

Research Article

S-Mixing Tuple of Operators on Banach Spaces

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We consider the question: what is the appropriate formulation of Godefroy-Shapiro criterion for tuples of operators? We also introduce a new notion about tuples of operators, S-mixing, which lies between mixing and weakly mixing. We also obtain a sufficient condition to ensure a tuple of operators to be S-mixing. Moreover, we study some new properties of S-mixing operators on several concrete Banach spaces.

1. Introduction

A continuous linear operator T acting on a topological vector space X is called hypercyclic, if there exists a vector $x \in X$ such that the orbit of x under T is dense in X ; that is, $\text{orb}(x, T) = \{x, Tx, T^2x, \dots\} = X$. Such a vector x is called a hypercyclic vector for T and the set of hypercyclic vectors for T will be denoted by $\text{HC}(T)$.

Rolewicz [1] and Kitai [2] proved that there does not exist hypercyclic operator on finite-dimensional space. This implies that hypercyclic phenomenon only appears in infinite dimensional space for a single operator. For more details on this topic, one can refer to [3, 4].

In 2006, Feldman provided a new class of hypercyclic operators, tuples of operators, and showed us a different direction from the previous results. This gives us hypercyclic phenomenon in both separable infinite dimensional space and finite-dimensional space. The first example that the author showed to us is that there exists an $(n + 1)$ -tuple of diagonal matrices that is hypercyclic on \mathbb{C}^n .

For an m -tuple of operators we mean a finite sequence of length m of mutually commuting continuous linear operators on a Banach space X . Let $\mathcal{T} = (T_1, T_2, \dots, T_m)$ be an m -tuple acting on a Banach space X (more general complete topology vector space), and \mathcal{F} is the semigroup generated by T ; that is,

$$\mathcal{F} = \langle \mathcal{T} \rangle = \{T_1^{k_1} T_2^{k_2} \dots T_m^{k_m} : k_i \in \mathbb{N}, i = 1, \dots, m\}, \quad (1)$$

which is a finitely generated abelian semigroup.

In [5, 6], Feldman extended the definitions of hypercyclic and topologically transitive of single bounded linear operator to tuples of operators. This started a new study in the field of hypercyclicity. In [7, 8], Yousefi gave the definition of mixing tuples of operators, and he obtained some sufficient conditions for a tuple of continuous operators to be hereditarily transitive and investigated the relationships between hypercyclicity and d -dense orbits of a tuple of operators.

In the past few years, the authors have studied the following dynamical properties of tuples \mathcal{T} :

- (1) hypercyclicity: $\text{orb}(x, \mathcal{F}) = \{Sx : S \in \mathcal{F}\} = X$;
- (2) chaotic (Devaney) [9]: \mathcal{T} is hypercyclic and has a dense set of periodic points;
- (3) mixing: for any pair U, V of nonempty open subsets of X , there exists some positive integer N such that

$$T_1^{k_1} \dots T_m^{k_m}(U) \cap V \neq \emptyset \quad \forall k_j \geq N, j = 1, \dots, m; \quad (2)$$

- (4) weakly mixing: for a tuple of operators \mathcal{T} and subsets $A, B \subset X$ define the return set as

$$N_{\mathcal{T}}(A, B) = \{(k_1, \dots, k_m) \in \mathbb{N}_0^m : T_1^{k_1} \dots T_m^{k_m}(A) \cap B \neq \emptyset\}. \quad (3)$$

Then \mathcal{T} is called weakly mixing if for every quadruple of nonempty open subsets $U_1, U_2, V_1, V_2 \subset X$ we have that $N_{\mathcal{T}}(U_1, V_1) \cap N_{\mathcal{T}}(U_2, V_2) \neq \emptyset$.

From the above results, one can deduce that

$$\text{mixing} \implies \text{weakly mixing} \implies \text{hypercyclic}. \quad (4)$$

Corresponding to Birkhoff's transitive theorem of single continuous map which was proposed by Birkhoff in 1920, Feldman [6] and Yousefi and Ershad [10], respectively, showed that hypercyclicity and topologically transitivity of tuples of operators are equivalent on separable infinite dimension Banach space. Here we describe one of the versions of the extended Birkhoff's transitive theorem as follows.

Theorem 1. *Let X be a separable infinite dimensional Banach space X and (T_1, T_2) the pair of operators T_1 and T_2 . The following assertions are equivalent:*

- (1) (T_1, T_2) is hypercyclic;
- (2) (T_1, T_2) is topologically transitive.

If one of these conditions holds then the set of points in X with dense orbit is a dense G_δ -set.

As we know, for a single continuous linear operator, we can use its point spectra and corresponding eigenvalues to judge the dynamical system properties of the operator. In [11], Godefroy and Shapiro gave a criterion. The following theorem we cited was settled by Grosse-Erdmann and Peris in their book [4].

Theorem 2 ((Godefroy-Shapiro criterion) [4, Theorem 3.1]). *Let T be an operator. Suppose that the subspaces*

$$X_0 := \text{span} \{x \in X : Tx = \lambda x \text{ for some } \lambda \in \mathbb{K} \text{ with } |\lambda| < 1\}, \quad (5)$$

$$Y_0 := \text{span} \{x \in X : Tx = \lambda x \text{ for some } \lambda \in \mathbb{K} \text{ with } |\lambda| > 1\}$$

are dense in X . Then T is mixing and in particular hypercyclic. Moreover, if X is a complex space and also the subspace

$$Z_0 := \text{span} \{x \in X : Tx = e^{\alpha i} x \text{ for some } \alpha \in \mathbb{Q}\} \quad (6)$$

is dense in X , then T is chaotic.

In our another unpublished paper, we established a generalization version of Godefroy-Shapiro criterion for the tuples of operators. Now we state this result as follows.

Theorem 3. *Let (T_1, T_2) be a pair of commuting continuous linear operators on a separable infinite Banach space X . Suppose that the subspaces*

$$X_0 := \text{span} \{x \in X : T_1 x = \lambda_1 x, T_2 x = \lambda_2 x \text{ for some } \lambda_1, \lambda_2 \in \mathbb{K} \text{ with } |\lambda_1| < 1, |\lambda_2| < 1\}, \quad (7)$$

$$Y_0 := \text{span} \{x \in X : T_1 x = \lambda_1 x, T_2 x = \lambda_2 x \text{ for some } \lambda_1, \lambda_2 \in \mathbb{K} \text{ with } |\lambda_1| > 1, |\lambda_2| > 1\}$$

are dense in X . Then (T_1, T_2) is mixing and in particular hypercyclic.

Moreover, if X is a complex space and also the subspace

$$\begin{aligned} Z_0 &:= \text{span} \{x \in X : T_1 x = |\lambda_1| e^{\alpha i} x, T_2 x \\ &= |\lambda_2| e^{\beta i} x \text{ for some } \lambda_1, \lambda_2 \in \mathbb{K} \text{ with } |\lambda_1 \lambda_2| \\ &= 1, \alpha, \beta \in \mathbb{Q}\} \end{aligned} \quad (8)$$

is dense in X , then (T_1, T_2) is chaotic.

Comparing the Godefroy-Shapiro criterion for a single operator and Theorem 3 above, it is natural to try to extend G-S criterion via the product of the joint point spectra of the tuple. However, when considering the product of the joint point spectra that satisfies the conditions in Theorem 3, we find that our idea mentioned above does not work; see Example 9 for the details. Therefore, we turn to consider some special joint point spectra, namely, some special subgroup of the abelian semigroup \mathcal{F} generated by the operator \mathcal{T} . To do so, we need to introduce the following definition.

Definition 4. The m -tuple (T_1, \dots, T_m) of commuting operators on X is S -mixing if for any nonempty open sets $U, V \subset X$, and for any m -tuple of sequences of positive integers $(k_{1,n}, \dots, k_{m,n})_{n \in \mathbb{N}}$ such that each sequence $(k_{j,n})_{n \in \mathbb{N}}$ is strictly increasing, $j = 1, \dots, m$, and

$$\sup \{|k_{i,n} - k_{j,n}| : n \in \mathbb{N}, 1 \leq i, j \leq m\} < \infty, \quad (9)$$

there exists $N \in \mathbb{N}$ such that

$$T^{k_{1,n}} \dots T^{k_{m,n}}(U) \cap V \neq \emptyset \quad \forall n \geq N. \quad (10)$$

Remark 5. We will give some examples which are related to the above definition in Section 3. According to the definition of mixing of a single operator T , we have that for any nonempty open sets U, V , $N(U, V)$ is cofinite. However, if \mathcal{T} is mixing or S -mixing, then $\mathbb{N}_0^2 \setminus N_{\mathcal{T}}(U, V)$ is infinite. Obviously, if \mathcal{T} is mixing, then it must be S -mixing, and an S -mixing tuple of operators \mathcal{T} must be hypercyclic.

From the definition of S -mixing and the remark above, one may ask the following question.

Question 1. Is the S -mixing (for the tuple of operators) stronger than weakly mixing (for the tuple of operators) or is the inverse true?

The following two theorems are our main results in the present paper.

Theorem 6. *Let (T_1, T_2) be a pair of commuting continuous linear operators on a separable infinite Banach space X . Suppose that the subspaces*

$$X_0 := \text{span} \{x \in X : T_1 x = \lambda_1 x, T_2 x = \lambda_2 x \text{ for some } \lambda_1, \lambda_2 \in \mathbb{K} \text{ with } |\lambda_1 \lambda_2| < 1\}, \quad (11)$$

$$Y_0 := \text{span} \{x \in X : T_1 x = \lambda_1 x, T_2 x = \lambda_2 x \text{ for some } \lambda_1, \lambda_2 \in \mathbb{K} \text{ with } |\lambda_1 \lambda_2| > 1\}$$

are dense in X . Then (T_1, T_2) is S -mixing and in particular hypercyclic.

Moreover, if X is a complex space and also the subspace

$$\begin{aligned} Z_0 &:= \text{span} \{x \in X : T_1 x = |\lambda_1| e^{\alpha \pi i} x, T_2 x \\ &= |\lambda_2| e^{\beta \pi i} x \text{ for some } \lambda_1, \lambda_2 \in \mathbb{K} \text{ with } |\lambda_1 \lambda_2| \\ &= 1, \alpha, \beta \in \mathbb{Q}\} \end{aligned} \quad (12)$$

is dense in X , then (T_1, T_2) is chaotic.

In the preceding theorem, we consider the product of the joint point spectra to extend the G-S criterion, and it is not hard to find that the pair of operators (T_1, T_2) is not mixing. We will give an example (Example 18) below. In the view of preserving the property of mixing, Theorem 3 is more suitable than Theorem 6 as a generalization of G-S criterion for tuples of operators.

The following theorem gives a positive answer to Question 1.

Theorem 7. *Let (T_1, T_2) be a pair of commuting continuous linear operators on a separable infinite Banach space X . If (T_1, T_2) is S -mixing, then (T_1, T_2) must be weakly mixing. However, the inverse is not true.*

Based on Theorem 7, we obtain a new notion of a dynamic system between mixing and weakly mixing. Let $((T_1, T_2), X)$ be a dynamic system; then we have

$$\begin{aligned} \text{mixing} &\implies S\text{-mixing} \implies \text{weakly mixing} \\ &\implies \text{hypercyclic.} \end{aligned} \quad (13)$$

2. Proofs of Main Results

Throughout this paper, let \mathbb{D} denote the unit disk $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and \mathbb{T} denote the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. $H(\mathbb{C})$ denotes the entire function space on \mathbb{C} .

Although the proof of Theorem 6 is simple and similar to the proof of single mixing operator, we still give a brief proof for the sake of completeness.

Proof of Theorem 6. Let U, V be a pair of nonempty open subsets of X ; by the hypothesis, we can find $x \in X_0 \cap U$ and $y \in Y_0 \cap V$. Hence these vectors can be expressed in the form

$$\begin{aligned} x &= \sum_{j=1}^m a_j x_j, \\ y &= \sum_{j=1}^m b_j y_j, \end{aligned} \quad (14)$$

where $T_1 x_j = \lambda_{j,1} x_j, T_2 x_j = \lambda_{j,2} x_j, T_1 y_j = \mu_{j,1} y_j$, and $T_2 y_j = \mu_{j,2} y_j$, for certain scalars $a_j, b_j, \lambda_{j,1}, \lambda_{j,2}, \mu_{j,1}, \mu_{j,2} \in \mathbb{K}$ with $|\lambda_{j,1} \lambda_{j,2}| < 1, |\mu_{j,1} \mu_{j,2}| > 1, j = 1, \dots, m$.

For any pair of sequences of positive integers $(k_{1,n}, k_{2,n})_{n \in \mathbb{N}}$ such that each sequence $(k_{j,n})_{n \in \mathbb{N}}$ is strictly increasing, $j = 1, 2$, and

$$\sup \{|k_{1,n} - k_{2,n}| : n \in \mathbb{N}\} < \infty,$$

$$\begin{aligned} T_1^{k_{1,n}} T_2^{k_{2,n}} x &= \sum_{j=1}^m a_j \lambda_{j,1}^{k_{1,n}} \lambda_{j,2}^{k_{2,n}} x_j \\ &= \sum_{j=1}^m a_j \lambda_{j,1}^{k_{1,n} - k_{2,n}} (\lambda_{j,1} \lambda_{j,2})^{k_{2,n}} x_j \longrightarrow 0, \end{aligned} \quad (15)$$

$$\begin{aligned} u_{k_{1,n}, k_{2,n}} &:= \sum_{j=1}^m b_j \frac{1}{\mu_{j,1}^{k_{1,n}} \mu_{j,2}^{k_{2,n}}} y_j \\ &= \sum_{j=1}^m b_j \frac{1}{\mu_{j,1}^{k_{1,n} - k_{2,n}} (\mu_{j,1} \mu_{j,2})^{k_{2,n}}} y_j \longrightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ and

$$T_1^{k_{1,n}} T_2^{k_{2,n}} u_{k_{1,n}, k_{2,n}} = \sum_{j=1}^m b_j y_j = y \quad (16)$$

for all $k_{1,n}, k_{2,n} \geq 0$, then one has

$$\begin{aligned} x + u_{k_{1,n}, k_{2,n}} &\in U, \\ T_1^{k_{1,n}} T_2^{k_{2,n}} (x + u_{k_{1,n}, k_{2,n}}) &\in V. \end{aligned} \quad (17)$$

It follows that (T_1, T_2) is S -mixing and also hypercyclic.

Moreover, if X is a complex space, then iterations of the pair of operators show to us that

$$\begin{aligned} T_1^{k_{1,n}} T_2^{k_{2,n}} x &= |\lambda_1|^{k_{1,n}} e^{k_{1,n} \alpha \pi i} |\lambda_2|^{k_{2,n}} e^{k_{2,n} \beta \pi i} x \\ &= |\lambda_1|^{k_{1,n}} |\lambda_2|^{k_{2,n}} e^{(k_{1,n} \alpha + k_{2,n} \beta) \pi i} x. \end{aligned} \quad (18)$$

Since $\alpha, \beta \in \mathbb{Q}$, there exists $k_{1,n} = k_{2,n}$ such that $T_1^{k_{1,n}} T_2^{k_{2,n}} = e^{k_{1,n} (\alpha + \beta) \pi i}$. Hence Z_0 is precisely the set of periodic points of the operator (T_1, T_2) . Thus (T_1, T_2) is chaotic whenever Z_0 is also dense in X . This completes the proof. \square

Remark 8. For a pair of operators (T_1, T_2) , if it is S -mixing, one considers $k_{2,n} = k_{1,n+1}$ for any $n \in \mathbb{N}$; then the bounded sup condition is exactly the fact that the sequence $(k_{1,n})_n$ (and thus $(k_{2,n})_n$) is syndetic. For single operators, the notion of syndetical hypercyclicity was studied in [12, 13].

Next, we give an example that satisfies the conditions of Theorem 6, but not mixing.

Example 9. Let $(\mu_1 B, \mu_2 B)$ be a pair of multiple of backward shift operators acting on any space $X = l^p(\mathbb{N}), 1 \leq p < \infty$, or $c_0(\mathbb{N})$, where $\mu_1, \mu_2 \in \mathbb{C} \setminus \{0\}$ and $|\mu_1 \mu_2| > 1$. Then $(\mu_1 B, \mu_2 B)$ is S -mixing and chaotic, but not mixing.

One can easily determine the eigenvectors of B as being the nonzero multiples of the sequence

$$e_\lambda := (\lambda, \lambda^2, \lambda^3, \dots), \quad |\lambda| < 1 \quad (19)$$

with corresponding eigenvalue λ , and $|\lambda| < 1$ ensures that $e_\lambda \in X$. Hence, e_λ is an eigenvector of $\mu_i B$ ($i = 1, 2$) to the eigenvalue $\mu_i \lambda$, and this also ensures that

$$\{x \in X : \mu_1 Bx = \lambda_1 x, \mu_2 Bx = \lambda_2 x \text{ for some } \lambda_1, \lambda_2 \in \mathbb{K}\} \quad (20)$$

is nonempty.

According to [14, Sublemma 7], for any nonempty subset Λ of the unit disk \mathbb{D} , the set $\text{span}\{e_\lambda; \lambda \in \Lambda\}$ is dense in X . Now we consider the following set:

$$\begin{aligned} X_0 &= \text{span} \{x \in X : \mu_1 Bx = \lambda_1 x, \mu_2 Bx \\ &= \lambda_2 x \text{ for } \lambda_1, \lambda_2 \in \mathbb{K} \text{ with } |\lambda_1 \lambda_2| < 1\} \end{aligned} \quad (21)$$

contains $\text{span}\{e_\lambda : |\lambda| < 1/|\mu_1 \mu_2|^{1/2}\}$. Since $|\mu_1 \mu_2| > 1$ and $|\lambda| < 1$, we have

$$\left\{ \lambda \in \mathbb{D} : |\lambda| < \frac{1}{|\mu_1 \mu_2|^{1/2}} \right\} \quad (22)$$

is a nonempty open subset of \mathbb{D} and thus X_0 is dense in X . Since

$$\begin{aligned} Y_0 &= \text{span} \{x \in X : \mu_1 Bx = \lambda_1 x, \mu_2 Bx \\ &= \lambda_2 x \text{ for } \lambda_1, \lambda_2 \in \mathbb{K} \text{ with } |\lambda_1 \lambda_2| > 1\} \end{aligned} \quad (23)$$

contains $\text{span}\{e_\lambda : 1/|\mu_1 \mu_2|^{1/2} < |\lambda| < 1\}$ and

$$\begin{aligned} Z_0 &= \text{span} \{x \in X : \mu_1 Bx = |\lambda_1| e^{\alpha \pi i} x, \mu_2 Bx \\ &= |\lambda_2| e^{\beta \pi i} x \text{ for some } \lambda_1, \lambda_2 \in \mathbb{K} \text{ with } |\lambda_1 \lambda_2| \\ &= 1, \alpha, \beta \in \mathbb{Q}\} \end{aligned} \quad (24)$$

contains $\text{span}\{e_\lambda : |\lambda| = 1/|\mu_1 \mu_2|^{1/2}\}$, one can show that Y_0 and Z_0 are both dense in X . Therefore, $(\mu_1 B, \mu_2 B)$ is S-mixing and chaotic.

Now we are going to show that $(\mu_1 B, \mu_2 B)$ is not mixing. To do this, it sufficient for us to construct a counterexample. By the hypothesis, $|\mu_1 \mu_2| > 1$, let $|\mu_1| = 3$, $|\mu_2| = 1/2$; there exist large enough positive integers α, β such that $3^\alpha (1/2)^\beta \leq 1$. Assume on the contrary that $(3B, (1/2)B)$ is mixing, according to the definition of mixing tuple of operators, for any pair U, V of nonempty open subsets of X , there exists some positive integer N such that

$$\left((3B)^{k_1} \left(\frac{1}{2} B \right)^{k_2} \right) (U) \cap V \neq \emptyset \quad (25)$$

for all $k_j \geq N$, $j = 1, 2$. Let $\alpha, \beta \geq N$; one has that

$$\left((3B)^\alpha \left(\frac{1}{2} B \right)^\beta \right) (U) \cap V \neq \emptyset. \quad (26)$$

Then $\|(3B)^\alpha ((1/2)B)^\beta\| > 1$; that is, $3^\alpha (1/2)^\beta > 1$; this is a contradiction.

The following corollary follows easily from Theorem 6, which tells us that S-mixing is stronger than weakly mixing.

Corollary 10. *Let (T_1, T_2) be a pair of commuting bounded linear operators on a separable infinite Banach space X . If (T_1, T_2) satisfies Theorem 6, then (T_1, T_2) must be weakly mixing.*

Before giving the proof of Theorem 7, we need another theorem.

Theorem 11. *Let (T_1, \dots, T_m) be an m -tuple on a separable infinite Banach space X . If (T_1, \dots, T_m) is S-mixing, then the abelian semigroup \mathcal{F} generated by (T_1, \dots, T_m) contains a mixing operator. Moreover, the m -tuple (T_1, \dots, T_m) is S-mixing if and only if the operator $T_1 \cdots T_m$ is mixing.*

Proof. If (T_1, \dots, T_m) is S-mixing, then for any nonempty open subsets $U, V \subset X$ and for any tuple of sequences of positive integers $(k_{1,n}, \dots, k_{m,n})_{n \in \mathbb{N}}$ such that each sequence $(k_{j,n})_{n \in \mathbb{N}}$ is strictly increasing, $j = 1, \dots, m$, and

$$\sup \{ |k_{i,n} - k_{j,n}| : n \in \mathbb{N}, 1 \leq i, j \leq m \} < \infty, \quad (27)$$

there exists a positive integer N such that

$$T_1^{k_{1,n}} \cdots T_m^{k_{m,n}} (U) \cap V \neq \emptyset \quad (28)$$

for all $n \geq N$. Take $k_{1,n} = \dots = k_{m,n}$; then one has

$$(T_1 \cdots T_m)^{k_{1,n}} (U) \cap V \neq \emptyset \quad (29)$$

for all $n \geq N$. This implies that $T_1 \cdots T_m$ is mixing.

Next, we need to show that the operator $T_1 \cdots T_m$ is mixing that implies the m -tuple (T_1, \dots, T_m) is S-mixing. For any m -tuple of sequences of positive integers $(k_{1,n}, \dots, k_{m,n})_{n \in \mathbb{N}}$ such that each sequence $(k_{j,n})_{n \in \mathbb{N}}$ is strictly increasing, $j = 1, \dots, m$, and

$$\sup \{ |k_{i,n} - k_{j,n}| : n \in \mathbb{N}, 1 \leq i, j \leq m \} < \infty, \quad (30)$$

we may assume that $k_{1,n} \leq k_{j,n}$, $j = 2, \dots, m$. Since the operator $T_1 \cdots T_m$ is mixing, for any nonempty open subsets $U, V \subset X$ and for any tuple of sequences of positive integers $(k_{1,n}, \dots, k_{m,n})_{n \in \mathbb{N}}$ above, we have that

$$(T_1 \cdots T_m)^{k_{1,n}} (U) \cap \left(T_2^{k_{1,n} - k_{2,n}} \cdots T_m^{k_{1,n} - k_{m,n}} \right) (V) \neq \emptyset \quad (31)$$

for all $n \geq N$. Clearly, (31) is equivalent to

$$U \cap (T_1 \cdots T_m)^{-k_{1,n}} \left(T_2^{k_{1,n} - k_{2,n}} \cdots T_m^{k_{1,n} - k_{m,n}} \right) (V) \neq \emptyset \quad (32)$$

for all $n \geq N$. Since T_1, \dots, T_m are mutually commutative operators, we obtain that

$$U \cap T_1^{-k_{1,n}} \cdots T_m^{-k_{m,n}} (V) \neq \emptyset \quad (33)$$

for all $n \geq N$. It follows that

$$T_1^{k_{1,n}} \cdots T_m^{k_{m,n}} (U) \cap V \neq \emptyset \quad (34)$$

for all $n \geq N$. This completes the proof. \square

Now we are ready to give the proof of Theorem 7.

Proof of Theorem 7. According to Theorem 11, the semigroup \mathcal{F} generated by (T_1, T_2) contains a mixing operator $\tilde{S} = T_1 T_2$, for any nonempty open subsets $U, V, U_1, U_2, V_1, V_2 \subset X$; there exists a positive integer \tilde{N} such that $\tilde{S}^n(U) \cap V \neq \emptyset$ for any $n \geq \tilde{N}$.

Let $S = \tilde{S}^{n_1}$, $n_1 \geq \tilde{N}$; then S is a continuous map that commutes with (T_1, T_2) and satisfies $S(U) \cap V \neq \emptyset$ for any nonempty open subsets $U, V \subset X$. Then we can find nonempty open sets $U'_1 \subset U_1, V'_1 \subset V_1$ such that $S(U'_1) \subset U_2, S(V'_1) \subset V_2$.

Since (T_1, T_2) is S -mixing operator, the return set $N(U'_1, V'_1) \neq \emptyset$. Suppose that $(k_1, k_2) \in N(U'_1, V'_1)$; then there exists $x \in U'_1$ with $T_1^{k_1} T_2^{k_2} x \in V'_1$. Therefore,

$$\begin{aligned} Sx &\in U_2, \\ T_1^{k_1} T_2^{k_2} Sx &= S T_1^{k_1} T_2^{k_2} x \in V_2, \end{aligned} \quad (35)$$

which yields that $(k_1, k_2) \in N(U_2, V_2)$. It follows that $N(U'_1, V'_1) \subset N(U_2, V_2)$. As $N(U'_1, V'_1) \subset N(U_1, V_1)$, we have that

$$\emptyset \neq N(U'_1, V'_1) \subset N(U_1, V_1) \cap N(U_2, V_2). \quad (36)$$

Hence (T_1, T_2) is weakly mixing. For the other direction, we will present a counterexample in Example 18. Thus we complete the proof. \square

3. Applications

In this section, we will give some applications of Theorems 6 and 11. For the pair of classical operators, we can easily obtain the following three results via Theorem 11.

Proposition 12. *Let $(\mu_1 B, \mu_2 B)$ be a pair of multiple backward shift operators acting on any space $l^p(\mathbb{N})$ ($1 \leq p < \infty$) or $c_0(\mathbb{N})$, where $\mu_1, \mu_2 \in \mathbb{C} \setminus \{0\}$ and $\mu_1 \neq \mu_2$; then $(\mu_1 B, \mu_2 B)$ is S -mixing if and only if $|\mu_1 \mu_2| > 1$.*

Proposition 13. *Let $(T_{a_1}, T_{a_2}) : H(\mathbb{C}) \rightarrow H(\mathbb{C})$ be a pair of translation operators, where $T_{a_i} : H(\mathbb{C}) \rightarrow H(\mathbb{C})$, $i = 1, 2$, defined by $T_{a_i} f(z) = f(z + a_i)$, for any $f \in H(\mathbb{C})$. Then (T_{a_1}, T_{a_2}) is S -mixing if and only if $a_1 + a_2 \neq 0$.*

Proposition 14. *Let $H^2(\mathbb{D})$ be a Hardy-Hilbert space of holomorphic function on the complex unit disk \mathbb{D} . Let φ_i ($i = 1, 2$) be a nonconstant bounded holomorphic function, M_{φ_i} a multiplication operator, and $M_{\varphi_i}^*$ the corresponding adjoint multiplication operator. Let $(M_{\varphi_1}^*, M_{\varphi_2}^*)$ be a pair of adjoint multiplication operators acting on $H^2(\mathbb{D})$; then $(M_{\varphi_1}^*, M_{\varphi_2}^*)$ is S -mixing if and only if $(\varphi_1 \varphi_2)(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.*

Using the G-S criterion (Theorem 2), Godefroy and Shapiro in [11] characterized the hypercyclicity of adjoint multipliers, excluding constant multipliers because their adjoint multiplication operators are multipliers of the identity. In [4], the theorem was settled as follows.

Theorem 15 (see [4, Theorem 4.42]). *Let φ be a nonconstant bounded holomorphic function on \mathbb{D} and let M_{φ}^* be the corresponding adjoint multiplier on $H^2(\mathbb{D})$. Then the following assertions are equivalent:*

- (1) M_{φ}^* is hypercyclic;
- (2) M_{φ}^* is mixing;
- (3) M_{φ}^* is chaotic;
- (4) $\varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.

Feldman in [15] focused on pair or tuple of adjoint multiplication operators on the Hardy-Hilbert space; he showed that most pairs of coanalytic Toeplitz operators are hypercyclic. More precisely, the author proved the following theorem.

Theorem 16 (see [15, Theorem 3.1]). *Let $f, g \in H^{\infty}(\mathbb{D}) \setminus \{0\}$ satisfy $|f(z)| > 1$ for all $z \in \mathbb{D}$ and $|g(z)| > 1$ for all $z \in \mathbb{D}$, and let M_f and M_g be the corresponding multiplication operators on $H^2(\mathbb{D})$. Then neither M_f^* nor M_g^* is hypercyclic on $H^2(\mathbb{D})$, but the following assertions are equivalent:*

- (1) the pair (M_f^*, M_g^*) is hypercyclic on $H^2(\mathbb{D})$;
- (2) the semigroup \mathcal{F} generated by the pair (M_f^*, M_g^*) contains a hypercyclic operator;
- (3) there exist $n, k \in \mathbb{N}$ such that $f^n g^k$ is nonconstant and $f^n g^k(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$;
- (4) there does not exist $p > 0$ and $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} / g^p(z)$ for all $z \in \mathbb{D}$.

Motivated by Theorems 15 and 16, we deduce the following corollary from Theorems 6 and 11.

Corollary 17. *Let $H^2(\mathbb{D})$ be a Hardy-Hilbert space of holomorphic function on the complex unit disk \mathbb{D} . Let φ_i ($i = 1, \dots, m$) be a nonconstant bounded holomorphic function, M_{φ_i} a multiplication operator, and $M_{\varphi_i}^*$ the corresponding adjoint multiplication operator. Let $(M_{\varphi_1}^*, \dots, M_{\varphi_m}^*)$ be a tuple of adjoint multiplication operators acting on $H^2(\mathbb{D})$; the following assertions are equivalent:*

- (1) $(M_{\varphi_1}^*, \dots, M_{\varphi_m}^*)$ is S -mixing;
- (2) $M_{\varphi_1}^* \cdots M_{\varphi_m}^*$ is mixing;
- (3) $M_{\varphi_1}^* \cdots M_{\varphi_m}^*$ is chaotic;
- (4) $(\varphi_1 \cdots \varphi_m)(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.

Next, we give an example to show that a pair of commutative operators is weakly mixing, but not S -mixing.

Example 18. Let $(2B, (1/3)B)$ be a pair of multiple backward shift operators acting on any space $l^p(\mathbb{N})$, $1 \leq p < \infty$, or $c_0(\mathbb{N})$, and then $(2B, (1/3)B)$ is hypercyclic also weakly mixing, but not S -mixing.

Assume on the contrary that $(2B, (1/3)B)$ is S -mixing; then for any nonempty open subsets $U, V \subset X$,

$$2^{k_1} \frac{1}{3^{k_2}} B^{k_1+k_2}(U) \cap V \neq \emptyset \quad \text{for } \sup_{k_1, k_2 \geq N} |k_1 - k_2| < \infty. \quad (37)$$

Take $m_1 = m_2 \geq N$; we have

$$2^{m_1} \frac{1}{3^{m_2}} B^{m_1+m_2}(U) \cap V \neq \emptyset. \quad (38)$$

Equivalently,

$$\left(\frac{2}{3}B^2\right)^{m_2}(U) \cap V \neq \emptyset \quad \forall m_2 \geq N. \quad (39)$$

Hence the operator $(2/3)B^2$ is mixing, but this is a contradiction.

Next, we check that the operator $(2B, (1/3)B)$ satisfies hypercyclic criterion of tuple of operators [6, Proposition 2.5]. We know that the finite sequences (i.e., sequences of the form $(x_1, \dots, x_n, 0, \dots)$, $n \geq 1$) constitute a dense subset of X .

Let X_0 be the finite sequences, and let $(n_j)_{j \geq 1}, (k_j)_{j \geq 1}$ be increasing sequences of positive integers and satisfy $(n_j - k_j) \rightarrow +\infty$ as $j \rightarrow \infty$. Let $S_j : X_0 \rightarrow X_0$, defined by $S_j = (1/2)^{n_j} 3^{k_j} F^{n_j+k_j}$; then for any $x \in X_0$, $(2B)^{n_j}((1/3)B)^{k_j} x \rightarrow 0$, $(2B)^{n_j}((1/3)B)^{k_j} S_j \rightarrow I$,

$$\|S_j x\| = \left(\frac{1}{2}F\right)^{n_j} (3F)^{k_j} x \leq \frac{1}{2^{n_j}} 3^{k_j} \|x\| \rightarrow 0 \quad (40)$$

as $j \rightarrow \infty$. Hence $(2B, (1/3)B)$ is weakly mixing and in particular hypercyclic. Therefore, $(2B, (1/3)B)$ is hypercyclic and weakly mixing, but not S -mixing.

Proposition 19. *Under the same assumptions in Proposition 12, if $(\mu_1 B, \mu_2 B)$ is S -mixing, then the semigroup \mathcal{F} generated by $(\mu_1 B, \mu_2 B)$ contains a mixing operator. But, the inverse is not true.*

Proof. Since $(\mu_1 B, \mu_2 B)$ is an S -mixing operator, according to Proposition 12, we have that $|\mu_1 \mu_2| > 1$ and $\mu_1 \mu_2 B^2$ is a mixing operator. Hence the semigroup \mathcal{F} generated by $(\mu_1 B, \mu_2 B)$ contains a mixing operator. Based on Example 18, we can find that the inverse of this is not true. Thus we complete the proof. \square

Remark 20. There are examples of tuples (T_1, T_2) in which every single operator $T_1^n T_2^m$ with $n, m \geq 0$, $n + m > 0$, is mixing (thus the tuple is S -mixing), but the corresponding semigroup generated by the tuple (T_1, T_2) is not mixing. Consider the example of translation semigroup on a sector of the complex plane given in Example 4.15 in [16]. The subsemigroup generated by the tuple of translations (T_1, T_2) of the translation semigroup in this example, where $T_1 = T_{1+i}$ and $T_2 = T_{1-i}$, is an example of this type, if we follow the argument given there.

For single operator, the relationships between mixing operators and chaotic operators have been studied very

clearly; next we give a brief discussion on the relationships between S -mixing tuples and chaotic tuples. Firstly, we present a proposition on the pair of multiple backward shift operators.

Proposition 21. *Under the same assumptions in Proposition 12, then the following assertions are equivalent:*

- (1) $(\mu_1 B, \mu_2 B)$ is chaotic;
- (2) $(\mu_1 B, \mu_2 B)$ is hypercyclic;
- (3) $N(U, V) \neq \emptyset$ for any nonempty open subsets $U, V \subset X$;
- (4) $\max\{|\mu_1|, |\mu_2|\} > 1$;
- (5) the semigroup \mathcal{F} generated by $(\mu_1 B, \mu_2 B)$ contains a hypercyclic operator;
- (6) the semigroup \mathcal{F} generated by $(\mu_1 B, \mu_2 B)$ contains a chaotic operator;
- (7) $(\mu_1 B, \mu_2 B)$ has a dense set of periodic points and $N(U, V) \neq \emptyset$ for any pair of nonempty open subsets $U, V \subset X$.

Proof. (1) \Rightarrow (2), (2) \Leftrightarrow (3), and (5) \Rightarrow (2) are trivial.

(2) \Rightarrow (4): by hypothesis, $(\mu_1 B, \mu_2 B)$ is hypercyclic. According to Theorem 1, we have that the set $\text{HC}((\mu_1 B, \mu_2 B))$ of all hypercyclic vectors of $(\mu_1 B, \mu_2 B)$ is a dense G_δ -set in X . Let $\mathcal{B}_X = \{x \in X : \|x\| < 1\}$; we have

$$\mathcal{B}_X \cap \text{HC}((\mu_1 B, \mu_2 B)) \neq \emptyset. \quad (41)$$

Then there exists some $y \in \mathcal{B}_X \cap \text{HC}((\mu_1 B, \mu_2 B))$ such that $\text{orb}((\mu_1 B, \mu_2 B), y)$ is dense in X . Assume on the contrary that $\max\{|\mu_1|, |\mu_2|\} \leq 1$; then we have

$$\text{orb}((\mu_1 B, \mu_2 B), y) \subseteq \mathcal{B}_X, \quad (42)$$

because

$$\begin{aligned} \left\| (\mu_1 B)^{k_1} (\mu_2 B)^{k_2} y \right\| &\leq \|\mu_1 B\|^{k_1} \|\mu_2 B\|^{k_2} \|y\| \\ &\leq |\mu_1|^{k_1} |\mu_2|^{k_2} \|y\| < 1, \end{aligned} \quad (43)$$

for all $k_1, k_2 \in \mathbb{N}$. This is a contradiction. Therefore, $\max\{|\mu_1|, |\mu_2|\} > 1$.

(4) \Rightarrow (5): Without loss of generality, we may assume that $|\mu_1| > 1$, then $\mu_1 B$ is hypercyclic. Hence the semigroup \mathcal{F} generated by $(\mu_1 B, \mu_2 B)$ contains a hypercyclic operator. Therefore, we get

$$(2) \Leftrightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (2). \quad (44)$$

Based on the proof above, we can easily obtain (1) \Rightarrow (4).

(4) \Rightarrow (6): without loss of generality, we assume that $|\mu_1| > 1$; then $\mu_1 B$ is a chaotic operator. Therefore, the semigroup \mathcal{F} generated by $(\mu_1 B, \mu_2 B)$ contains a chaotic operator.

(6) \Rightarrow (1): by hypothesis, the semigroup \mathcal{F} generated by $(\mu_1 B, \mu_2 B)$ contains a chaotic operator T . Let $T = \mu_1^{\alpha_1} \mu_2^{\alpha_2} B^{\alpha_1+\alpha_2}$, $\alpha_1, \alpha_2 \in \mathbb{N}$; then T has a dense set of periodic points and there exists $x \in X$ such that $\text{orb}(x, T)$ is dense in

X . For any $x \in \text{Per}(T)$, there exists a positive integer N such that $T^N x = x$; that is,

$$\mu_1^{\alpha_1 N} \mu_2^{\alpha_2 N} B^{\alpha_1 N + \alpha_2 N} x = x. \quad (45)$$

Hence $x \in \text{Per}((\mu_1 B, \mu_2 B))$; then we have

$$\begin{aligned} \text{Per}(T) &\subseteq \text{Per}((\mu_1 B, \mu_2 B)), \\ \text{orb}(x, T) &\subseteq \text{orb}((\mu_1 B, \mu_2 B), x). \end{aligned} \quad (46)$$

Therefore, $(\mu_1 B, \mu_2 B)$ has a dense set of periodic points and $\text{orb}((\mu_1 B, \mu_2 B), x)$ is dense in X . Hence $(\mu_1 B, \mu_2 B)$ is chaotic.

Since (7) \Leftrightarrow (1), we have

$$(1) \implies (4) \implies (6) \implies (1) \iff (7). \quad (47)$$

Thus we complete the proof. \square

Remark 22. Based on Propositions 12 and 21, we can see that the operator $(2B, (1/3)B)$ in Example 18 is chaotic, but not S-mixing.

Competing Interests

The authors declare that they have no competing interests.

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