{(\lambda_1-\varepsilon/p)-1} = [|m-\xi|+(m-\xi)\cos\alpha]^{\widetilde{\lambda}_1-\varepsilon-1} \quad (|m|\in\mathbf{N}),$$
$$\widetilde{b}_n := [|n-\eta|+(n-\eta)\cos\beta]^{(\lambda_2-\varepsilon/q)-1} = [|n-\eta|+(n-\eta)\cos\beta]^{\widetilde{\lambda}_2-1} \quad (|n|\in\mathbf{N}). \quad (34)$$

Then by (22) and (10), we find

$$\widetilde{I}_1 := \left\{\sum_{|m|=1}^{\infty}[|m-\xi|+(m-\xi)\cos\alpha]^{p(1-\lambda_1)-1}\widetilde{a}_m^p\right\}^{1/p}\cdot\left\{\sum_{|n|=1}^{\infty}[|n-\eta|+(n-\eta)\cos\beta]^{q(1-\lambda_2)-1}\widetilde{b}_n^q\right\}^{1/q}$$
$$= \left\{\sum_{|m|=1}^{\infty}\frac{1}{[|m-\xi|+(m-\xi)\cos\alpha]^{1+\varepsilon}}\right\}^{1/p}\cdot\left\{\sum_{|n|=1}^{\infty}\frac{1}{[|n-\eta|+(n-\eta)\cos\beta]^{1+\varepsilon}}\right\}^{1/q}$$
$$= \frac{1}{\varepsilon}\left[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}}+\frac{1}{(1-\cos\alpha)^{1+\varepsilon}}\right]^{1/p}\cdot\left[\frac{1}{(1+\cos\beta)^{1+\varepsilon}}+\frac{1}{(1-\cos\beta)^{1+\varepsilon}}\right]^{1/q}(1+o_1(1))^{1/p}\cdot(1+o_2(1))^{1/q},$$

$$\widetilde{I} := \sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty} k(m,n)\widetilde{a}_m\widetilde{b}_n = \sum_{|m|=1}^{\infty}\sum_{|m|=1}^{\infty} k(m,n) \cdot \frac{[|m-\xi| + (m-\xi)\cos\alpha]^{\widetilde{\lambda}_1-\varepsilon-1}}{[|n-\eta| + (n-\eta)\cos\beta]^{1-\widetilde{\lambda}_2}}$$
$$= \sum_{|m|=1}^{\infty} \frac{\omega(\widetilde{\lambda}_2, m)}{[|m-\xi| + (m-\xi)\cos\alpha]^{\varepsilon+1}} \geq k_\beta(\widetilde{\lambda}_1) \cdot \sum_{|m|=1}^{\infty} \frac{1-\theta(\widetilde{\lambda}_2, m)}{[|m-\xi| + (m-\xi)\cos\alpha]^{\varepsilon+1}} = k_\beta(\widetilde{\lambda}_1)$$
$$\cdot \left\{ \sum_{|m|=1}^{\infty} \frac{1}{[|m-\xi| + (m-\xi)\cos\alpha]^{\varepsilon+1}} - \sum_{|m|=1}^{\infty} \frac{1}{O\left([|m-\xi| + (m-\xi)\cos\alpha]^{(\varepsilon/p+\lambda_2)+1}\right)} \right\}$$
$$= \frac{k_\beta(\widetilde{\lambda}_1)}{\varepsilon} \left\{ \left[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}} + \frac{1}{(1-\cos\alpha)^{1+\varepsilon}} \right] (1 + o_1(1)) - \varepsilon O(1) \right\}. \tag{35}$$

If there exists a constant $k \leq k(\lambda_1)$, such that (26) is valid when replacing $k(\lambda_1)$ by k, then, in particular, we have $\varepsilon\widetilde{I} < k\widetilde{I}_1$; namely,

$$k_\beta(\widetilde{\lambda}_1) \cdot \left\{ \left[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}} + \frac{1}{(1-\cos\alpha)^{1+\varepsilon}} \right] (1 + o_1(1)) - \varepsilon O(1) \right\} < k \left[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}} + \frac{1}{(1-\cos\alpha)^{1+\varepsilon}} \right]^{1/p} (1 + o_1(1))^{1/p} \cdot \left[\frac{1}{(1+\cos\beta)^{1+\varepsilon}} + \frac{1}{(1-\cos\beta)^{1+\varepsilon}} \right]^{1/q} (1 + o_2(1))^{1/q}. \tag{36}$$

It follows that

$$4B(\lambda_1, \lambda_2)\csc^2\beta\csc^2\alpha \leq 2k\csc^{2/p}\alpha\csc^{2/q}\beta \quad (\varepsilon \longrightarrow 0^+); \tag{37}$$

namely, $k(\lambda_1) = 2B(\lambda_1,\lambda_2)\csc^{2/p}\beta\csc^{2/q}\alpha \leq k$. Hence, $k = k(\lambda_1)$ is the best possible constant factor of (26). The constant factor $k(\lambda_1)$ in (27) is still the best possible one. Otherwise, we would reach a contradiction by (30) that the constant factor in (26) is not the best possible one. □

We set functions $\Phi(m)$ and $\Psi(n)$ as follows:

$$\Phi(m) := [|m-\xi| + (m-\xi)\cos\alpha]^{p(1-\lambda_1)-1} \quad (|m| \in \mathbf{N}),$$
$$\Psi(n) := [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\lambda_2)-1} \quad (|n| \in \mathbf{N}), \tag{38}$$

wherefrom $\Psi^{1-p}(n) = [|n-\eta| + (n-\eta)\cos\beta]^{p\lambda_2-1}$ $(|n| \in \mathbf{N})$. We also set the following weight normed spaces:

$$l_{p,\Phi} := \left\{ a = \{a_m\}_{|m|=1}^{\infty};\ \|a\|_{p,\Phi} = \left(\sum_{|m|=1}^{\infty} \Phi(m)|a_m|^p \right)^{1/p} < \infty \right\},$$
$$l_{q,\Psi} := \left\{ b = \{b_n\}_{|n|=1}^{\infty};\ \|b\|_{q,\Psi} = \left(\sum_{|n|=1}^{\infty} \Psi(n)|b_n|^q \right)^{1/q} < \infty \right\}, \tag{39}$$
$$l_{p,\Psi^{1-p}} := \left\{ c = \{c_n\}_{|n|=1}^{\infty};\ \|c\|_{p,\Psi^{1-p}} = \left(\sum_{|n|=1}^{\infty} \Psi^{1-p}(n)|c_n|^p \right)^{1/p} < \infty \right\}.$$

Then, for $a = \{a_m\}_{|m|=1}^{\infty} \in l_{p,\Phi}$, $c = \{c_n\}_{|n|=1}^{\infty}$, and $c_n = \sum_{|m|=1}^{\infty} k(m,n)a_m$, in view of (27), we have $\|c\|_{p,\Psi^{1-p}} < k(\lambda_1)\|a\|_{p,\Phi}$; namely, $c \in l_{p,\Psi^{1-p}}$.

Definition 7. Define a Hilbert-type operator $T : l_{p,\Phi} \to l_{p,\Psi^{1-p}}$ as follows: for any $a = \{a_m\}_{|m|=1}^{\infty} \in l_{p,\Phi}$, there exists a unique representation $c = Ta \in l_{p,\Psi^{1-p}}$. One also defines the formal inner product of Ta and $b = \{b_n\}_{|n|=1}^{\infty} \in l_{q,\Psi}$ $(b_n \geq 0)$ as follows:

$$(Ta, b) := \sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty} k(m,n)a_m b_n. \tag{40}$$

Then for $a_m \geq 0$ $(|m| \in \mathbf{N})$, we may rewrite (26) and (27) as follows:

$$(Ta, b) < k(\lambda_1)\|a\|_{p,\Phi}\|b\|_{q,\Psi}, \tag{41}$$
$$\|Ta\|_{p,\Psi^{1-p}} < k(\lambda_1)\|a\|_{p,\Phi}. \tag{42}$$

We define the norm of operator T as follows:

$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\Phi}} \frac{\|Ta\|_{p,\Psi^{1-p}}}{\|a\|_{p,\Phi}}. \tag{43}$$

Since, by Theorem 6, the constant factor $k(\lambda_1)$ in (42) is the best possible one, we have

$$\|T\| = k(\lambda_1) = 2B(\lambda_1, \lambda_2) \csc^{2/p}\beta \csc^{2/q}\alpha. \quad (44)$$

Remark 8. (i) For $\xi = \eta = 0$, (26) reduces to

$$\sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \frac{1}{(|m| + m\cos\alpha + |n| + n\cos\beta)^{\lambda}} a_m b_n < k(\lambda_1) \left[\sum_{|m|=1}^{\infty} (|m| + m\cos\alpha)^{p(1-\lambda_1)-1} a_m^p \right]^{1/p} \cdot \left[\sum_{|n|=1}^{\infty} (|n| + n\cos\beta)^{q(1-\lambda_2)-1} b_n^q \right]^{1/q}. \quad (45)$$

Hence, (26) is a more accurate inequality of (45). In particular, for $\alpha = \beta = \pi/2$ in (45), we have the following new inequality:

$$\sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \frac{a_m b_n}{(|m| + |n|)^{\lambda}} < 2B(\lambda_1, \lambda_2) \cdot \left[\sum_{|m|=1}^{\infty} |m|^{p(1-\lambda_1)-1} a_m^p \right]^{1/p} \cdot \left[\sum_{|n|=1}^{\infty} |n|^{q(1-\lambda_2)-1} b_n^q \right]^{1/q}. \quad (46)$$

(ii) For $\alpha = \beta = \pi/2$ in (26), we have (5). For $a_{-m} = a_m$ and $b_{-n} = b_n$ $(m, n \in \mathbf{N})$, (5) reduces to

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{1}{(m+n-\xi-\eta)^{\lambda}} + \frac{1}{(m+n+\xi-\eta)^{\lambda}} + \frac{1}{(m+n+\xi+\eta)^{\lambda}} + \frac{1}{(m+n-\xi+\eta)^{\lambda}} \right] a_m b_n < 2B(\lambda_1, \lambda_2) \cdot \left\{ \sum_{m=1}^{\infty} \left[(m-\xi)^{p(1-\lambda_1)-1} + (m+\xi)^{p(1-\lambda_1)-1} \right] a_m^p \right\}^{1/p} \cdot \left\{ \sum_{n=1}^{\infty} \left[(n-\eta)^{q(1-\lambda_2)-1} + (n+\eta)^{q(1-\lambda_2)-1} \right] b_n^q \right\}^{1/q}. \quad (47)$$

In particular, for $\xi = \eta \in [0, 1/2]$, we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{1}{(m+n-2\eta)^{\lambda}} + \frac{2}{(m+n)^{\lambda}} + \frac{1}{(m+n+2\eta)^{\lambda}} \right] a_m b_n < 2B(\lambda_1, \lambda_2) \cdot \left\{ \sum_{m=1}^{\infty} \left[(m-\eta)^{p(1-\lambda_1)-1} + (m+\eta)^{p(1-\lambda_1)-1} \right] \cdot a_m^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} \left[(n-\eta)^{q(1-\lambda_2)-1} + (n+\eta)^{q(1-\lambda_2)-1} \right] \cdot b_n^q \right\}^{1/q}. \quad (48)$$

If $\eta = 0$, then (48) reduces to

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1, \lambda_2) \left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p \right]^{1/p} \cdot \left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q \right]^{1/q}, \quad (49)$$

which is an extension of (1).

Competing Interests

The authors declare that they have no competing interests.

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