Research Article

Integral Transforms on a Function Space with Change of Scales Using Multivariate Normal Distributions

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Using simple formulas for generalized conditional Wiener integrals on a function space which is an analogue of Wiener space, we evaluate two generalized analytic conditional Wiener integrals of a generalized cylinder function which is useful in Feynman integration theories and quantum mechanics. We then establish various integral transforms over continuous paths with change of scales for the generalized analytic conditional Wiener integrals. In these evaluation formulas and integral transforms we use multivariate normal distributions so that the orthonormalization process of projection vectors which are needed to establish the conditional Wiener integrals can be removed in the existing change of scale transforms. Consequently the transforms in the present paper can be expressed in terms of the generalized cylinder function itself.

1. Introduction

Let $C_0[0, T]$ denote the classical Wiener space, the space of continuous real-valued functions x on $[0, T]$ with $x(0) = 0$. It is well known that Wiener measure and Wiener measurability are not invariant under change of scale and under translation [1, 2]. As integral transforms, change of scale formulas for Wiener integrals of various functions was developed on the classical and abstract Wiener spaces [3–7]. Further change of scale formulas for conditional Wiener integrals was introduced by the author and his coauthors [8–10]. In fact change of scale formulas for conditional Wiener integrals was established on $C_0[0, T]$, on the infinite dimensional Wiener space, and on $C[0, T]$, an analogue of Wiener space [11] which is the space of real-valued continuous paths on $[0, T]$. Some difficulties in studying the transforms for the conditional Wiener integrals of cylinder functions which play important roles in Feynman integration theories are that they cannot be expressed in terms of the original cylinder functions. To avoid these difficulties, modified cylinder functions expressed by a polygonal function with projection vectors satisfying orthonormality were used to derive the change of scale transforms [8–10]. In this paper we use multivariate normal distributions so that the orthonormalization process

of projection vectors which are needed to establish the conditional Wiener integrals can be removed in the existing change of scale transforms. Consequently the transforms in the present paper can be expressed by the cylinder function itself and generalize the results in [8–10].

Let $a \in C[0, T]$ and let h be of bounded variation with $h \neq 0$ a.e. on $[0, T]$. Define a stochastic process $Z : C[0, T] \times [0, T] \rightarrow \mathbb{R}$ by

$$Z(x, t) = \int_0^t h(s) dx(s) + x(0) + a(t) \quad (1)$$

for $x \in C[0, T]$ and $t \in [0, T]$, where the integral denotes a generalized Paley-Wiener-Zygmund stochastic integral. For a partition $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = T$ of $[0, T]$ define random vectors $Z_n : C[0, T] \rightarrow \mathbb{R}^{n+1}$ and $Z_{n+1} : C[0, T] \rightarrow \mathbb{R}^{n+2}$ by

$$\begin{aligned} Z_n(x) &= (Z(x, t_0), Z(x, t_1), \dots, Z(x, t_n)), \\ Z_{n+1}(x) &= (Z(x, t_0), Z(x, t_1), \dots, Z(x, t_n), Z(x, t_{n+1})). \end{aligned} \quad (2)$$

Using simple formulas for generalized conditional Wiener integrals on $C[0, T]$ with the conditioning functions Z_n and

Z_{n+1} [12] we evaluate generalized analytic conditional Wiener integrals of the following generalized cylinder function:

$$F_Z(x) = f\left(\int_0^T v_1(s) dZ(x, s), \dots, \int_0^T v_r(s) dZ(x, s)\right), \quad (3)$$

where $f \in L_p(\mathbb{R}^r)$ with $1 \leq p \leq \infty$ and $\{v_1, \dots, v_r\}$ is an orthonormal subset of $L_2[0, T]$. We then establish various change of scale transforms for the generalized analytic conditional Wiener integrals of F_Z with Z_n and Z_{n+1} . In these evaluation formulas and scale transforms we use multivariate normal distributions so that Gram-Schmidt orthonormalization process of $\{\mathcal{P}^\perp h v_1, \dots, \mathcal{P}^\perp h v_r\}$ can be removed in the existing change of scale transforms for a suitable orthogonal projection $\mathcal{P}^\perp$ on $L_2[0, T]$. In contrast with the existing change of scale transforms in [8–10], the transforms in this paper are expressed in terms of the cylinder function F_Z itself and generalize some results in those references.

2. An Analogue of Wiener Space and Preliminary Results

In this section we will introduce the analogue of Wiener space $C[0, T]$ and the preliminary results which are needed in the following sections.

For a positive real T let $C[0, T]$ denote the space of real-valued continuous functions on the time interval $[0, T]$ with the supremum norm. For $\vec{t} = (t_0, t_1, \dots, t_n)$ with $0 = t_0 < t_1 < \dots < t_n \leq T$ let $J_{\vec{t}} : C[0, T] \rightarrow \mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}}(x) = (x(t_0), x(t_1), \dots, x(t_n)). \quad (4)$$

For B_j ($j = 0, 1, \dots, n$) in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0, T]$ is called an interval and let $\mathcal{I}$ be the set of all such intervals. For a probability measure φ on $\mathcal{B}(\mathbb{R})$ let

$$m_\varphi \left[J_{\vec{t}}^{-1} \left(\prod_{j=0}^n B_j \right) \right] = \left[\prod_{j=1}^n \frac{1}{2\pi(t_j - t_{j-1})} \right]^{1/2} \cdot \int_{B_0} \int_{\prod_{j=1}^n B_j} W_n(\vec{t}, \vec{u}, u_0) d\vec{u} d\varphi(u_0), \quad (5)$$

where for $\vec{t} = (t_0, t_1, \dots, t_n)$ and $\vec{u} = (u_1, \dots, u_n)$

$$W_n(\vec{t}, \vec{u}, u_0) = \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}} \right\}. \quad (6)$$

$\mathcal{B}(C[0, T])$, the Borel σ -algebra of $C[0, T]$, coincides with the smallest σ -algebra generated by $\mathcal{I}$ and there exists a unique probability measure w_φ on $C[0, T]$ such that $w_\varphi(I) = m_\varphi(I)$ for all $I \in \mathcal{I}$. This measure w_φ is called an analogue of Wiener measure associated with the probability measure φ [11]. Let $\{e_k : k = 1, 2, \dots\}$ be a complete orthonormal subset of $L_2[0, T]$ such that each e_k is of bounded variation. For $v \in L_2[0, T]$ and x in $C[0, T]$ let

$$(v, x) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_0^T \langle v, e_k \rangle e_k(t) dx(t) \quad (7)$$

if the limit exists, where $\langle \cdot, \cdot \rangle$ denotes the inner product over $L_2[0, T]$. (v, x) is called the Paley-Wiener-Zygmund integral of v associated with x .

Let $\mathbb{C}$ and $\mathbb{C}_+$ denote the sets of complex numbers and complex numbers with positive real parts, respectively. Let $F : C[0, T] \rightarrow \mathbb{C}$ be integrable and let X be a random vector on $C[0, T]$ assuming that the value space of X is a normed space with the Borel σ -algebra. Then we have the conditional expectation $E[F | X]$ of F given X from a well-known probability theory. Furthermore there exists a P_X -integrable complex valued function ψ on the value space of X such that $E[F | X](x) = (\psi \circ X)(x)$ for w_φ a.e. $x \in C[0, T]$, where P_X is the probability distribution of X . The function ψ is called the conditional w_φ -integral of F given X and it is also denoted by $E[F | X]$.

Let $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = T$ be a partition of $[0, T]$, where n is a fixed nonnegative integer. Let $h \in L_2[0, T]$ be of bounded variation with $h \neq 0$ a.e. on $[0, T]$. For $j = 1, \dots, n, n+1$ let

$$\alpha_j = \frac{1}{\|\chi_{(t_{j-1}, t_j]}\|} \chi_{(t_{j-1}, t_j]} h, \quad (8)$$

let V be the subspace of $L_2[0, T]$ generated by $\{\alpha_1, \dots, \alpha_n, \alpha_{n+1}\}$, and let $V^\perp$ be the orthogonal complement of V . Let $\mathcal{P} : L_2[0, T] \rightarrow V$ be the orthogonal projection given by $\mathcal{P}v = \sum_{j=1}^{n+1} \langle v, \alpha_j \rangle \alpha_j$ and let $\mathcal{P}^\perp : L_2[0, T] \rightarrow V^\perp$ be an orthogonal projection. Let $a \in C[0, T]$ and let it be absolutely continuous. Define stochastic processes $X, Z : C[0, T] \times [0, T] \rightarrow \mathbb{R}$ by

$$\begin{aligned} X(x, t) &= (h\chi_{[0, t]}), \\ Z(x, t) &= (h\chi_{[0, t]}, x) + x(0) + a(t) \end{aligned} \quad (9)$$

for $x \in C[0, T]$ and $t \in [0, T]$. Define random vectors $Z_n : C[0, T] \rightarrow \mathbb{R}^{n+1}$ and $Z_{n+1} : C[0, T] \rightarrow \mathbb{R}^{n+2}$ by

$$\begin{aligned} Z_n(x) &= (Z(x, t_0), Z(x, t_1), \dots, Z(x, t_n)), \\ Z_{n+1}(x) &= (Z(x, t_0), Z(x, t_1), \dots, Z(x, t_n), Z(x, t_{n+1})) \end{aligned} \quad (10)$$

for $x \in C[0, T]$. For $t \in [0, T]$ let $b(t) = \int_0^t [h(s)]^2 ds$ and for any function f on $[0, T]$ define the polygonal function $P_{b, n+1}(f)$ of f by

$$\begin{aligned} P_{b, n+1}(f)(t) &= \sum_{j=1}^{n+1} \left[\frac{b(t_j) - b(t)}{b(t_j) - b(t_{j-1})} f(t_{j-1}) \right. \\ &\quad \left. + \frac{b(t) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} f(t_j) \right] \chi_{(t_{j-1}, t_j]}(t) + f(0) \chi_{\{0\}}(t) \end{aligned} \quad (11)$$

for $t \in [0, T]$, where $\chi_{(t_{j-1}, t_j]}$ and $\chi_{\{0\}}$ denote the indicator functions on the interval $[0, T]$. For $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2}$ define the polygonal function $P_{b, n+1}(\vec{\xi}_{n+1})$ of $\vec{\xi}_{n+1}$ by (11) with $f(t_j)$ replaced by ξ_j for $j = 0, 1, \dots, n, n+1$.

If $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, $P_{b,n}(\vec{\xi}_n)$ is interpreted as $P_{b,n+1}(\vec{\xi}_{n+1})\chi_{[0,t_n]}$ on $[0, T]$. For $x \in C[0, T]$ and $t \in [0, T]$ let

$$A(t) = a(t) - P_{b,n+1}(a)(t), \quad (12)$$

$$X_{b,n+1}(x, t) = X(x, t) - P_{b,n+1}(X(x, \cdot))(t), \quad (13)$$

$$Z_{b,n+1}(x, t) = Z(x, t) - P_{b,n+1}(Z(x, \cdot))(t). \quad (14)$$

For $\vec{a}, \vec{u} \in \mathbb{R}^r$, $\lambda \in \mathbb{C}$, and any nonsingular positive $r \times r$ matrix A_r on $\mathbb{R}$ let

$$\begin{aligned} \Psi_r(\lambda, \vec{a}, A_r, \vec{u}) &= \left[\frac{\lambda^r}{(2\pi)^r |A_r|} \right]^{1/2} \\ &\cdot \exp \left\{ -\frac{\lambda}{2} \langle A_r^{-1}(\vec{u} - \vec{a}), \vec{u} - \vec{a} \rangle_{\mathbb{R}} \right\}, \end{aligned} \quad (15)$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ denotes the dot product on $\mathbb{R}^r$. For a function $F : C[0, T] \rightarrow \mathbb{C}$ let $F_Z(x) = F(Z(x, \cdot))$ for $x \in C[0, T]$. By Theorems 6 and 7 in [12], we have the following theorems.

Theorem 1. Let F be a complex valued function on $C[0, T]$ and let F_Z be integrable over $C[0, T]$. Then for $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$\begin{aligned} E[F_Z | Z_{n+1}](\vec{\xi}_{n+1}) \\ = E[F(Z_{b,n+1}(x, \cdot) + P_{b,n+1}(\vec{\xi}_{n+1}))], \end{aligned} \quad (16)$$

where $Z_{b,n+1}$ is given by (14), $P_{Z_{n+1}}$ is the probability distribution of Z_{n+1} on $(\mathbb{R}^{n+2}, \mathcal{B}(\mathbb{R}^{n+2}))$, and the expectation on the right hand side of the equation is taken over the variable x .

Theorem 2. Let F_Z be integrable over $C[0, T]$ and let P_{Z_n} be the probability distribution of Z_n on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$. Then for P_{Z_n} a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} E[F_Z | Z_n](\vec{\xi}_n) &= \int_{\mathbb{R}} \Psi_1(1, a(T) - a(t_n), b(T) \\ &\quad - b(t_n), \xi_{n+1} - \xi_n) \\ &\quad \cdot E[F(Z_{b,n+1}(x, \cdot) + P_{b,n+1}(\vec{\xi}_{n+1}))] d\xi_{n+1}, \end{aligned} \quad (17)$$

where $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$ and Ψ_1 is given by (15) with $r = 1$.

For $\lambda > 0$ and $x \in C[0, T]$ let $F_Z^\lambda(x) = F_Z(\lambda^{-1/2}x)$, $Z_n^\lambda(x) = Z_n(\lambda^{-1/2}x)$, and $Z_{n+1}^\lambda(x) = Z_{n+1}(\lambda^{-1/2}x)$. Suppose that $E[F_Z^\lambda]$ exists. By Theorem 1

$$\begin{aligned} E[F_Z^\lambda | Z_{n+1}^\lambda](\vec{\xi}_{n+1}) \\ = E[F(\lambda^{-1/2}X_{b,n+1}(x, \cdot) + A + P_{b,n+1}(\vec{\xi}_{n+1}))] \end{aligned} \quad (18)$$

for $P_{Z_{n+1}^\lambda}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, where A and $X_{b,n+1}$ are given by (12) and (13), respectively, and $P_{Z_{n+1}^\lambda}$ is the probability distribution

of Z_{n+1}^λ on $(\mathbb{R}^{n+2}, \mathcal{B}(\mathbb{R}^{n+2}))$. By Theorem 2 we also have for $P_{Z_n^\lambda}$ a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} E[F_Z^\lambda | Z_n^\lambda](\vec{\xi}_n) &= \int_{\mathbb{R}} \Psi_1(\lambda, a(T) - a(t_n), b(T) \\ &\quad - b(t_n), \xi_{n+1} - \xi_n) E[F(\lambda^{-1/2}X_{b,n+1}(x, \cdot) + A \\ &\quad + P_{b,n+1}(\vec{\xi}_{n+1}))] d\xi_{n+1}, \end{aligned} \quad (19)$$

where $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$ and $P_{Z_n^\lambda}$ is the probability distribution of Z_n^λ on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$.

Let $I_{F_Z}^\lambda(\vec{\xi}_{n+1})$ and $K_{F_Z}^\lambda(\vec{\xi}_n)$ be the right hand sides of (18) and (19), respectively. If $I_{F_Z}^\lambda(\vec{\xi}_{n+1})$ has an analytic extension $J_\lambda^*(F_Z)(\vec{\xi}_{n+1})$ on $\mathbb{C}_+$, then it is called the conditional analytic Wiener ω_φ -integral of F_Z given Z_{n+1} with the parameter λ and is denoted by

$$E^{amw_\lambda}[F_Z | Z_{n+1}](\vec{\xi}_{n+1}) = J_\lambda^*(F_Z)(\vec{\xi}_{n+1}) \quad (20)$$

for $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$. Moreover if, for nonzero real q , $E^{amw_\lambda}[F_Z | Z_{n+1}](\vec{\xi}_{n+1})$ has a limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman ω_φ -integral of F_Z given Z_{n+1} with the parameter q and is denoted by

$$\begin{aligned} E^{anf_q}[F_Z | Z_{n+1}](\vec{\xi}_{n+1}) \\ = \lim_{\lambda \rightarrow -iq} E^{amw_\lambda}[F_Z | Z_{n+1}](\vec{\xi}_{n+1}). \end{aligned} \quad (21)$$

Replacing $I_{F_Z}^\lambda(\vec{\xi}_{n+1})$ by $K_{F_Z}^\lambda(\vec{\xi}_n)$, we define $E^{amw_\lambda}[F_Z | Z_n](\vec{\xi}_n)$ and $E^{anf_q}[F_Z | Z_n](\vec{\xi}_n)$. If $E[F_Z^\lambda]$ exists for $\lambda > 0$ and it has an analytic extension $J_\lambda^*(F_Z)$ on $\mathbb{C}_+$, then we call $J_\lambda^*(F_Z)$ the analytic Wiener ω_φ -integral of F over $C[0, T]$ with parameter λ and it is denoted by

$$E^{amw_\lambda}[F_Z] = J_\lambda^*(F_Z). \quad (22)$$

$E^{anf_q}[F_Z]$ is also defined by

$$E^{anf_q}[F_Z] = \lim_{\lambda \rightarrow -iq} E^{amw_\lambda}[F_Z], \quad (23)$$

if it exists, where the limit is taken through $\mathbb{C}_+$.

Applying Theorem 2.3 in [13], we can easily prove the following theorem.

Theorem 3. Let $\{h_1, h_2, \dots, h_r\}$ be an orthonormal system of $L_2[0, T]$. Then $(h_1, \cdot), \dots, (h_r, \cdot)$ are independent and each $(h_i, \cdot)$ has the standard normal distribution. Moreover if $f : \mathbb{R}^{r+1} \rightarrow \mathbb{R}$ is Borel measurable, then

$$\begin{aligned} \int_{C[0,T]} f(x(0), (h_1, x), \dots, (h_r, x)) d\omega_\varphi(x) \\ \doteq \int_{\mathbb{R}} \int_{\mathbb{R}^r} f(u_0, \vec{u}) \Psi_r(1, \vec{0}, I_r, \vec{u}) d\vec{u} d\varphi(u_0), \end{aligned} \quad (24)$$

where I_r is the identity matrix on $\mathbb{R}^r$ and $\doteq$ means that if either side exists then both sides exist and they are equal.

Since φ is a probability measure on $\mathbb{R}$ we have the following corollary.

Corollary 4. *Under the assumptions as given in Theorem 3*

$$\begin{aligned} & \int_{C[0,T]} f((h_1, x), \dots, (h_r, x)) dw_\varphi(x) \\ & \stackrel{*}{=} \int_{\mathbb{R}^r} f(\vec{u}) \Psi_r(1, \vec{0}, I_r, \vec{u}) d\vec{u} \end{aligned} \quad (25)$$

if $f: \mathbb{R}^r \rightarrow \mathbb{R}$ is Borel measurable.

The following lemmas are useful to prove the results in the next sections and their proofs are simple.

Lemma 5. *Let $v \in L_2[0, T]$. Then for w_φ a.e. $x \in C[0, T]$*

$$(v, P_{b,n+1}(X(x, \cdot))) = (\mathcal{P}M_h v, x), \quad (26)$$

where $M_h: L_2[0, T] \rightarrow L_2[0, T]$ is the multiplication operator defined by

$$M_h u = hu, \quad \text{for } u \in L_2[0, T]. \quad (27)$$

Lemma 6. *Let $v \in L_2[0, T]$, $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2}$, and*

$$(v, P_{b,n}(\vec{\xi}_n)) = \sum_{j=1}^n \langle v\alpha_j, \alpha_j \rangle (\xi_j - \xi_{j-1}), \quad (28)$$

where $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n)$. Then

$$\begin{aligned} (v, P_{b,n+1}(\vec{\xi}_{n+1})) &= \sum_{j=1}^{n+1} \langle v\alpha_j, \alpha_j \rangle (\xi_j - \xi_{j-1}) \\ &= (v, P_{b,n}(\vec{\xi}_n)) \\ &\quad + \langle v\alpha_{n+1}, \alpha_{n+1} \rangle (\xi_{n+1} - \xi_n). \end{aligned} \quad (29)$$

Remark 7. (1) The multiplication operator M_h in Lemma 5 is well defined because h is of bounded variation which implies the boundedness of h . M_h will denote the operator as given in the lemma unless otherwise specified.

(2) For $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$ it is possible that $P_{b,n}(\vec{\xi}_n) \notin C[0, T]$ if $\xi_n \neq 0$. In this case the symbol $(v, P_{b,n}(\vec{\xi}_n))$ does not mean the Paley-Wiener-Zygmund integral of $v \in L_2[0, T]$. It is only a formal expression for $\sum_{j=1}^n \langle v\alpha_j, \alpha_j \rangle (\xi_j - \xi_{j-1})$ which is as given in Lemma 6.

3. Multivariate Normal Distributions

In this section we derive a multivariate normal distribution which will be needed in the next section.

Lemma 8. *Let $\{v_1, \dots, v_r\}$ be a set of independent vectors in $L_2[0, T]$. Then the covariance matrix $\Sigma = [a_{ij}]_{r \times r}$ of the random variables $(v_l, \cdot)$, $l = 1, \dots, r$, exists and is positive definite. Moreover a_{ij} is given by*

$$a_{ij} = \langle v_i, v_j \rangle \quad (30)$$

and the determinant $|\Sigma|$ of Σ is positive so that Σ is nonsingular and the inverse matrix Σ^{-1} of Σ is also positive definite.

Proof. By Theorem 3

$$\begin{aligned} & \|v_i\|^2 + 2 \int_{C[0,T]} (v_i, x)(v_j, x) dw_\varphi(x) + \|v_j\|^2 \\ &= \int_{C[0,T]} (v_i + v_j, x)^2 dw_\varphi(x) \\ &= \|v_i\|^2 + 2 \langle v_i, v_j \rangle + \|v_j\|^2 \end{aligned} \quad (31)$$

so that the covariance a_{ij} of $(v_i, \cdot)$ and $(v_j, \cdot)$ is given by $a_{ij} = \langle v_i, v_j \rangle$ which proves (30). We have for $\vec{c} = (c_1, \dots, c_r) \in \mathbb{R}^r$

$$\langle \Sigma \vec{c}, \vec{c} \rangle_{\mathbb{R}} = \sum_{l=1}^r \sum_{j=1}^r a_{lj} c_l c_j = \left\| \sum_{j=1}^r c_j v_j \right\|^2 \geq 0. \quad (32)$$

Moreover if $\langle \Sigma \vec{c}, \vec{c} \rangle_{\mathbb{R}} = 0$, then $\sum_{j=1}^r c_j v_j = 0$ which implies $c_j = 0$ for $j = 1, \dots, r$ by the assumption. Thus the covariance matrix Σ is positive definite. Since Σ is symmetric and positive definite, the eigenvalues $\lambda_1, \dots, \lambda_r$ of Σ are real and positive. Since $|\Sigma| = \prod_{j=1}^r \lambda_j > 0$, Σ is invertible. Since

$$\langle \Sigma^{-1} \vec{c}, \vec{c} \rangle_{\mathbb{R}} = \langle \vec{c}, \Sigma^{-1} \vec{c} \rangle_{\mathbb{R}} = \langle \Sigma \Sigma^{-1} \vec{c}, \Sigma^{-1} \vec{c} \rangle_{\mathbb{R}} \geq 0, \quad (33)$$

$\langle \Sigma^{-1} \vec{c}, \vec{c} \rangle_{\mathbb{R}} = 0$ implies $\Sigma^{-1} \vec{c} = \vec{0}$; that is, $\vec{c} = \vec{0}$ and Σ^{-1} is positive definite. $\square$

For simplicity let

$$(\vec{v}, x) = ((v_1, x), \dots, (v_r, x)) \quad (34)$$

for $x \in C[0, T]$ and $\{v_1, \dots, v_r\} \subseteq L_2[0, T]$.

By Lemma 8, Theorem 4 of [14], and the change of variable theorem, we have the following theorem.

Theorem 9. *Let the assumptions and notations be as given in Lemma 8. Then for every Borel measurable function $f: \mathbb{R}^{r+1} \rightarrow \mathbb{C}$*

$$\begin{aligned} & \int_{C[0,T]} f(x(0), (\vec{v}, x)) dw_\varphi(x) \\ & \stackrel{*}{=} \int_{\mathbb{R}} \int_{\mathbb{R}^r} f(u_0, \vec{u}) \Psi_r(1, \vec{0}, \Sigma, \vec{u}) d\vec{u} d\varphi(u_0) \\ & \stackrel{*}{=} \int_{\mathbb{R}} \int_{\mathbb{R}^r} f(u_0, \Sigma^{1/2} \vec{u}) \Psi_r(1, \vec{0}, I_r, \vec{u}) d\vec{u} d\varphi(u_0), \end{aligned} \quad (35)$$

where Ψ_r is given by (15), I_r is the identity matrix on $\mathbb{R}$, and $\Sigma^{1/2}$ is the positive definite matrix satisfying $(\Sigma^{1/2})^2 = \Sigma$.

By the same process as used in Lemma 2.1 of [15], Theorem 9, and the change of variable theorem, we have the following corollary.

Corollary 10. Let $\{v_1, \dots, v_r\}$ be a subset of $L_2[0, T]$ and suppose that $\{M_h v_1, \dots, M_h v_r\}$ is an independent set. Then the random vector $(\vec{v}, Z(x, \cdot))$ has the multivariate normal distribution with mean vector $(\vec{v}, a)$ and covariance matrix $\Sigma_{M_h} = [\langle M_h v_i, M_h v_j \rangle]_{r \times r}$. Moreover, for any Borel measurable function $f: \mathbb{R}^{r+1} \rightarrow \mathbb{C}$, we have

$$\begin{aligned} & \int_{C[0, T]} f(x(0), (\vec{v}, Z(x, \cdot))) d\omega_\varphi(x) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^r} f(u_0, \vec{u}) \\ & \quad \cdot \Psi_r(1, (\vec{v}, a), \Sigma_{M_h}, \vec{u}) d\vec{u} d\varphi(u_0) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^r} f(u_0, \Sigma_{M_h}^{1/2} \vec{u} + (\vec{v}, a)) \\ & \quad \cdot \Psi_r(1, \vec{0}, I_r, \vec{u}) d\vec{u} d\varphi(u_0). \end{aligned} \quad (36)$$

4. Analytic Feynman Integrals and Conditional Analytic Feynman Integrals

We begin this section with introducing the cylinder function on the analogue of Wiener space. Let $\{v_1, v_2, \dots, v_r\}$ be an orthonormal subset of $L_2[0, T]$, let r be any positive integer, let $1 \leq p \leq \infty$, and let F be given by

$$F(x) = f((\vec{v}, x)) \quad (37)$$

for w_φ a.e. $x \in C[0, T]$, where $f \in L_p(\mathbb{R}^r)$. Without loss of generality we can take f to be Borel measurable. In the following theorem we evaluate the Wiener and Feynman integrals of F_Z .

Theorem 11. Let F be given by (37) with $1 \leq p \leq \infty$ and suppose that $\{M_h v_1, \dots, M_h v_r\}$ is an independent subset of $L_2[0, T]$. Then for $\lambda \in \mathbb{C}_+$

$$\begin{aligned} E^{anw_\lambda}[F_Z] &= \int_{\mathbb{R}^r} f(\vec{u}) \Psi_r(\lambda, (\vec{v}, a), \Sigma_{M_h}, \vec{u}) d\vec{u} \\ &= \int_{\mathbb{R}^r} f(\Sigma_{M_h}^{1/2} \vec{u} + (\vec{v}, a)) \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) d\vec{u}, \end{aligned} \quad (38)$$

where $\Sigma_{M_h} = [\langle M_h v_i, M_h v_j \rangle]_{r \times r}$ and Ψ_r is given by (15). Moreover if $p = 1$, then, for a nonzero real q , $E^{anf_q}[F_Z]$ is given by the right hand side of (38) with replacing λ by $-iq$.

Proof. Let $\lambda > 0$. Replacing h by $\lambda^{-1/2}h$ we have by Corollary 10

$$\begin{aligned} & \int_{C[0, T]} F_Z^\lambda(x) d\omega_\varphi(x) \\ &= \int_{\mathbb{R}^r} f(\vec{u}) \Psi_r(\lambda, (\vec{v}, a), \Sigma_{M_h}, \vec{u}) d\vec{u} \\ &= \int_{\mathbb{R}^r} f(\Sigma_{M_h}^{1/2} \vec{u} + (\vec{v}, a)) \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) d\vec{u}. \end{aligned} \quad (39)$$

For any real $N > 0$

$$\begin{aligned} & \exp \left\{ -\frac{N}{2} \left\langle \Sigma_{M_h}^{-1} (\vec{u} - (\vec{v}, a)), \vec{u} - (\vec{v}, a) \right\rangle_{\mathbb{R}} \right\} \\ &= \exp \left\{ -\frac{N}{2} \left\| \Sigma_{M_h}^{-1/2} (\vec{u} - (\vec{v}, a)) \right\|_{\mathbb{R}}^2 \right\} \end{aligned} \quad (40)$$

which is integrable over $\vec{u}$, so that we have the theorem by the change of variable theorem, Morera's theorem, the uniqueness of analytic continuation, and the dominated convergence theorem. $\square$

Let $\{e_{11}, \dots, e_{1r}\}$ be the orthonormal set obtained from $\{M_h v_1, \dots, M_h v_r\}$ by the Gram-Schmidt orthonormalization process. For $l = 1, \dots, r$ let $M_h v_l = \sum_{j=1}^r \alpha_{lj} e_{1j}$ be the linear combination of e_{1j} 's and let $B_1 = [\alpha_{lj}]_{r \times r}$ be the coefficient matrix of the combinations. Since

$$\langle M_h v_i, M_h v_j \rangle = \sum_{l=1}^r \sum_{k=1}^r \langle \alpha_{il} e_{1l}, \alpha_{jk} e_{1k} \rangle = \sum_{l=1}^r \alpha_{il} \alpha_{jl} \quad (41)$$

we have $\Sigma_{M_h} = B_1 B_1^T$, where B_1^T is the transpose of B_1 .

We now have the following corollary by (41), Theorem 11, and the change of variable theorem.

Corollary 12. Under the assumptions as given in Theorem 11

$$\begin{aligned} E^{anw_\lambda}[F_Z] &= \int_{\mathbb{R}^r} f(\vec{u}) \Psi_r(\lambda, (\vec{v}, a), B_1 B_1^T, \vec{u}) d\vec{u} \\ &= \int_{\mathbb{R}^r} f(B_1 \vec{u} + (\vec{v}, a)) \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) d\vec{u}. \end{aligned} \quad (42)$$

Moreover if $p = 1$, then $E^{anf_q}[F_Z]$ is given by the right hand side of (42) with replacing λ by $-iq$.

Theorem 13. Let F be given by (37) with $1 \leq p \leq \infty$ and suppose that $\{\mathcal{P}^\perp M_h v_1, \dots, \mathcal{P}^\perp M_h v_r\}$ is an independent subset of $L_2[0, T]$. Then for $\lambda \in \mathbb{C}_+$

$$\begin{aligned} E^{anw_\lambda}[F_Z | Z_{n+1}](\vec{\xi}_{n+1}) &= \int_{\mathbb{R}^r} f(\vec{u}) \\ & \quad \cdot \Psi_r(\lambda, (\vec{v}, A + P_{b,n+1}(\vec{\xi}_{n+1})), \Sigma_{\mathcal{P}^\perp}, \vec{u}) d\vec{u} \\ &= \int_{\mathbb{R}^r} f(\Sigma_{\mathcal{P}^\perp}^{1/2} \vec{u} + (\vec{v}, A + P_{b,n+1}(\vec{\xi}_{n+1}))) \\ & \quad \cdot \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) d\vec{u} \end{aligned} \quad (43)$$

for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, where $\Sigma_{\mathcal{P}^\perp} = [\langle \mathcal{P}^\perp M_h v_i, \mathcal{P}^\perp M_h v_j \rangle]_{r \times r}$ and A is given by (12). Moreover if $p = 1$, then, for a nonzero real q , $E^{anf_q}[F_Z | Z_{n+1}]$ is given by the right hand side of (43) with replacing λ by $-iq$.

Proof. For $j = 1, \dots, r$ and w_φ a.e. $x \in C[0, T]$ we have by Lemma 5

$$\begin{aligned} (v_j, X_{b,n+1}(x, \cdot)) &= (v_j, X(x, \cdot) - P_{b,n+1}(X(x, \cdot))) \\ &= (M_h v_j - \mathcal{P} M_h v_j, x) \\ &= (\mathcal{P}^\perp M_h v_j, x) \end{aligned} \quad (44)$$

so that for $\lambda > 0$ and $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ we have by Corollary 10

$$\begin{aligned}
I_{F_Z}^\lambda(\vec{\xi}_{n+1}) &= \int_{C[0,T]} F(\lambda^{-1/2} X_{b,n+1}(x, \cdot) + A \\
&\quad + P_{b,n+1}(\vec{\xi}_{n+1})) dw_\varphi(x) \\
&= \int_{C[0,T]} f(\lambda^{-1/2} ((\mathcal{P}^\perp M_h v_1, x), \dots, \\
&\quad (\mathcal{P}^\perp M_h v_r, x)) + (\vec{v}, A \\
&\quad + P_{b,n+1}(\vec{\xi}_{n+1}))) dw_\varphi(x) \\
&= \int_{\mathbb{R}^r} f(\vec{u}) \Psi_r(\lambda, (\vec{v}, A + P_{b,n+1}(\vec{\xi}_{n+1})), \Sigma_{\mathcal{P}^\perp}, \\
&\quad \vec{u}) d\vec{u} = \int_{\mathbb{R}^r} f(\Sigma_{\mathcal{P}^\perp}^{1/2} \vec{u} + (\vec{v}, A \\
&\quad + P_{b,n+1}(\vec{\xi}_{n+1}))) \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) d\vec{u},
\end{aligned} \tag{45}$$

where Ψ_r is given by (15). By the Morera and the dominated convergence theorem we have the theorem. $\square$

Let $\{e_{21}, \dots, e_{2r}\}$ be the orthonormal set obtained from $\{\mathcal{P}^\perp M_h v_1, \dots, \mathcal{P}^\perp M_h v_r\}$ by the Gram-Schmidt orthonormalization process. For $l = 1, \dots, r$ let $\mathcal{P}^\perp M_h v_l = \sum_{j=1}^r \beta_{lj} e_{2j}$ be the linear combination of e_{2j} 's and let $B_2 = [\beta_{lj}]_{r \times r}$ be the coefficient matrix of the combinations.

We now have the following corollary by a similar calculation of (41).

Corollary 14. *Under the assumptions as given in Theorem 13*

$$\begin{aligned}
E^{anw_\lambda}[F_Z | Z_{n+1}](\vec{\xi}_{n+1}) &= \int_{\mathbb{R}^r} f(\vec{u}) \\
&\quad \cdot \Psi_r(\lambda, (\vec{v}, A + P_{b,n+1}(\vec{\xi}_{n+1})), B_2 B_2^T, \vec{u}) d\vec{u} \\
&= \int_{\mathbb{R}^r} f(B_2 \vec{u} + (\vec{v}, A + P_{b,n+1}(\vec{\xi}_{n+1}))) \\
&\quad \cdot \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) d\vec{u}.
\end{aligned} \tag{46}$$

Moreover if $p = 1$, then $E^{anf_q}[F_Z | Z_{n+1}]$ is given by the right hand side of (46) with replacing λ by $-iq$.

For $\vec{v} = (v_1, \dots, v_r)$, $\vec{u} \in \mathbb{R}^r$, $\lambda \in \mathbb{C}$, and any nonsingular positive $r \times r$ matrix A_r on $\mathbb{R}$ let

$$\begin{aligned}
\langle \vec{v} \alpha_{n+1}, \alpha_{n+1} \rangle &= (\langle v_1 \alpha_{n+1}, \alpha_{n+1} \rangle, \dots, \langle v_r \alpha_{n+1}, \alpha_{n+1} \rangle), \\
\Lambda(A_r) &= 1 + (b(T) - b(t_n)) \|A_r^{-1} \langle \vec{v} \alpha_{n+1}, \alpha_{n+1} \rangle\|_{\mathbb{R}}^2, \\
\Phi_r(\lambda, \vec{v}, A_r, \vec{u}) &= \left[\frac{1}{\Lambda(A_r)} \right]^{1/2}
\end{aligned} \tag{47}$$

$$\begin{aligned}
&\cdot \exp \left\{ \frac{\lambda}{2\Lambda(A_r)} (b(T) - b(t_n)) \right. \\
&\quad \left. \cdot \langle \vec{u}, A_r^{-1} \langle \vec{v} \alpha_{n+1}, \alpha_{n+1} \rangle \rangle_{\mathbb{R}}^2 \right\}.
\end{aligned}$$

With the above notations, we have the following theorem.

Theorem 15. *Let F be given by (37) with $1 \leq p \leq \infty$. Then for $\lambda \in \mathbb{C}_+$*

$$\begin{aligned}
E^{anw_\lambda}[F_Z | Z_n](\vec{\xi}_n) &= \int_{\mathbb{R}^r} f(\Sigma_{\mathcal{P}^\perp}^{1/2} \vec{u} + (\vec{v}, A) \\
&\quad + (\vec{v}, P_{b,n}(\vec{\xi}_n)) + (a(T) - a(t_n)) \langle \vec{v} \alpha_{n+1}, \alpha_{n+1} \rangle) \\
&\quad \cdot \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) \Phi_r(\lambda, \vec{v}, \Sigma_{\mathcal{P}^\perp}^{1/2}, \vec{u}) d\vec{u}
\end{aligned} \tag{48}$$

for a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, where $(\vec{v}, P_{b,n}(\vec{\xi}_n)) = ((v_1, P_{b,n}(\vec{\xi}_n)), \dots, (v_r, P_{b,n}(\vec{\xi}_n)))$. If $p = 1$, then, for a nonzero real q , $E^{anf_q}[F_Z | Z_n]$ is given by the right hand side of the above equality with replacing λ by $-iq$. Moreover the matrix $\Sigma_{\mathcal{P}^\perp}^{1/2}$ can be replaced by B_2 in the above results.

Proof. For $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$ let $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$, where $\xi_{n+1} \in \mathbb{R}$. For $\lambda > 0$ we have by Lemma 6, Theorem 13, and the change of variable theorem

$$\begin{aligned}
K_{F_Z}^\lambda(\vec{\xi}_n) &= \int_{\mathbb{R}} \int_{\mathbb{R}^r} \Psi_1(\lambda, a(T) - a(t_n), b(T) \\
&\quad - b(t_n), \xi_{n+1} - \xi_n) f(\Sigma_{\mathcal{P}^\perp}^{1/2} \vec{u} \\
&\quad + (\vec{v}, A + P_{b,n+1}(\vec{\xi}_{n+1}))) \\
&\quad \cdot \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) d\vec{u} d\xi_{n+1} \\
&= \int_{\mathbb{R}^r} \int_{\mathbb{R}} \Psi_1(\lambda, a(T) - a(t_n), b(T) - b(t_n), \xi_{n+1} \\
&\quad - \xi_n) f(\Sigma_{\mathcal{P}^\perp}^{1/2} \vec{u} + (\vec{v}, A) + (\vec{v}, P_{b,n}(\vec{\xi}_n)) \\
&\quad + (\xi_{n+1} - \xi_n) \langle \vec{v} \alpha_{n+1}, \alpha_{n+1} \rangle) \\
&\quad \cdot \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) d\xi_{n+1} d\vec{u} \\
&= \int_{\mathbb{R}^r} \int_{\mathbb{R}} \Psi_1(\lambda, 0, b(T) - b(t_n), \xi_{n+1}) f(\Sigma_{\mathcal{P}^\perp}^{1/2} \vec{u} \\
&\quad + (\vec{v}, A) + (\vec{v}, P_{b,n}(\vec{\xi}_n)) \\
&\quad + (a(T) - a(t_n)) \langle \vec{v} \alpha_{n+1}, \alpha_{n+1} \rangle \\
&\quad + \langle \vec{v} \alpha_{n+1}, \alpha_{n+1} \rangle \xi_{n+1}) \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) d\xi_{n+1} d\vec{u}.
\end{aligned} \tag{49}$$

Using the same method as used in the proof of Theorem 3.2 in [9]

$$\begin{aligned}
E^{anw_\lambda}[F_Z | Z_n](\vec{\xi}_n) &= \int_{\mathbb{R}^r} f(\Sigma_{\mathcal{P}^\perp}^{1/2} \vec{u} + (\vec{v}, A) \\
&\quad + (\vec{v}, P_{b,n}(\vec{\xi}_n)) + (a(T) - a(t_n)) \langle \vec{v} \alpha_{n+1}, \alpha_{n+1} \rangle) \\
&\quad \cdot \Psi_r(\lambda, \vec{0}, I_r, \vec{u}) \Phi_r(\lambda, \vec{v}, \Sigma_{\mathcal{P}^\perp}^{1/2}, \vec{u}) d\vec{u}
\end{aligned} \tag{50}$$

by the Schwarz inequality and the Morera theorem. The final results follow from the dominated convergence theorem and by a similar calculation of (41). $\square$

Remark 16. (1) An orthonormal subset $\{v_1, v_2, \dots, v_r\}$ of $L_2[0, T]$ such that both $\{M_h v_1, \dots, M_h v_r\}$ and $\{\mathcal{P}^\perp M_h v_1, \dots, \mathcal{P}^\perp M_h v_r\}$ are independent sets exists.

(2) It does not mean that $B_1 = \Sigma_{M_h}^{1/2}$ in the equations of Corollary 12 and $B_2 = \Sigma_{\mathcal{P}^\perp}^{1/2}$ in the equations of Corollary 14 and of Theorem 15. They satisfy only the following equations:

$$\begin{aligned} B_1 B_1^T &= \Sigma_{M_h} = \left(\Sigma_{M_h}^{1/2} \right)^2, \\ B_2 B_2^T &= \Sigma_{\mathcal{P}^\perp} = \left(\Sigma_{\mathcal{P}^\perp}^{1/2} \right)^2. \end{aligned} \quad (51)$$

5. Integral Transforms with Change of Scales

In this section we derive change of scale transforms for the generalized conditional Wiener integrals of the function F_Z which is introduced in the previous section. To derive these scale transforms we use multivariate normal distributions so that the orthonormalization process of projection vectors can be removed from the change of scale transforms in [8–10] and the transforms are expressed in terms of F_Z itself.

For $\lambda \in \mathbb{C}$ and $x \in C[0, T]$ let

$$\begin{aligned} K_1(\lambda, x) &= \exp \left\{ \frac{1-\lambda}{2} \left\| \Sigma_{M_h}^{-1/2} (\vec{v}, X(x, \cdot)) \right\|_{\mathbb{R}}^2 \right\} \\ &= \exp \left\{ \frac{1-\lambda}{2} \left\| B_1^{-1} (\vec{v}, X(x, \cdot)) \right\|_{\mathbb{R}}^2 \right\}. \end{aligned} \quad (52)$$

We now have the following theorem.

Theorem 17. Let $1 \leq p \leq \infty$ and let F be given by (37). Then for $\lambda \in \mathbb{C}_+$

$$E^{anw_\lambda} [F_Z] = \lambda^{r/2} \int_{C[0, T]} K_1(\lambda, x) F_Z(x) dw_\varphi(x), \quad (53)$$

where K_1 is given by (52). If $p = 1$ and q is a nonzero real number, then

$$\begin{aligned} E^{anf_q} [F_Z] \\ = \lim_{m \rightarrow \infty} \lambda_m^{r/2} \int_{C[0, T]} K_1(\lambda_m, x) F_Z(x) dw_\varphi(x) \end{aligned} \quad (54)$$

for any sequence $\{\lambda_m\}_{m=1}^\infty$ in $\mathbb{C}_+$ converging to $-iq$ as m approaches to ∞ .

Proof. For $\lambda \in \mathbb{C}_+$ we have by (41), Corollary 10, and Theorem 11

$$\begin{aligned} \lambda^{r/2} \int_{C[0, T]} K_1(\lambda, x) F_Z(x) dw_\varphi(x) &= \lambda^{r/2} \int_{C[0, T]} f(\vec{v}, Z(x, \cdot)) \exp \left\{ \frac{1-\lambda}{2} \left\langle \Sigma_{M_h}^{-1} (\vec{v}, X(x, \cdot)), (\vec{v}, X(x, \cdot)) \right\rangle_{\mathbb{R}} \right\} dw_\varphi(x) \\ &= \lambda^{r/2} \int_{C[0, T]} f(\vec{v}, Z(x, \cdot)) \exp \left\{ \frac{1-\lambda}{2} \left\langle \Sigma_{M_h}^{-1} ((\vec{v}, Z(x, \cdot)) - (\vec{v}, a)), (\vec{v}, Z(x, \cdot)) - (\vec{v}, a) \right\rangle_{\mathbb{R}} \right\} dw_\varphi(x) \\ &= \lambda^{r/2} \int_{\mathbb{R}^r} f(\vec{u}) \Psi_r(1, (\vec{v}, a), \Sigma_{M_h}, \vec{u}) \exp \left\{ \frac{1-\lambda}{2} \left\langle \Sigma_{M_h}^{-1} (\vec{u} - (\vec{v}, a)), \vec{u} - (\vec{v}, a) \right\rangle_{\mathbb{R}} \right\} d\vec{u} \\ &= \int_{\mathbb{R}^r} f(\vec{u}) \Psi_r(\lambda, (\vec{v}, a), \Sigma_{M_h}, \vec{u}) d\vec{u} = E^{anw_\lambda} [F_Z], \end{aligned} \quad (55)$$

where Ψ_r is given by (15). If $p = 1$, the final result immediately follows from the dominated convergence theorem. $\square$

For $\lambda \in \mathbb{C}$, $x \in C[0, T]$, and $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ let

$$\begin{aligned} K_2(\lambda, \vec{\xi}_{n+1}, x) &= \left(\frac{1}{|\Sigma_{\mathcal{P}^\perp}|} \right)^{1/2} \\ &\cdot \exp \left\{ -\frac{\lambda}{2} \left\| \Sigma_{\mathcal{P}^\perp}^{-1/2} (\vec{v}, Z(x, \cdot) - A - P_{b, n+1}(\vec{\xi}_{n+1})) \right\|_{\mathbb{R}}^2 \right\} \\ &= \frac{1}{|B_2|} \\ &\cdot \exp \left\{ -\frac{\lambda}{2} \left\| B_2^{-1} (\vec{v}, Z(x, \cdot) - A - P_{b, n+1}(\vec{\xi}_{n+1})) \right\|_{\mathbb{R}}^2 \right\}. \end{aligned} \quad (56)$$

We now have the following theorem.

Theorem 18. Let $1 \leq p \leq \infty$ and let F be given by (37). Then for $\lambda \in \mathbb{C}_+$

$$\begin{aligned} E^{anw_\lambda} [F_Z | Z_{n+1}] (\vec{\xi}_{n+1}) &= \left(\lambda^r |\Sigma_{M_h}| \right)^{1/2} \\ &\cdot \int_{C[0, T]} K_1(0, x) K_2(\lambda, \vec{\xi}_{n+1}, x) F_Z(x) dw_\varphi(x) \end{aligned} \quad (57)$$

for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, where K_2 is given by (56). If $p = 1$ and q is a nonzero real number, then

$$\begin{aligned} E^{anf_q} [F_Z | Z_{n+1}] (\vec{\xi}_{n+1}) &= \lim_{m \rightarrow \infty} \left(\lambda_m^r |\Sigma_{M_h}| \right)^{1/2} \\ &\cdot \int_{C[0, T]} K_1(0, x) K_2(\lambda_m, \vec{\xi}_{n+1}, x) F_Z(x) dw_\varphi(x) \end{aligned} \quad (58)$$

for any sequence $\{\lambda_m\}_{m=1}^\infty$ in $\mathbb{C}_+$ converging to $-iq$ as m approaches to ∞ . Moreover $|M_h|$ can be replaced by $|B_1|^2$ in the above equalities.

Proof. For $\lambda > 0$ and a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ we have by Corollary 10

$$\begin{aligned}
 & (\lambda^r |\Sigma_{M_h}|)^{1/2} \int_{C[0,T]} K_1(0, x) K_2(\lambda, \vec{\xi}_{n+1}, x) F_Z(x) dw_\varphi(x) = \left(\frac{\lambda^r |\Sigma_{M_h}|}{|\Sigma_{\mathcal{D}^\perp}|} \right)^{1/2} \int_{C[0,T]} f(\vec{v}, Z(x, \cdot)) \\
 & \cdot \exp \left\{ \frac{1}{2} \left\langle \Sigma_{M_h}^{-1} ((\vec{v}, Z(x, \cdot)) - (\vec{v}, a)), (\vec{v}, Z(x, \cdot)) - (\vec{v}, a) \right\rangle_{\mathbb{R}} \right. \\
 & - \frac{\lambda}{2} \left\langle \Sigma_{\mathcal{D}^\perp}^{-1} (\vec{v}, Z(x, \cdot) - A - P_{b,n+1}(\vec{\xi}_{n+1})), (\vec{v}, Z(x, \cdot) - A - P_{b,n+1}(\vec{\xi}_{n+1})) \right\rangle_{\mathbb{R}} \left. \right\} dw_\varphi(x) = \left(\frac{\lambda^r |\Sigma_{M_h}|}{|\Sigma_{\mathcal{D}^\perp}|} \right)^{1/2} \\
 & \cdot \int_{\mathbb{R}^r} f(\vec{u}) \Psi_r(1, (\vec{v}, a), \Sigma_{M_h}, \vec{u}) \exp \left\{ \frac{1}{2} \left\langle \Sigma_{M_h}^{-1} (\vec{u} - (\vec{v}, a)), \vec{u} - (\vec{v}, a) \right\rangle_{\mathbb{R}} \right. \\
 & - \frac{\lambda}{2} \left\langle \Sigma_{\mathcal{D}^\perp}^{-1} (\vec{u} - (\vec{v}, A + P_{b,n+1}(\vec{\xi}_{n+1}))), \vec{u} - (\vec{v}, A + P_{b,n+1}(\vec{\xi}_{n+1})) \right\rangle_{\mathbb{R}} \left. \right\} d\vec{u} = \int_{\mathbb{R}^r} f(\vec{u}) \Psi_r(\lambda, (\vec{v}, A + P_{b,n+1}(\vec{\xi}_{n+1})), \\
 & \Sigma_{\mathcal{D}^\perp}, \vec{u}) d\vec{u}.
 \end{aligned} \tag{59}$$

By (41), the analytic continuation, the dominated convergence theorem, and Theorem 13 we have the theorem. $\square$

By Theorems 15 and 18 we have the final theorem.

Theorem 19. Let $1 \leq p \leq \infty$ and let F be given by (37). Then for $\lambda \in \mathbb{C}_+$

$$\begin{aligned}
 E^{amw_\lambda} [F_Z | Z_n] (\vec{\xi}_n) &= (\lambda^r |\Sigma_{M_h}|)^{1/2} \\
 & \cdot \int_{\mathbb{R}} \int_{C[0,T]} \Psi_1(\lambda, a(T) - a(t_n), b(T) \\
 & - b(t_n), \xi_{n+1} - \xi_n) K_1(0, x) K_2(\lambda, \vec{\xi}_{n+1}, x) \\
 & \cdot F_Z(x) dw_\varphi(x) d\vec{\xi}_{n+1}
 \end{aligned} \tag{60}$$

for a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, where $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$ and Ψ_1, K_2 are given by (15) and (56), respectively. If $p = 1$ and q is a nonzero real number, then

$$\begin{aligned}
 E^{anf_q} [F_Z | Z_n] (\vec{\xi}_n) &= \lim_{m \rightarrow \infty} (\lambda_m^r |\Sigma_{M_h}|)^{1/2} \\
 & \cdot \int_{\mathbb{R}} \int_{C[0,T]} \Psi_1(\lambda_m, a(T) - a(t_n), b(T) \\
 & - b(t_n), \xi_{n+1} - \xi_n) K_1(0, x) K_2(\lambda_m, \vec{\xi}_{n+1}, x) \\
 & \cdot F_Z(x) dw_\varphi(x) d\vec{\xi}_{n+1}
 \end{aligned} \tag{61}$$

for any sequence $\{\lambda_m\}_{m=1}^\infty$ in $\mathbb{C}_+$ converging to $-iq$ as m approaches to ∞ . Moreover $|M_h|$ can be replaced by $|B_1|^2$ in the above equalities.

Remark 20. (1) Letting $\lambda = \rho^{-2}$ in the theorems of this section, where $\rho > 0$, we have the change of scale formulas for $E[F_Z(\rho \cdot)]$, $E[F_Z(\rho \cdot) | Z_n(\rho \cdot)]$, and $E[F_Z(\rho \cdot) | Z_{n+1}(\rho \cdot)]$ as integral transforms.

(2) If $a = 0$ and $h = 1$ a.e., then we can obtain Theorems 5.1 and 5.2 in [9].

(3) If $a = 0$, $h = 1$ a.e., and $\varphi = \delta_0$ which is the Dirac measure concentrated at 0, then we can obtain the change of scale transforms in [10].

(4) The results in this paper are independent of a particular choice of the initial distribution φ .

Competing Interests

The author declares that they have no competing interests.

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