

Research Article

A New Extension of Hardy-Hilbert's Inequality in the Whole Plane

Bicheng Yang¹ and Qiang Chen²

¹Department of Mathematics, Guangdong University of Education, Guangzhou, Guangdong 510303, China

²Department of Computer Science, Guangdong University of Education, Guangzhou, Guangdong 510303, China

Correspondence should be addressed to Bicheng Yang; bcyang818@163.com

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By the use of weight coefficients and Hermite-Hadamard's inequality, a new extension of Hardy-Hilbert's inequality in the whole plane with multiparameters and a best possible constant factor is given. The equivalent forms, the operator expressions, and a few particular inequalities are considered.

1. Introduction

Suppose that $p > 1$, $1/p + 1/q = 1$, $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$, $\|a\|_p = (\sum_{m=1}^{\infty} a_m^p)^{1/p} > 0$, and $\|b\|_q > 0$. We have the following well known Hardy-Hilbert's inequality:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q, \quad (1)$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible one (cf. [1]). The more accurate form of (1) was given as follows (cf. [2], Theorem 323):

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n-2\alpha} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q \quad (2)$$

$$\left(0 \leq \alpha \leq \frac{1}{2} \right),$$

where the constant factor $\pi/\sin(\pi/p)$ is still the best possible one. For $\alpha = 0$, inequality (2) reduces to (1).

In 2011, Yang gave an extension of (2) as follows (cf. [3]): If $0 < \lambda_1, \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = \lambda$, $a_m, b_n \geq 0$, $\|a\|_{p,\varphi} = \{\sum_{m=1}^{\infty} (m-\alpha)^{p(1-\lambda_1)-1} a_m^p\}^{1/p} \in (0, \infty)$, and $\|b\|_{q,\psi} = \{\sum_{n=1}^{\infty} (n-\alpha)^{q(1-\lambda_2)-1} b_n^q\}^{1/q} \in (0, \infty)$, then

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n-2\alpha)^{\lambda}} < B(\lambda_1, \lambda_2) \|a\|_{p,\varphi} \|b\|_{q,\psi} \quad (3)$$

$$\left(0 \leq \alpha \leq \frac{1}{2} \right),$$

where the constant factor $B(\lambda_1, \lambda_2)$ is the best possible one and $B(u, v)$ is the beta function defined by (cf. [4])

$$B(u, v) := \int_0^{\infty} \frac{1}{(1+t)^{u+v}} t^{u-1} dt \quad (u, v > 0). \quad (4)$$

For $\lambda = 1$, $\lambda_1 = 1/q$, and $\lambda_2 = 1/p$, (3) reduces to (2). Some other results related to (1)–(3) are provided by [5–22].

In this paper, by the use of weight coefficients and Hermite-Hadamard's inequality, an extension of (3) in the

whole plane is given as follows: For $\xi, \eta \in [0, 1/2]$, $a_m, b_n \geq 0$, $\sum_{|m|=1}^{\infty} |m - \xi|^{p(1-\lambda_1)-1} a_m^p \in (0, \infty)$, and $\sum_{|n|=1}^{\infty} |n - \eta|^{q(1-\lambda_2)-1} b_n^q \in (0, \infty)$, we have

$$\begin{aligned} & \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \frac{1}{(|m - \xi| + |n - \eta|)^{\lambda}} a_m b_n < 2B(\lambda_1, \lambda_2) \\ & \cdot \left[\sum_{|m|=1}^{\infty} |m - \xi|^{p(1-\lambda_1)-1} a_m^p \right]^{1/p} \\ & \cdot \left[\sum_{|n|=1}^{\infty} |n - \eta|^{q(1-\lambda_2)-1} b_n^q \right]^{1/q}. \end{aligned} \quad (5)$$

Moreover, a generation of (5) with multiparameters and a best possible constant factor is proved. The equivalent forms, the operator expressions, and a few particular inequalities are also considered.

2. Some Lemmas

First, we make appointment that $\mathbf{N} = \{1, 2, \dots\}$, $p > 1$, $1/p + 1/q = 1$, $\alpha, \beta \in (0, \pi)$, $\xi, \eta \in [0, 1/2]$, $\lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 = \lambda$, and, for $|x|, |y| > 1/2$,

$$\begin{aligned} k(x, y) \\ := \frac{1}{[|x - \xi| + (x - \xi) \cos \alpha + |y - \eta| + (y - \eta) \cos \beta]^{\lambda}}. \end{aligned} \quad (6)$$

In particular, for $\alpha = \beta = \pi/2$, we indicate

$$h(x, y) := \frac{1}{(|x - \xi| + |y - \eta|)^{\lambda}}, \quad |x|, |y| > \frac{1}{2}. \quad (7)$$

Definition 1. Define the following weight coefficients:

$$\begin{aligned} \omega(\lambda_2, m) &:= \sum_{|n|=1}^{\infty} k(m, n) \frac{[|m - \xi| + (m - \xi) \cos \alpha]^{\lambda_1}}{[(n - \eta) + (n - \eta) \cos \beta]^{1-\lambda_2}}, \\ &\quad |m| \in \mathbf{N}, \end{aligned} \quad (8)$$

$$\begin{aligned} \omega(\lambda_1, n) \\ := \sum_{|m|=1}^{\infty} k(m, n) \frac{[(n - \eta) + (n - \eta) \cos \beta]^{\lambda_2}}{[|m - \xi| + (m - \xi) \cos \alpha]^{1-\lambda_1}}, \\ &\quad |n| \in \mathbf{N}, \end{aligned} \quad (9)$$

where $\sum_{|j|=1}^{\infty} \dots = \sum_{j=-1}^{-\infty} \dots + \sum_{j=1}^{\infty} \dots$ ($j = m, n$).

Lemma 2. If $\lambda_2 \leq 1$, then, for $k_{\beta}(\lambda_1) := 2B(\lambda_1, \lambda_2) \csc^2 \beta$, one has

$$\begin{aligned} k_{\beta}(\lambda_1) (1 - \theta(\lambda_2, m)) &< \omega(\lambda_2, m) < k_{\beta}(\lambda_1), \\ &\quad |m| \in \mathbf{N}, \end{aligned} \quad (10)$$

where

$$\begin{aligned} \theta(\lambda_2, m) &:= \frac{1}{B(\lambda_1, \lambda_2)} \\ &\cdot \int_0^{(1+\eta)(1+\cos \beta)/(|m - \xi| + (m - \xi) \cos \alpha)} \frac{u^{\lambda_2-1}}{(1+u)^{\lambda}} du \\ &= O\left(\frac{1}{[|m - \xi| + (m - \xi) \cos \alpha]^{\lambda_2}}\right) \in (0, 1), \\ &\quad |m| \in \mathbf{N}. \end{aligned} \quad (11)$$

Proof. For $|x| > 1/2$, we set

$$\begin{aligned} k^{(1)}(x, y) \\ := \frac{1}{[|x - \xi| + (x - \xi) \cos \alpha + (y - \eta) (\cos \beta - 1)]^{\lambda}}, \\ &\quad y < -\frac{1}{2}, \\ k^{(2)}(x, y) \\ := \frac{1}{[|x - \xi| + (x - \xi) \cos \alpha + (y - \eta) (1 + \cos \beta)]^{\lambda}}, \\ &\quad y > \frac{1}{2}, \end{aligned} \quad (12)$$

and then, for $y > 1/2$,

$$\begin{aligned} k^{(1)}(x, -y) \\ = \frac{1}{[|x - \xi| + (x - \xi) \cos \alpha + (y + \eta) (1 - \cos \beta)]^{\lambda}}. \end{aligned} \quad (13)$$

We find

$$\begin{aligned} \omega(\lambda_2, m) \\ = \sum_{n=-1}^{-\infty} k^{(1)}(m, n) \frac{[|m - \xi| + (m - \xi) \cos \alpha]^{\lambda_1}}{[(n - \eta) (\cos \beta - 1)]^{1-\lambda_2}} \\ + \sum_{n=1}^{\infty} k^{(2)}(m, n) \frac{[|m - \xi| + (m - \xi) \cos \alpha]^{\lambda_1}}{[(n - \eta) (1 + \cos \beta)]^{1-\lambda_2}} \\ = \frac{[|m - \xi| + (m - \xi) \cos \alpha]^{\lambda_1}}{(1 - \cos \beta)^{1-\lambda_2}} \sum_{n=1}^{\infty} \frac{k^{(1)}(m, -n)}{(n + \eta)^{1-\lambda_2}} \\ + \frac{[|m - \xi| + (m - \xi) \cos \alpha]^{\lambda_1}}{(1 + \cos \beta)^{1-\lambda_2}} \sum_{n=1}^{\infty} \frac{k^{(2)}(m, n)}{(n - \eta)^{1-\lambda_2}}. \end{aligned} \quad (14)$$

It is evident that, for fixed $m \in \mathbf{N}$, $\lambda_2 \leq 1$ ($\lambda > 0$), both $k^{(1)}(m, -y)/(y + \eta)^{1-\lambda_2}$ and $k^{(2)}(m, y)/(y - \eta)^{1-\lambda_2}$ are strictly

decreasing and strictly convex with respect to $y \in (1/2, \infty)$, satisfying

$$\begin{aligned} \frac{d}{dy} \frac{k^{(i)}(m, (-1)^i y)}{\left[y + (-1)^i \eta\right]^{1-\lambda_2}} &< 0, \\ \frac{d^2}{dy^2} \frac{k^{(i)}(m, (-1)^i y)}{\left[y + (-1)^i \eta\right]^{1-\lambda_2}} &> 0, \end{aligned} \quad (i = 1, 2). \quad (15)$$

By Hermite-Hadamard's inequality (cf. [23]), we find

$$\begin{aligned} \omega(\lambda_2, m) &< \frac{[|m - \xi| + (m - \xi) \cos \alpha]^{\lambda_1}}{(1 - \cos \beta)^{1-\lambda_2}} \int_{1/2}^{\infty} \frac{k^{(1)}(m, -y)}{(y + \eta)^{1-\lambda_2}} dy \\ &\quad + \frac{[|m - \xi| + (m - \xi) \cos \alpha]^{\lambda_1}}{(1 + \cos \beta)^{1-\lambda_2}} \int_{1/2}^{\infty} \frac{k^{(2)}(m, y)}{(y - \eta)^{1-\lambda_2}} dy \\ &\leq \frac{[|m - \xi| + (m - \xi) \cos \alpha]^{\lambda_1}}{(1 - \cos \beta)^{1-\lambda_2}} \int_{-\eta}^{\infty} \frac{k^{(1)}(m, -y)}{(y + \eta)^{1-\lambda_2}} dy \\ &\quad + \frac{[|m - \xi| + (m - \xi) \cos \alpha]^{\lambda_1}}{(1 + \cos \beta)^{1-\lambda_2}} \int_{\eta}^{\infty} \frac{k^{(2)}(m, y)}{(y - \eta)^{1-\lambda_2}} dy. \end{aligned} \quad (16)$$

Setting $u = ((y + \eta)(1 - \cos \beta)/(|m - \xi| + (m - \xi) \cos \alpha))((y - \eta)(1 + \cos \beta)/(|m - \xi| + (m - \xi) \cos \alpha))$ in the above first (second) integral, by simplification, we find

$$\begin{aligned} \omega(\lambda_2, m) &< \left(\frac{1}{1 - \cos \beta} + \frac{1}{1 + \cos \beta} \right) \int_0^{\infty} \frac{u^{\lambda_2-1}}{(1 + u)^{\lambda}} du \\ &= 2B(\lambda_1, \lambda_2) \csc^2 \beta = k_{\beta}(\lambda_1). \end{aligned} \quad (17)$$

By (14), since both $k^{(1)}(m, -y)/(y + \eta)^{1-\lambda_2}$ and $k^{(2)}(m, y)/(y - \eta)^{1-\lambda_2}$ are strictly decreasing, we still have

$$\begin{aligned} \omega(\lambda_2, m) &> \frac{[|m - \xi| + (m - \xi) \cos \alpha]^{\lambda_1}}{(1 - \cos \beta)^{1-\lambda_2}} \\ &\quad \cdot \int_1^{\infty} \frac{k^{(1)}(m, -y)}{(y + \eta)^{1-\lambda_2}} dy \\ &\quad + \frac{[|m - \xi| + (m - \xi) \cos \alpha]^{\lambda_1}}{(1 + \cos \beta)^{1-\lambda_2}} \int_1^{\infty} \frac{k^{(2)}(m, y)}{(y - \eta)^{1-\lambda_2}} dy \\ &\geq \frac{1}{1 - \cos \beta} \end{aligned}$$

$$\begin{aligned} &\cdot \int_{(1+\eta)(1+\cos\beta)/(|m-\xi|+(m-\xi)\cos\alpha)}^{\infty} \frac{u^{\lambda_2-1}}{(1+u)^{\lambda}} du \\ &+ \frac{1}{1 + \cos \beta} \\ &\cdot \int_{(1+\eta)(1+\cos\beta)/(|m-\xi|+(m-\xi)\cos\alpha)}^{\infty} \frac{u^{\lambda_2-1}}{(1+u)^{\lambda}} du \\ &= k_{\beta}(\lambda_2) (1 - \theta(\lambda_2, m)) > 0. \end{aligned} \quad (18)$$

We obtain

$$\begin{aligned} 0 < \theta(\lambda_2, m) &= \frac{1}{B(\lambda_1, \lambda_2)} \\ &\cdot \int_0^{(1+\eta)(1+\cos\beta)/(|m-\xi|+(m-\xi)\cos\alpha)} \frac{u^{\lambda_2-1}}{(1+u)^{\lambda}} du \\ &< \frac{1}{B(\lambda_1, \lambda_2)} \int_0^{(1+\eta)(1+\cos\beta)/(|m-\xi|+(m-\xi)\cos\alpha)} u^{\lambda_2-1} du \\ &= \frac{1}{\lambda_2 B(\lambda_1, \lambda_2)} \left[\frac{(1 + \eta)(1 + \cos \beta)}{|m - \xi| + (m - \xi) \cos \alpha} \right]^{\lambda_2}, \end{aligned} \quad (19)$$

and then we have (10) and (11). \square

In the same way, we still have the following.

Lemma 3. If $\lambda_1 \leq 1$, then, for $k_{\alpha}(\lambda_1) = 2B(\lambda_1, \lambda_2) \csc^2 \alpha$, one has

$$k_{\alpha}(\lambda_1) (1 - \vartheta(\lambda_1, n)) < \omega(\lambda_1, n) < k_{\alpha}(\lambda_1), \quad |n| \in \mathbb{N}, \quad (20)$$

where

$$\begin{aligned} \vartheta(\lambda_1, n) &:= \frac{1}{B(\lambda_1, \lambda_2)} \\ &\cdot \int_0^{(1+\xi)(1+\cos\alpha)/(|n-\eta|+(n-\eta)\cos\beta)} \frac{u^{\lambda_1-1}}{(1+u)^{\lambda}} du \\ &= O\left(\frac{1}{[|n - \eta| + (n - \eta) \cos \beta]^{\lambda_1}}\right) \in (0, 1), \end{aligned} \quad (21)$$

$$|n| \in \mathbb{N}.$$

Lemma 4. If $\zeta \in [0, 1/2]$ and $\theta \in (0, \pi)$, then, for $\rho > 0$,

$$\begin{aligned} H_{\rho}(\zeta, \theta) &:= \sum_{|n|=1}^{\infty} \frac{1}{[|n - \zeta| + (n - \zeta) \cos \theta]^{1+\rho}} \\ &= \frac{1 + o(1)}{\rho} \left[\frac{1}{(1 + \cos \theta)^{1+\rho}} + \frac{1}{(1 - \cos \theta)^{1+\rho}} \right] \quad (22) \\ &\quad (\rho \rightarrow 0^+). \end{aligned}$$

Proof. We have

$$\begin{aligned}
H_\rho(\zeta, \theta) &= \sum_{n=-1}^{-\infty} \frac{1}{[(n-\zeta)(\cos \theta - 1)]^{1+\rho}} \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{[(n-\zeta)(\cos \theta + 1)]^{1+\rho}} \\
&= \frac{1}{(1-\cos \theta)^{1+\rho}} \sum_{n=1}^{\infty} \frac{1}{(n+\zeta)^{1+\rho}} \\
&\quad + \frac{1}{(1+\cos \theta)^{1+\rho}} \sum_{n=1}^{\infty} \frac{1}{(n-\zeta)^{1+\rho}}.
\end{aligned} \tag{23}$$

For $a = 1/(1-\zeta)^{1+\rho}$, we find

$$\begin{aligned}
H_\rho(\zeta, \theta) &\leq \left[\frac{1}{(1-\cos \theta)^{1+\rho}} + \frac{1}{(1+\cos \theta)^{1+\rho}} \right] \left[a \right. \\
&\quad \left. + \sum_{n=2}^{\infty} \frac{1}{(n-\zeta)^{1+\rho}} \right] < \left[\frac{1}{(1-\cos \theta)^{1+\rho}} \right. \\
&\quad \left. + \frac{1}{(1+\cos \theta)^{1+\rho}} \right] \left[a + \int_1^\infty \frac{dy}{(y-\zeta)^{1+\rho}} \right] \\
&= \frac{1}{\rho} \left[\frac{1}{(1-\cos \theta)^{1+\rho}} + \frac{1}{(1+\cos \theta)^{1+\rho}} \right] \left\{ 1 \right. \\
&\quad \left. + \left[a\rho + \frac{1}{(1-\zeta)^\rho} - 1 \right] \right\}, \\
H_\rho(\zeta, \theta) &\geq \left[\frac{1}{(1-\cos \theta)^{1+\rho}} + \frac{1}{(1+\cos \theta)^{1+\rho}} \right] \\
&\quad \cdot \sum_{n=1}^{\infty} \frac{1}{(n+\zeta)^{1+\rho}} > \left[\frac{1}{(1-\cos \theta)^{1+\rho}} \right. \\
&\quad \left. + \frac{1}{(1+\cos \theta)^{1+\rho}} \right] \int_1^\infty \frac{dy}{(y+\zeta)^{1+\rho}} \\
&= \frac{1 + [(1+\zeta)^{-\rho} - 1]}{\rho} \left[\frac{1}{(1-\cos \theta)^{1+\rho}} \right. \\
&\quad \left. + \frac{1}{(1+\cos \theta)^{1+\rho}} \right].
\end{aligned} \tag{24}$$

Hence, we have (22). \square

3. Main Results and Operation Expressions

Theorem 5. If $\lambda_1, \lambda_2 \leq 1$, $a_m, b_n \geq 0$ ($|m|, |n| \in \mathbb{N}$),

$$\begin{aligned}
0 &< \sum_{|m|=1}^{\infty} [|m-\xi| + (m-\xi) \cos \alpha]^{p(1-\lambda_1)-1} a_m^p \\
&< \infty,
\end{aligned}$$

$$\begin{aligned}
0 &< \sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta) \cos \beta]^{q(1-\lambda_2)-1} b_n^q \\
&< \infty,
\end{aligned}$$

$$\begin{aligned}
k(\lambda_1) &:= k_\beta^{1/p}(\lambda_1) k_\alpha^{1/q}(\lambda_1) \\
&= 2B(\lambda_1, \lambda_2) \csc^{2/p} \beta \csc^{2/q} \alpha,
\end{aligned} \tag{25}$$

then one has the following equivalent inequalities:

$$\begin{aligned}
I &:= \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} k(m, n) a_m b_n < k(\lambda_1) \\
&\quad \cdot \left\{ \sum_{|m|=1}^{\infty} [|m-\xi| + (m-\xi) \cos \alpha]^{p(1-\lambda_1)-1} a_m^p \right\}^{1/p}
\end{aligned} \tag{26}$$

$$\begin{aligned}
J &:= \sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta) \cos \beta]^{p\lambda_2-1} \\
&\quad \cdot \left(\sum_{|m|=1}^{\infty} k(m, n) a_m \right)^p < k(\lambda_1) \\
&\quad \cdot \left\{ \sum_{|m|=1}^{\infty} [|m-\xi| + (m-\xi) \cos \alpha]^{p(1-\lambda_1)-1} a_m^p \right\}^{1/p}.
\end{aligned} \tag{27}$$

Proof. By Hölder's inequality (cf. [23]) and (9), we have

$$\begin{aligned}
\left(\sum_{|m|=1}^{\infty} k(m, n) a_m \right)^p &= \left\{ \sum_{|m|=1}^{\infty} k(m, n) \right. \\
&\quad \left. \cdot \frac{[|m-\xi| + (m-\xi) \cos \alpha]^{(1-\lambda_1)/q}}{[|n-\eta| + (n-\eta) \cos \beta]^{(1-\lambda_2)/p}} a_m^p \right. \\
&\quad \left. \cdot \frac{[|n-\eta| + (n-\eta) \cos \beta]^{(1-\lambda_2)/p}}{[|m-\xi| + (m-\xi) \cos \alpha]^{(1-\lambda_1)/q}} \right\}^p \\
&\leq \sum_{|m|=1}^{\infty} k(m, n) \\
&\quad \cdot \frac{[|m-\xi| + (m-\xi) \cos \alpha]^{(1-\lambda_1)p/q}}{[|n-\eta| + (n-\eta) \cos \beta]^{1-\lambda_2}} a_m^p \\
&\quad \cdot \left\{ \sum_{|m|=1}^{\infty} k(m, n) \frac{[|n-\eta| + (n-\eta) \cos \beta]^{(1-\lambda_2)q/p}}{[|m-\xi| + (m-\xi) \cos \alpha]^{1-\lambda_1}} \right\}^{p-1} \\
&= \frac{(\omega(\lambda_1, n))^{p-1}}{[|n-\eta| + (n-\eta) \cos \beta]^{p\lambda_2-1}} \sum_{|m|=1}^{\infty} k(m, n) \\
&\quad \cdot \frac{[|m-\xi| + (m-\xi) \cos \alpha]^{(1-\lambda_1)p/q}}{[|n-\eta| + (n-\eta) \cos \beta]^{1-\lambda_2}} a_m^p.
\end{aligned} \tag{28}$$

By (20), we have

$$\begin{aligned}
J &< k_{\alpha}^{1/q}(\lambda_1) \left\{ \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} k(m, n) \right. \\
&\quad \cdot \frac{[|m - \xi| + (m - \xi) \cos \alpha]^{(1-\lambda_1)p/q}}{[|n - \eta| + (n - \eta) \cos \beta]^{1-\lambda_2}} a_m^p \left. \right\}^{1/p} \\
&= k_{\alpha}^{1/q}(\lambda_1) \left\{ \sum_{|m|=1}^{\infty} \sum_{|n|=1}^{\infty} k(m, n) \right. \\
&\quad \cdot \frac{[|m - \xi| + (m - \xi) \cos \alpha]^{(1-\lambda_1)p/q}}{[|n - \eta| + (n - \eta) \cos \beta]^{1-\lambda_2}} a_m^p \left. \right\}^{1/p} \\
&= k_{\alpha}^{1/q}(\lambda_1) \left\{ \sum_{|m|=1}^{\infty} \omega(\lambda_2, m) [|m - \xi| + (m - \xi) \right. \\
&\quad \cdot \cos \alpha]^{p(1-\lambda_1)-1} a_m^p \left. \right\}^{1/p}. \tag{29}
\end{aligned}$$

By (10), we have (27).

By Hölder's inequality (cf. [23]), we have

$$\begin{aligned}
I &= \sum_{|n|=1}^{\infty} \left\{ [|n - \eta| + (n - \eta) \cos \beta]^{\lambda_2-1/p} \right. \\
&\quad \cdot \left. \sum_{|m|=1}^{\infty} k(m, n) a_m \right\} [|n - \eta| + (n - \eta) \cos \beta]^{1/p-\lambda_2} b_n \tag{30} \\
&\leq J \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + (n - \eta) \cos \beta]^{q(1-\lambda_2)-1} b_n^q \right\}^{1/q}.
\end{aligned}$$

Then by (27), we have (26).

On the other hand, assuming that (26) is valid, we set

$$\begin{aligned}
b_n &:= [|n - \eta| + (n - \eta) \cos \beta]^{p\lambda_2-1} \\
&\quad \cdot \left(\sum_{|m|=1}^{\infty} k(m, n) a_m \right)^{p-1}, \quad |n| \in \mathbb{N}. \tag{31}
\end{aligned}$$

Then it follows that

$$J = \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + (n - \eta) \cos \beta]^{q(1-\lambda_2)-1} b_n^q \right\}^{1/p}. \tag{32}$$

By (29), we find $J < \infty$. If $J = 0$, then (27) is evidently valid; if $J > 0$, then, by (26), we have

$$\begin{aligned}
0 &< \sum_{|n|=1}^{\infty} [|n - \eta| + (n - \eta) \cos \beta]^{q(1-\lambda_2)-1} b_n^q = J^p = I \\
&< k(\lambda_1) \cdot \left\{ \sum_{|m|=1}^{\infty} [|m - \xi| + (m - \xi) \cos \alpha]^{p(1-\lambda_1)-1} a_m^p \right\}^{1/p} \\
&\quad \cdot \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + (n - \eta) \cos \beta]^{q(1-\lambda_2)-1} b_n^q \right\}^{1/q}, \\
J &= \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + (n - \eta) \cos \beta]^{q(1-\lambda_2)-1} b_n^q \right\}^{1/p} \\
&< k(\lambda_1) \\
&\quad \cdot \left\{ \sum_{|m|=1}^{\infty} [|m - \xi| + (m - \xi) \cos \alpha]^{p(1-\lambda_1)-1} a_m^p \right\}^{1/p}; \tag{33}
\end{aligned}$$

namely, (27) follows, which is equivalent to (26). \square

Theorem 6. As regards the assumptions of Theorem 5, the constant factor $k(\lambda_1)$ in (26) and (27) is the best possible one.

Proof. For any $\varepsilon \in (0, q\lambda_2)$, we set $\tilde{\lambda}_1 = \lambda_1 + \varepsilon/q$, $\tilde{\lambda}_2 = \lambda_2 - \varepsilon/q$ ($\in (0, 1)$), and

$$\begin{aligned}
\tilde{a}_m &:= [|m - \xi| + (m - \xi) \cos \alpha]^{(\lambda_1-\varepsilon/p)-1} \\
&= [|m - \xi| + (m - \xi) \cos \alpha]^{\tilde{\lambda}_1-\varepsilon-1} \quad (|m| \in \mathbb{N}), \\
\tilde{b}_n &:= [|n - \eta| + (n - \eta) \cos \beta]^{(\lambda_2-\varepsilon/q)-1} \\
&= [|n - \eta| + (n - \eta) \cos \beta]^{\tilde{\lambda}_2-\varepsilon-1} \quad (|n| \in \mathbb{N}). \tag{34}
\end{aligned}$$

Then by (22) and (10), we find

$$\begin{aligned}
\tilde{I}_1 &:= \left\{ \sum_{|m|=1}^{\infty} [|m - \xi| + (m - \xi) \cos \alpha]^{p(1-\lambda_1)-1} \tilde{a}_m^p \right\}^{1/p} \\
&\quad \cdot \left\{ \sum_{|n|=1}^{\infty} [|n - \eta| + (n - \eta) \cos \beta]^{q(1-\lambda_2)-1} \tilde{b}_n^q \right\}^{1/q} \\
&= \left\{ \sum_{|m|=1}^{\infty} \frac{1}{[|m - \xi| + (m - \xi) \cos \alpha]^{1+\varepsilon}} \right\}^{1/p} \\
&\quad \cdot \left\{ \sum_{|n|=1}^{\infty} \frac{1}{[|n - \eta| + (n - \eta) \cos \beta]^{1+\varepsilon}} \right\}^{1/q} \\
&= \frac{1}{\varepsilon} \left[\frac{1}{(1 + \cos \alpha)^{1+\varepsilon}} + \frac{1}{(1 - \cos \alpha)^{1+\varepsilon}} \right]^{1/p} \\
&\quad \cdot \left[\frac{1}{(1 + \cos \beta)^{1+\varepsilon}} + \frac{1}{(1 - \cos \beta)^{1+\varepsilon}} \right]^{1/q} (1 + o_1(1))^{1/p} \\
&\quad \cdot (1 + o_2(1))^{1/q},
\end{aligned}$$

$$\begin{aligned}
\tilde{I} &:= \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} k(m, n) \tilde{a}_m \tilde{b}_n = \sum_{|m|=1}^{\infty} \sum_{|m|=1}^{\infty} k(m, n) \\
&\cdot \frac{[|m - \xi| + (m - \xi) \cos \alpha]^{\tilde{\lambda}_1 - \varepsilon - 1}}{[|n - \eta| + (n - \eta) \cos \beta]^{1 - \tilde{\lambda}_2}} \\
&= \sum_{|m|=1}^{\infty} \frac{\omega(\tilde{\lambda}_2, m)}{[|m - \xi| + (m - \xi) \cos \alpha]^{\varepsilon + 1}} \geq k_{\beta}(\tilde{\lambda}_1) \\
&\cdot \sum_{|m|=1}^{\infty} \frac{1 - \theta(\tilde{\lambda}_2, m)}{[|m - \xi| + (m - \xi) \cos \alpha]^{\varepsilon + 1}} = k_{\beta}(\tilde{\lambda}_1) \\
&\cdot \left\{ \sum_{|m|=1}^{\infty} \frac{1}{[|m - \xi| + (m - \xi) \cos \alpha]^{\varepsilon + 1}} \right. \\
&- \left. \sum_{|m|=1}^{\infty} \frac{1}{O([|m - \xi| + (m - \xi) \cos \alpha]^{(\varepsilon/p + \lambda_2) + 1})} \right\} \\
&= \frac{k_{\beta}(\tilde{\lambda}_1)}{\varepsilon} \left\{ \left[\frac{1}{(1 + \cos \alpha)^{1+\varepsilon}} + \frac{1}{(1 - \cos \alpha)^{1+\varepsilon}} \right] (1 \right. \\
&\left. + o_1(1)) - \varepsilon O(1) \right\}. \tag{35}
\end{aligned}$$

If there exists a constant $k \leq k(\lambda_1)$, such that (26) is valid when replacing $k(\lambda_1)$ by k , then, in particular, we have $\varepsilon \tilde{I} < k \tilde{I}_1$; namely,

$$\begin{aligned}
&k_{\beta}(\tilde{\lambda}_1) \\
&\cdot \left\{ \left[\frac{1}{(1 + \cos \alpha)^{1+\varepsilon}} + \frac{1}{(1 - \cos \alpha)^{1+\varepsilon}} \right] (1 + o_1(1)) \right. \\
&\left. - \varepsilon O(1) \right\} < k \left[\frac{1}{(1 + \cos \alpha)^{1+\varepsilon}} \right. \\
&\left. + \frac{1}{(1 - \cos \alpha)^{1+\varepsilon}} \right]^{1/p} (1 + o_1(1))^{1/p} \\
&\cdot \left[\frac{1}{(1 + \cos \beta)^{1+\varepsilon}} + \frac{1}{(1 - \cos \beta)^{1+\varepsilon}} \right]^{1/q} (1 \\
&+ o_2(1))^{1/q}. \tag{36}
\end{aligned}$$

It follows that

$$4B(\lambda_1, \lambda_2) \csc^2 \beta \csc^2 \alpha \leq 2k \csc^{2/p} \alpha \csc^{2/q} \beta \tag{37}$$

$(\varepsilon \rightarrow 0^+);$

namely, $k(\lambda_1) = 2B(\lambda_1, \lambda_2) \csc^{2/p} \beta \csc^{2/q} \alpha \leq k$. Hence, $k = k(\lambda_1)$ is the best possible constant factor of (26). The constant factor $k(\lambda_1)$ in (27) is still the best possible one. Otherwise, we would reach a contradiction by (30) that the constant factor in (26) is not the best possible one. \square

We set functions $\Phi(m)$ and $\Psi(n)$ as follows:

$$\begin{aligned}
\Phi(m) &:= [|m - \xi| + (m - \xi) \cos \alpha]^{p(1 - \lambda_1) - 1} \\
&\quad (|m| \in \mathbb{N}), \tag{38}
\end{aligned}$$

$$\begin{aligned}
\Psi(n) &:= [|n - \eta| + (n - \eta) \cos \beta]^{q(1 - \lambda_2) - 1} \\
&\quad (|n| \in \mathbb{N}),
\end{aligned}$$

wherefrom $\Psi^{1-p}(n) = [|n - \eta| + (n - \eta) \cos \beta]^{p\lambda_2 - 1}$ ($|n| \in \mathbb{N}$). We also set the following weight normed spaces:

$$\begin{aligned}
l_{p,\Phi} &:= \left\{ a = \{a_m\}_{|m|=1}^{\infty}; \|a\|_{p,\Phi} \right. \\
&= \left(\sum_{|m|=1}^{\infty} \Phi(m) |a_m|^p \right)^{1/p} < \infty \left. \right\}, \\
l_{q,\Psi} &:= \left\{ b = \{b_n\}_{|n|=1}^{\infty}; \|b\|_{q,\Psi} = \left(\sum_{|n|=1}^{\infty} \Psi(n) |b_n|^q \right)^{1/q} \right. \\
&< \infty \left. \right\}, \\
l_{p,\Psi^{1-p}} &:= \left\{ c = \{c_n\}_{|n|=1}^{\infty}; \|c\|_{p,\Psi^{1-p}} \right. \\
&= \left(\sum_{|n|=1}^{\infty} \Psi^{1-p}(n) |c_n|^p \right)^{1/p} < \infty \left. \right\}.
\end{aligned} \tag{39}$$

Then, for $a = \{a_m\}_{|m|=1}^{\infty} \in l_{p,\Phi}$, $c = \{c_n\}_{|n|=1}^{\infty}$, and $c_n = \sum_{|m|=1}^{\infty} k(m, n) a_m$, in view of (27), we have $\|c\|_{p,\Psi^{1-p}} < k(\lambda_1) \|a\|_{p,\Phi}$; namely, $c \in l_{p,\Psi^{1-p}}$.

Definition 7. Define a Hilbert-type operator $T : l_{p,\Phi} \rightarrow l_{p,\Psi^{1-p}}$ as follows: for any $a = \{a_m\}_{|m|=1}^{\infty} \in l_{p,\Phi}$, there exists a unique representation $c = Ta \in l_{p,\Psi^{1-p}}$. One also defines the formal inner product of Ta and $b = \{b_n\}_{|n|=1}^{\infty} \in l_{q,\Psi}$ ($b_n \geq 0$) as follows:

$$(Ta, b) := \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} k(m, n) a_m b_n. \tag{40}$$

Then for $a_m \geq 0$ ($|m| \in \mathbb{N}$), we may rewrite (26) and (27) as follows:

$$(Ta, b) < k(\lambda_1) \|a\|_{p,\Phi} \|b\|_{q,\Psi}, \tag{41}$$

$$\|Ta\|_{p,\Psi^{1-p}} < k(\lambda_1) \|a\|_{p,\Phi}. \tag{42}$$

We define the norm of operator T as follows:

$$\|T\| := \sup_{a(\neq 0) \in l_{p,\Phi}} \frac{\|Ta\|_{p,\Psi^{1-p}}}{\|a\|_{p,\Phi}}. \tag{43}$$

Since, by Theorem 6, the constant factor $k(\lambda_1)$ in (42) is the best possible one, we have

$$\|T\| = k(\lambda_1) = 2B(\lambda_1, \lambda_2) \csc^{2/p} \beta \csc^{2/q} \alpha. \quad (44)$$

Remark 8. (i) For $\xi = \eta = 0$, (26) reduces to

$$\begin{aligned} & \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \frac{1}{(|m| + m \cos \alpha + |n| + n \cos \beta)^{\lambda}} a_m b_n \\ & < k(\lambda_1) \left[\sum_{|m|=1}^{\infty} (|m| + m \cos \alpha)^{p(1-\lambda_1)-1} a_m^p \right]^{1/p} \quad (45) \\ & \cdot \left[\sum_{|n|=1}^{\infty} (|n| + n \cos \beta)^{q(1-\lambda_2)-1} b_n^q \right]^{1/q}. \end{aligned}$$

Hence, (26) is a more accurate inequality of (45). In particular, for $\alpha = \beta = \pi/2$ in (45), we have the following new inequality:

$$\begin{aligned} & \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \frac{a_m b_n}{(|m| + |n|)^{\lambda}} < 2B(\lambda_1, \lambda_2) \\ & \cdot \left[\sum_{|m|=1}^{\infty} |m|^{p(1-\lambda_1)-1} a_m^p \right]^{1/p} \quad (46) \\ & \cdot \left[\sum_{|n|=1}^{\infty} |n|^{q(1-\lambda_2)-1} b_n^q \right]^{1/q}. \end{aligned}$$

(ii) For $\alpha = \beta = \pi/2$ in (26), we have (5). For $a_{-m} = a_m$ and $b_{-n} = b_n$ ($m, n \in \mathbb{N}$), (5) reduces to

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{1}{(m+n-\xi-\eta)^{\lambda}} + \frac{1}{(m+n+\xi-\eta)^{\lambda}} \right. \\ & \left. + \frac{1}{(m+n+\xi+\eta)^{\lambda}} + \frac{1}{(m+n-\xi+\eta)^{\lambda}} \right] a_m b_n \\ & < 2B(\lambda_1, \lambda_2) \quad (47) \\ & \cdot \left\{ \sum_{m=1}^{\infty} [(m-\xi)^{p(1-\lambda_1)-1} + (m+\xi)^{p(1-\lambda_1)-1}] a_m^p \right\}^{1/p} \\ & \cdot \left\{ \sum_{n=1}^{\infty} [(n-\eta)^{q(1-\lambda_2)-1} + (n+\eta)^{q(1-\lambda_2)-1}] b_n^q \right\}^{1/q}. \end{aligned}$$

In particular, for $\xi = \eta \in [0, 1/2]$, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{1}{(m+n-2\eta)^{\lambda}} + \frac{2}{(m+n)^{\lambda}} \right. \\ & \left. + \frac{1}{(m+n+2\eta)^{\lambda}} \right] a_m b_n < 2B(\lambda_1, \lambda_2) \\ & \cdot \left\{ \sum_{m=1}^{\infty} [(m-\eta)^{p(1-\lambda_1)-1} + (m+\eta)^{p(1-\lambda_1)-1}] \right. \\ & \left. \cdot a_m^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} [(n-\eta)^{q(1-\lambda_2)-1} + (n+\eta)^{q(1-\lambda_2)-1}] \right. \\ & \left. \cdot b_n^q \right\}^{1/q}. \end{aligned} \quad (48)$$

If $\eta = 0$, then (48) reduces to

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1, \lambda_2) \left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p \right]^{1/p} \\ & \cdot \left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q \right]^{1/q}, \end{aligned} \quad (49)$$

which is an extension of (1).

Competing Interests

The authors declare that they have no competing interests.

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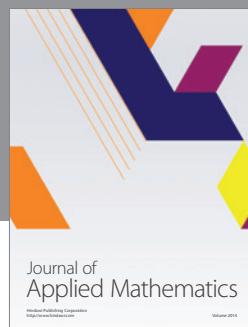
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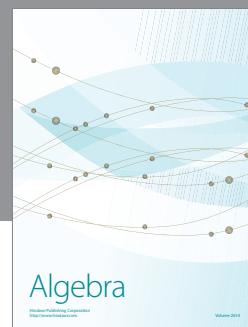
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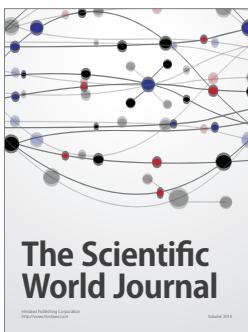
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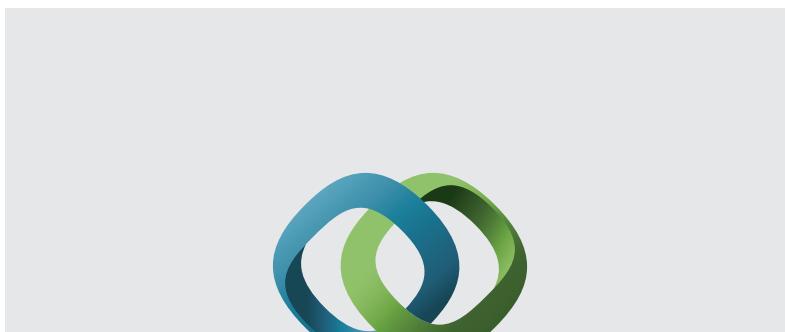
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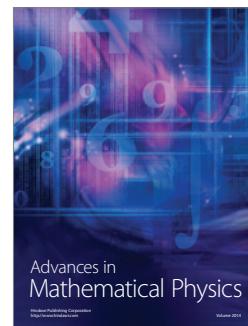


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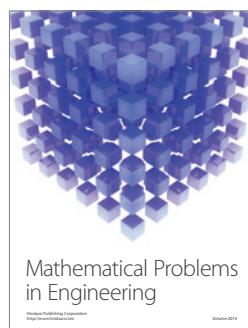
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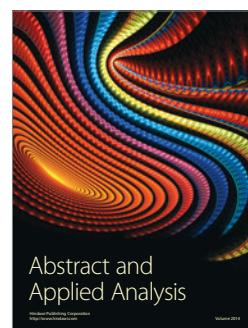
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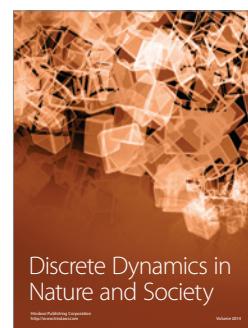
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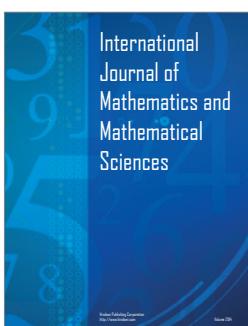
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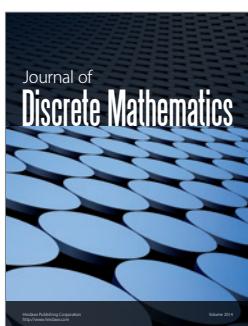
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