Research Article
Existence of Solutions to a Class of Semilinear Elliptic Problem with Nonlinear Singular Terms and Variable Exponent

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The authors of this paper prove the existence and regularity results for the homogeneous Dirichlet boundary value problem to the equation

\[- \text{div} (M(x) \nabla u) = \frac{f(x)}{u^{\alpha(x)}}, \quad x \in \Omega,\]

\[u = 0, \quad x \in \partial \Omega,\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N (N \geq 2)\) with smooth boundary \(\partial \Omega\), \(\alpha(x)\) is a continuous function on \(\Omega\), \(\alpha(x) > 0\), \(\alpha^+ = \sup_{x \in \Omega} \alpha(x)\), \(\alpha^- = \inf_{x \in \Omega} \alpha(x)\), \(f\) is a nonnegative function belonging to the Lebesgue space \(L^m(\Omega)\), for some suitable \(m > 1\), and \(M\) is a bounded positive definite matrix; that is, there exist \(0 < \gamma < \beta\) such that

\[\gamma |\eta|^2 \leq (M(x) \eta) \cdot \eta,\]

\[|M(x)| \leq \beta,\]

for every \(\eta \in \mathbb{R}^N\), for almost every \(x\) in \(\Omega\).

Problem (1) has been extensively studied in the past. In [5], Lazer and Mckenna dealt with model (1) with \(f\), a continuous function; they proved that the solution was in \(H^1_0(\Omega)\) if and only if \(\alpha < 3\), while it was not in \(C^1(\Omega)\) if \(\alpha > 1\).

Later, Lair and Shaker in [6] studied the existence of solutions to the elliptic equation

\[- \Delta u = f(x) u^{-\alpha}, \quad x \in \Omega,\]

\[u = 0, \quad x \in \partial \Omega,\]

They proved that problem (3) with \(0 < \alpha < 1\) has a unique weak positive solution in \(H^1_0(\Omega)\) if \(f(x)\) is a nonnegative nontrivial function in \(L^2(\Omega)\).

Moreover, the results of Lair and Shaker were generalized by Shi and Yao (see [7]); they studied the following problem:

\[- \Delta u = f(x, u), \quad x \in \Omega,\]

\[u = \varphi, \quad x \in \partial \Omega,\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N, N \geq 2\), \(\varphi \geq 0\) may take the value 0 on \(\partial \Omega\), and \(f(x, s)\) is possibly singular near \(s = 0\). They proved the existence and the uniqueness of positive solutions without assuming monotonicity or strict positivity on \(f(x, s)\).
Recently, Boccardo and Orsina in [8] studied the existence, regularity, and nonexistence of solutions for the following problem:

\[- \text{div} (M(x) \nabla u) = \frac{f(x)}{u^\alpha}, \quad x \in \Omega, \quad u = 0, \quad x \in \partial \Omega. \tag{5}\]

They discussed the dependence of the results on the summability of $f$ and the values of $\alpha$. For the other results of singular elliptic equations, see [9, 10]. In this paper, we generalize the results in [8] to the case when $\alpha$ is a variable exponent by applying the method of regularization, Schauder fixed point theorem, the integrability of solution to the approximate problem with $n = 1$, and a necessary compactness argument to overcome some difficulties arising from the singular terms with variable exponent.

2. Preliminaries

Firstly, we give the definition of weak solutions to problem (1).

**Definition 1.** A function $u \in H^1_0(\Omega)$ is called a weak solution of problem (1), if the following identity holds:

\[
\int_\Omega M(x) \nabla u \cdot \nabla \varphi \, dx = \int_\Omega \frac{f}{u^{\alpha(x)}} \varphi \, dx, \quad \forall \varphi \in C_0^1(\Omega). \tag{6}
\]

In order to prove our results, we will consider the following approximation problem:

\[- \text{div} (M(x) \nabla u_n) = \frac{f_n}{(u_n + 1/n)^{\alpha(x)}}, \quad x \in \Omega, \quad u_n = 0, \quad x \in \partial \Omega, \tag{7}\]

where $n \in \mathbb{N}$, $f_n(x) = \min\{f(x), n\}$.

**Lemma 2.** Problem (7) has a nonnegative solution $u_n$ in $H^1_0(\Omega) \cap L^{\infty}(\Omega)$.

**Proof.** Let $n \in \mathbb{N}$ be fixed and $\omega$ a function in $L^2(\Omega)$. It is not difficult to prove that the following problem has a unique solution $v \in H^1_0(\Omega) \cap L^{\infty}(\Omega)$ (see [11, 12]):

\[- \text{div} (M(x) \nabla v) = \frac{f_n}{(|\omega| + 1/n)^{\alpha(x)}}, \quad x \in \Omega, \quad v = 0, \quad x \in \partial \Omega. \tag{8}\]

So, for any $\omega \in L^2(\Omega)$, we define the mapping $\Gamma : L^2(\Omega) \to H^1_0(\Omega) \cap L^{\infty}(\Omega)$ as $\Gamma(\omega) = v$. Taking $v$ as a test function, we have, using (2),

\[
y \int_\Omega |\nabla v|^2 \, dx \leq \int_\Omega (M(x) \nabla v) \cdot \nabla v \, dx = \int_\Omega \frac{f_n v}{(|\omega| + 1/n)^{\alpha(x)}} \, dx \leq n^{1+\alpha} \int_\Omega |v|^2 \, dx.
\]

By Poincaré inequality (on the left hand side) and Hölder’s inequality (on the right hand side), we get that

\[
\int_\Omega |v|^2 \, dx \leq C n^{1+\alpha} \left( \int_\Omega |v|^2 \, dx \right)^{1/2}, \tag{10}
\]

for some constant $C$ independent of $\omega$. This implies that

\[
\|v\|_{L^2(\Omega)} \leq C n^{1+\alpha}. \tag{11}
\]

Therefore, the ball of $L^2(\Omega)$ of radius $C n^{1+\alpha}$ is invariant under the mapping $\Gamma$. Since the embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact, we obtain that $\Gamma$ is a compact operator and $\|\cdot\|_{L^2(\Omega)} \leq C$. It is also easy to prove that $\Gamma$ is continuous on $L^2(\Omega)$, by Schauder’s fixed point theorem, we get that there exists a function $u_n \in H^1_0(\Omega)$, for every fixed $n \in \mathbb{N}$, such that $u_n = S(u_n)$; that is, problem (7) has a solution. Since $f_n/(u_n + 1/n)^{\alpha(x)} > 0$, the maximum principle implies that $u_n > 0$. Since the right hand side of (7) belongs to $L^{\infty}(\Omega)$, the result of Theorem 4.2 in [13] implies that $u_n \in L^{\infty}(\Omega)$.

**Lemma 3.** The sequence $u_n$ is increasing with respect to $n$, $u_n > 0$ in $\Omega$, and for every $\Omega \subset \subset \Omega$ there exists $C_{\Omega'} > 0$ (independent of $n$) such that

\[
u(x) \geq C_{\Omega'} > 0 \tag{12}
\]

for every $x \in \Omega'$, for every $n \in \mathbb{N}$.

**Proof.** Due to $0 \leq f_n \leq f_{n+1}$ and $\alpha(x) > 0$, we have that

\[- \text{div} (M(x) \nabla u_n) = \frac{f_n}{(u_n + 1/n)^{\alpha(x)}} \leq \frac{f_{n+1}}{(u_n + 1/(n+1))^{\alpha(x)}}, \tag{13}\]

\[- \text{div} (M(x) \nabla u_{n+1}) = \frac{f_{n+1}}{(u_{n+1} + 1/(n+1))^{\alpha(x)}}, \tag{14}\]

so that

\[- \text{div} (M(x) \nabla (u_n - u_{n+1})) \leq \frac{1}{(u_n + 1/(n+1))^{\alpha(x)}} - \frac{1}{(u_{n+1} + 1/(n+1))^{\alpha(x)}} \tag{15}\]

Choosing $(u_n - u_{n+1})_+ = \max\{u_n - u_{n+1}, 0\}$ as a test function, observing that

\[
\left( \frac{1}{u_n + 1} \right)^{\alpha(x)}_+ - \left( \frac{1}{u_{n+1} + 1} \right)^{\alpha(x)}_+ \leq 0,
\]

we get

\[
\left( \int_\Omega |v|^2 \, dx \right)^{1/2} \leq C n^{1+\alpha} \left( \int_\Omega |v|^2 \, dx \right)^{1/2}.
\]
and applying (2), we get that
\[
0 \leq \gamma \int_{\Omega} |\nabla (u_n - u_{n+1})|_2^2 \, dx \leq 0. \tag{16}
\]
This implies that \((u_n - u_{n+1})_+ = 0\) a.e. in \(\Omega\); that is, \(u_n \leq u_{n+1}\) for every \(n \in N\). Since the sequence \(\{u_n\}\) is increasing with respect to \(n\), we only need to prove that (12) holds for \(u_1\).

Applying Lemma 2, we know that \(u_1 \in L^\infty(\Omega)\); that is, there exists a constant \(C\) (depending only on \(\Omega\) and \(N\)) such that
\[
\|u_1\|_{L^\infty(\Omega)} \leq C \|f\|_{L^\infty(\Omega)} \leq C, \tag{17}
\]
and then
\[
- \text{div} \left( M(x) \nabla u_1 \right) = \frac{f_1}{(u_1 + 1)\alpha(x)} \geq \frac{f_1}{(C + 1)\alpha(x)}. \tag{18}
\]
Due to \(f_1/(C + 1)^{\alpha(x)} \geq 0, f_1/(C + 1)^{\alpha(x)} \not\equiv 0\), the strong maximum principle implies that \(u_1 > 0\) in \(\Omega\) and (12) holds for \(u_1\). The monotonicity of \(u_n\) implies that (12) holds for \(u_n\).

\begin{proof}
Choosing \(u_n\) as a test function in (7), by Hölder’s inequality, (2), and the fact that \(f_n \leq f\), we get
\[
y \int_{\Omega} |\nabla u_n|^2 \, dx \leq \int_{\Omega} \frac{f_n u_n}{(u_n + 1/n)^{\alpha(x)}} \, dx
\]
\[
\leq \int_{\Omega} f_{u_n}^{1-\alpha(x)} \, dx \leq \int_{\Omega} f_{u_n}^{1-\alpha} \, dx + \int_{\Omega} f_{u_n}^{1-\alpha} \, dx
\]
\[
\leq \|f\|_{L^n(\Omega)} \left( \int_{\Omega} u_n^{(1-\alpha)m'} \, dx \right)^{1/m'} + \|f\|_{L^n(\Omega)}
\]
\[
\left( \int_{\Omega} u_n^{(1-\alpha)m'} \, dx \right)^{1/m'} \leq \|f\|_{L^n(\Omega)}
\]
\[
\left( \int_{\Omega} \left( 1-(1-\alpha)/(1-\alpha)' \right)^{1/m'} \right) \left( \int_{\Omega} u_n^{(1-\alpha)m'} \, dx \right)^{1/m'} + |\Omega|^{(\alpha-\alpha')/(1-\alpha)m'}
\]
\[
\|f\|_{L^n(\Omega)} \left( \int_{\Omega} u_n^{(1-\alpha)m'} \, dx \right)^{(1-\alpha')/(1-\alpha)m'}.
\]
By the assumption of \(m, \) we have \((1-\alpha)m' = 2\), and using Sobolev Embedding Theorem (on the left hand side), we have that
\[
yS \left( \int_{\Omega} u_n^{2/m} \, dx \right)^{2/m} \leq y \int_{\Omega} |\nabla u_n|^2 \, dx \leq \|f\|_{L^n(\Omega)}
\]
\[
\left( \int_{\Omega} u_n^{2/m} \, dx \right)^{1/m'} + |\Omega|^{(\alpha-\alpha')/(1-\alpha)m'} \|f\|_{L^n(\Omega)}
\]
\[
\left( \int_{\Omega} \left( 1-(1-\alpha)/(1-\alpha)' \right)^{1/m'} \right) \left( \int_{\Omega} u_n^{(1-\alpha)m'} \, dx \right)^{1/m'}
\]
that is,
\[
yS \left( \int_{\Omega} u_n^{2/m} \, dx \right)^{2/m} \leq \|f\|_{L^n(\Omega)} \left( \int_{\Omega} u_n^{2/m} \, dx \right)^{(1-\alpha')/2}
\]
\[
+ |\Omega|^{(\alpha-\alpha')/(1-\alpha)m'} \|f\|_{L^n(\Omega)} \left( \int_{\Omega} u_n^{2/m} \, dx \right)^{(1-\alpha')/2'}.
\]
Since \(1-\alpha' < 1 - \alpha < 2\), (22) yields the boundedness of \(u_n\) in \(L^2(\Omega)\). By this estimate and (22), the conclusion follows.
\end{proof}

Once we have the boundedness of \(u_n\), we can prove an existence result for (1).

\begin{theorem}
Suppose that \(f\) is a nonnegative function in \(L^n(\Omega)\) \((f \not\equiv 0)\), with \(m = 2N/(N+2)\) \((N-2)\alpha) = (2^*/(1-\alpha)'\), and let \(0 < \alpha \leq \alpha(x) \leq \alpha^+ < 1\). Then problem (1) has a solution \(u \in H^1_0(\Omega)\) satisfying (6).
\end{theorem}
Proof. Since \( u_n \) is bounded in \( H^1_0(\Omega) \) by Lemma 6 and \( u_n \) converges to \( u \) pointwise in \( \Omega \) (by Lemma 3), then we know that there exists \( u \in H^1_0(\Omega) \) such that

\[
\begin{align*}
\lim_{n \to \infty} u_n & \to u \\
& \quad \text{weakly in } H^1_0(\Omega) \quad \text{and strongly in } L^2(\Omega), \quad (24) \\
\n\n\end{align*}
\]

\( \nabla u_n \to \nabla u \) weakly in \( L^2(\Omega) \).

So we have that

\[
\begin{align*}
\lim_{n \to \infty} \int_{\Omega} (M(x) \nabla u_n) \cdot \nabla \varphi \, dx &= \int_{\Omega} (M(x) \nabla u) \cdot \nabla \varphi \, dx, \quad \forall \varphi \in C^1_0(\Omega). \
\end{align*}
\]

(25)

Since \( u_n \) satisfies (12), we get that

\[
0 \leq \frac{\int_{\Omega} f_n \varphi \, dx}{(u_n + 1/n)^{\alpha(x)}} \leq \frac{\|\varphi\|_{L^\infty(\Omega)} f_{\min} \{C^{\alpha}, C^{\alpha'}\}}, \quad (26)
\]

where \( \Omega^f = \{x : \varphi \neq 0\} \). Then by Lebesgue Dominated Convergence Theorem, we have that

\[
\begin{align*}
\lim_{n \to \infty} \int_{\Omega} \frac{f_n \varphi}{(u_n + 1/n)^{\alpha(x)}} \, dx &= \int_{\Omega} \frac{f \varphi}{u^{\alpha(x)}} \, dx, \quad \forall \varphi \in C^1_0(\Omega). \
\end{align*}
\]

(27)

Since \( u_n \) is a solution of (7), this implies that

\[
\begin{align*}
\int_{\Omega} (M(x) \nabla u_n) \cdot \nabla \varphi \, dx &= \int_{\Omega} \frac{f_n \varphi}{(u_n + 1/n)^{\alpha(x)}} \, dx, \quad \forall \varphi \in C^1_0(\Omega). \
\end{align*}
\]

(28)

Letting \( n \to \infty \), combining (25) with (27), we get that

\[
\int_{\Omega} (M(x) \nabla u) \cdot \nabla \varphi \, dx = \int_{\Omega} \frac{f \varphi}{u^{\alpha(x)}} \, dx,
\]

which proves that (1) has a solution \( u \) in \( H^1_0(\Omega) \).

The summability of \( u \) depends on the summability of \( f \), which is proved in the next Lemma.

**Lemma 8.** Suppose that \( f \in L^m(\Omega) \), \( m \geq 2N/(N + 2 + (N - 2)\alpha^\star) \), and let \( 0 < \alpha^\star \leq \alpha(x) \leq \alpha^\star < 1 \). Then the solution \( u \) of (1) given by Theorem 7 is such that

(i) if \( m > N/2 \), then \( u \in L^\infty(\Omega) \);

(ii) if \( 2N/(N + 2 + (N - 2)\alpha^\star) \leq m < N/2 \), then \( u \in L^m(\Omega) \), \( s = Nm(1 + \alpha^\star)/(N - 2m) \).

Proof. To prove (i), let \( k > 1 \) and define \( G_k(s) = (s - k)_+ \). Taking \( G_k(u_n) \) as a test function in (7), using (2), we get

\[
\begin{align*}
\gamma \int_{\Omega} \|\nabla G_k(u_n)\|^2 \, dx & \leq \int_{\Omega} (M(x) \nabla G_k(u_n)) \cdot \nabla G_k(u_n) \, dx \\
& = \int_{\Omega} \frac{f_n G_k(u_n)}{(u_n + 1/n)^{\alpha(x)}} \, dx.
\end{align*}
\]

(30)

Since \( G_k(u_n) \neq 0 \), it follows that

\[
\begin{align*}
\gamma \int_{\Omega} \|\nabla G_k(u_n)\|^2 \, dx & \leq \int_{\Omega} f G_k(u_n) \, dx. \
\end{align*}
\]

(31)

Starting from inequality (31), Theorem 4.2 in [13] shows that there exists a constant \( C \) (independent of \( n \)), such that

\[
\|u_n\|_{L^\infty(\Omega)} \leq C \|f\|_{L^m(\Omega)},
\]

(32)

which implies that \( u \) belongs to \( L^\infty(\Omega) \).

To prove (ii), noting that if \( m = 2N/(N + 2 + (N - 2)\alpha^\star) \), \( s = 2N/(N - 2) = 2^\star \), since \( u \in H^1_0(\Omega) \), the result when \( m = 2N/(N + 2 + (N - 2)\alpha^\star) \) is true by Sobolev Embedding Theorem. If \( 2N/(N + 2 + (N - 2)\alpha^\star) < m < N/2 \), letting \( \delta > 1 \) and choosing \( u_n^{2\delta - 1} \) as a test function in (7), using Hölder’s inequality, we get

\[
\begin{align*}
\gamma (2\delta - 1) \int_{\Omega} \|\nabla u_n\|^2 u_n^{2\delta - 2} \, dx & \leq \int_{\Omega} \frac{f u_n^{2\delta - 1}}{(u_n + 1/n)^{\alpha(x)}} \, dx \\
& \leq \int_{\{x \in \Omega \alpha(x) \geq 1\}} \frac{f u_n^{2\delta - 1}}{u_n^{\alpha(x)}} \, dx + \int_{\{x \in \Omega \alpha(x) < 1\}} \frac{f u_n^{2\delta - 1}}{u_n^{\alpha(x)}} \, dx \\
& \leq \int_{\Omega} \frac{f u_n^{2\delta - 1 - \alpha} \, dx}{u_n^{\alpha(x)}} + \int_{\Omega} \frac{f u_n^{2\delta - 1 - \alpha} \, dx}{u_n^{\alpha(x)}} \\
& \leq \left( \int_{\Omega} \frac{(2\delta - 1 - \alpha)^m \, dx}{u_n^{(2\delta - 1 - \alpha)\, dx}} \right)^{1/m} + \|f\|_{L^m(\Omega)} \\
& \leq \left( \int_{\Omega} \frac{(2\delta - 1 - \alpha)^m \, dx}{u_n^{(2\delta - 1 - \alpha)\, dx}} \right)^{1/m} \\
& \leq \left( \int_{\Omega} \frac{(2\delta - 1 - \alpha)^m \, dx}{u_n^{(2\delta - 1 - \alpha)\, dx}} \right)^{1/m} \\
& \leq |\Omega|^{(\alpha^\star - \alpha)/(2\delta - 1 - \alpha)}\, m^{1/m}
\end{align*}
\]

(33)

By Sobolev inequality (on the left hand side), we have that

\[
\begin{align*}
\int_{\Omega} \|\nabla u_n\|^2 u_n^{2\delta - 2} \, dx & \leq \frac{1}{\delta^2} \int_{\Omega} \|\nabla u_n\|^2 \, dx \\
& \geq \frac{S}{\delta^2} \left( \int_{\Omega} u_n^{2\delta} \, dx \right)^{2\delta - 1},
\end{align*}
\]
where $S$ is the constant of the Sobolev embedding; combining with (33) and (34), we have that

$$
\frac{Sy(2\delta - 1)}{\delta^2} \left( \int_{\Omega} \left( u_n^{*(2\delta - 1)} \right)^{1/m'} \right)^{2/2'} \leq \left\| f \right\|_{L^{m}(\Omega)}
$$

\begin{align}
+ \left| \Omega \right|^{(\alpha' - \alpha)/(2\delta - 1)} \left( \int_{\Omega} \left( u_n^{(2\delta - 1 - \alpha')/(2\delta - 1 - \alpha')} \right)^{1/m'} \right)^{2/2'}
\end{align}

which implies that

\begin{align}
\left( \int_{\Omega} \left( u_n \right)^{1/\alpha'} \left( 2^* \delta - 1 \right) \right)^{2/2'} \leq \frac{\delta^2}{Sy(2\delta - 1)} \left\| f \right\|_{L^{m}(\Omega)}
\end{align}

where $\epsilon = Sy(2\delta - 1)/2\delta^2 \| f \|_{L^{m}(\Omega)}$. Thus, we get that

\begin{align}
\left( \int_{\Omega} \left( u_n \right)^{1/\alpha'} \left( 2^* \delta - 1 \right) \right)^{2/2'} \leq \frac{2\delta^2}{Sy(2\delta - 1)} \left( \int_{\Omega} \left( u_n \right)^{\alpha' - \alpha'} \left( 2^* \delta - 1 \right) \right)^{2/2'}
\end{align}

Therefore, we know that $u_n$ is bounded in $L^q(\Omega)$, and so does $u \in L^q(\Omega)$.

**Theorem 9.** Suppose that $f \in L^m(\Omega)$, $(1 + \alpha')N/(1 + \alpha')((N - 2)\alpha') \leq m < 2N/(N + 2 + (N - 2)\alpha')$, and $0 < \alpha' \leq \alpha(x) < \alpha' < 1$. Then problem (1) has a solution $u \in W_0^{1,q}(\Omega)$, $q = Nm(1 + \alpha')/(N - m(1 - \alpha'))$.

**Proof.** The lines of our proof are that we can prove that $u_n$ is bounded in $L^q(\Omega)$ (with $q$ as in the statement), the existence of a solution $u$ in $W_0^{1,q}(\Omega)$ of (1) will be proved by passing to the limit in (7) as in the proof of Theorem 7. To prove that $u_n$ is bounded in $W_0^{1,q}(\Omega)$, we begin by proving that it is bounded in $L^q(\Omega)$, with $s = Nm(1 + \alpha')/(N - 2m)$. To attain this goal, we choose $u_n^{2^* - 1}$ as a test function in (7) as in the statement of Lemma 8, where $(1 + \alpha')/2 \leq \delta < 1$; however, $Vu_n^{2^* - 1}$ will be singular at $u_n = 0$, and therefore, we choose $(u_n + \epsilon)^{2^* - 1} - \epsilon^{2^* - 1}$ as a test function in (7), where $\epsilon < 1/n$ for $n$ fixed; by (2) and $f_n \leq f$, we have that

\begin{align}
\gamma(2\delta - 1) \int_{\Omega} \left| Vu_n \right|^2 (u_n + \epsilon)^{2^* - 2} dx
\end{align}

By Sobolev Embedding Theorem ($H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$) on the left hand side, it follows that

\begin{align}
\int_{\Omega} \left| Vu_n \right|^2 (u_n + \epsilon)^{2^* - 2} dx = \int_{\Omega} \left| \frac{\gamma}{(u_n + \epsilon)^{2^* - 2}} dx \right|^2 dx
\end{align}

\begin{align}
\leq \int_{\Omega} \gamma(2\delta - 1) \frac{Sy}{(2\delta - 1)} \left| Vu_n \right|^2 (u_n + \epsilon)^{2^* - 2} dx
\end{align}

\begin{align}
\leq \int_{\Omega} \left( u_n + \epsilon \right)^{2^* - 2} dx
\end{align}

\begin{align}
\geq S \frac{Sy}{(2\delta - 1)} \left( \left( u_n + \epsilon \right)^2 \right)^{2/2'} dx
\end{align}
where $S$ is the best constant of the Sobolev Embedding Theorem. Combining (41) with (42), we have that

$$
\frac{S \gamma (2\delta - 1)}{\delta^2} \left( \int_{\Omega} f \left( u_n^\delta - e^{\delta} \right)^{2*} \, dx \right)^{2/2*} \\
\leq \int_{\Omega} f \left( u_n + e \right)^{2\delta - 1 - \alpha} \, dx \\
+ \int_{\Omega} f \left( u_n + e \right)^{2\delta - 1 - \alpha^*} \, dx.
$$

(43)

Using Hölder’s inequality on the right hand side, we get

$$
\frac{S \gamma (2\delta - 1)}{\delta^2} \left( \int_{\Omega} f \left( u_n + e \right)^{2\delta - 1 - \alpha} \, dx \right)^{2/2*} \\
\leq \|f\|_{L^m(\Omega)} \left( \int_{\Omega} \left( u_n + e \right)^{2\delta - 1 - \alpha^*} \, dx \right)^{1/m'} \\
+ |\Omega|^{(\alpha - \alpha^*)/(2\delta - 1 - \alpha^*)} \|f\|_{L^m(\Omega)}^{1/m'} \\
\cdot \left( \int_{\Omega} \left( u_n + e \right)^{2\delta - 1 - \alpha^*} \, dx \right)^{(2\delta - 1 - \alpha^*)/(2\delta - 1 - \alpha^*)}.
$$

(44)

Letting $\varepsilon \to 0$, we get (35); that is,

$$
\left( \int_{\Omega} u_n^{2*\delta} \, dx \right)^{2/2*} \leq \frac{\delta^2}{S \gamma (2\delta - 1)} \|f\|_{L^m(\Omega)} \\
\cdot \left( \int_{\Omega} u_n^{2\delta - 1 - \alpha^*} \, dx \right)^{1/m'} \\
+ |\Omega|^{(\alpha - \alpha^*)/(2\delta - 1 - \alpha^*)} \|f\|_{L^m(\Omega)}^{1/m'} \\
\cdot \left( \int_{\Omega} u_n^{2\delta - 1 - \alpha^*} \, dx \right)^{(2\delta - 1 - \alpha^*)/(2\delta - 1 - \alpha^*)},
$$

(45)

where $\delta$ is chosen in such a way that $2*\delta = (2\delta - 1 - \alpha^*)m'$; that is,

$$
\delta = \frac{(1 + \alpha^-) (N - 2) m}{2 (N - 2m)}.
$$

(46)

If $m = (1 + \alpha^-)N/((1 + \alpha^-)(N - 2) + 2(1 + \alpha^*))$, choosing $\delta = (1 + \alpha^-)/2$ in (43), and letting $\varepsilon \to 0$, we have that

$$
\left( \int_{\Omega} u_n^{2*\delta} \, dx \right)^{2/2*} \leq \frac{\delta^2}{S \gamma (2\delta - 1)} \left( \int_{\Omega} f u_n^{2\delta - 1 - \alpha} \, dx + \int_{\Omega} f \, dx \right).
$$

(47)

Using Hölder’s inequality and Young’s inequality, we get

$$
\left( \int_{\Omega} u_n^{2*\delta} \, dx \right)^{2/2*} \leq \frac{\delta^2}{S \gamma (2\delta - 1)} \left( \int_{\Omega} f \, dx \right)^{1/m'} \\
\cdot \left( \int_{\Omega} u_n^{2\delta - 1 - \alpha^*} \, dx \right)^{1/m'} \\
\leq \frac{\delta^2}{S \gamma (2\delta - 1)} \left( \int_{\Omega} f \, dx \right)^{2/2*} + \left( \int_{\Omega} u_n^{2\delta - 1 - \alpha^*} \, dx \right)^{1/m'} \\
\cdot \left( \int_{\Omega} \left( u_n + e \right)^{2\delta - 1 - \alpha^*} \, dx \right)^{(2\delta - 1 - \alpha^*)/(2\delta - 1 - \alpha^*)}.
$$

(48)

where $\varepsilon = S \gamma (2\delta - 1)/2\delta^2 \|f\|_{L^m(\Omega)}$. Thus we have that

$$
\left( \int_{\Omega} u_n^{2*\delta} \, dx \right)^{2/2*} \leq \frac{2\delta^2}{S \gamma (2\delta - 1)} \left( \|f\|_{L^m(\Omega)} \right)^{1/m'} \\
\cdot \left( \int_{\Omega} u_n^{2\delta - 1 - \alpha^*} \, dx \right)^{2/2*} + \left( \int_{\Omega} \left( u_n + e \right)^{2\delta - 1 - \alpha^*} \, dx \right)^{1/m'}.
$$

(49)

Therefore we obtain that $u_n$ is bounded in $L^{N(1+\alpha^-)/(N-2)}(\Omega)$, where $N(1+\alpha^-)/(N-2)$ is the value of $s$ for $m = (1 + \alpha^-)N/((1 + \alpha^-)(N - 2) + 2(1 + \alpha^*))$.

If $(1 + \alpha^-)N/((1 + \alpha^-)(N - 2) + 2(1 + \alpha^*)) < m < 2N/(N + 2 + (N - 2)\alpha^-)$, it is clear that the inequality on $m$ holds true if and only if $(1 + \alpha^-)/2 < \delta < 1$, starting from (35) and arguing as in the proof of Lemma 8, we also get that $u_n$ is bounded in $L^s(\Omega)$ with $s =Nm(1+\alpha^-)/(N-2m)$.

The right hand side of (41) is bounded with respect to $n$ (and $\varepsilon$, which we take smaller than 1) by using the estimate on $u_n$ in $L^s(\Omega)$ and the choice of $\delta$.

Since $\delta < 1$,

$$
\int_{\Omega} \frac{|\nabla u_n|^2}{(u_n + \varepsilon)^{2-2\delta}} \, dx \leq \int_{\Omega} |\nabla u_n|^2 \left( u_n + \varepsilon \right)^{2\delta - 2} \, dx \leq C.
$$

(50)

If $q = Nm(1 + \alpha^-)/(N - m(1 + \alpha^-)) < 2$, by Hölder’s inequality, we have that

$$
\int_{\Omega} |\nabla u_n|^q \, dx = \int_{\Omega} \frac{|\nabla u_n|^q}{(u_n + \varepsilon)^{(1-\delta)q}} \left( u_n + \varepsilon \right)^{(1-\delta)q} \, dx \\
\leq \left( \int_{\Omega} \frac{|\nabla u_n|^q}{(u_n + \varepsilon)^{(1-\delta)q}} \, dx \right)^{q/2} \\
\cdot \left( \int_{\Omega} \left( u_n + \varepsilon \right)^{2(1-\delta)(2-q)/2} \, dx \right)^{1-q/2}.
$$
\[
\left( \frac{\|u_n\|^2_{H^1 \Omega}}{(u_n + \epsilon)^{2(1-\delta)}} \right)^{q/2} \cdot \left( \int_{\Omega} (u_n + \epsilon)^{2(1-\delta)(1/2-q)} \, dx \right)^{1-q/2} \leq C \left( \int_{\Omega} (u_n + \epsilon)^{2(1-\delta)(1/2-q)} \, dx \right)^{1-q/2}.
\]

(51)

The choice of \( \delta \) and the value of \( q \) are such that \( 2(1-\delta)q/(2-q) = s \), so that the right hand side of (51) is bounded with respect to \( n \) and \( \epsilon \). Hence, \( u_n \) is bounded in \( W^{1,q}_0(\Omega) \).

**Theorem 10.** Suppose that \( f \in L^m(\Omega) \), \( 1/(2\delta - \alpha^-) < m < ((1 + \alpha^+)
/N/(1 + \alpha^-))(N - 2) + 2(1 + \alpha^+) \) with \((1 + \alpha^-)/2 < \delta < (1 + \alpha^+)/2, \) and \( 0 < \alpha^- \leq \alpha(x) \leq \alpha^+ < 1 \). Then problem (1) has a solution \( u \in W^{1,q}_0(\Omega), q = Nm(1 + \alpha^-)/(N-m(1-\alpha^-)) \).

**Proof.** The lines of our proof are similar to that in the proof of Theorem 9. We also begin by proving that \( u_n \) is bounded in \( L^s(\Omega) \), with \( s = Nm(1 + \alpha^+)/(N - 2m) \). To this aim, we also choose \( (u_n + \epsilon)^{2\delta-1-\epsilon^{2\delta-1}} \) as a test function in (7), where \( (1 + \alpha^-)/2 < \delta < (1 + \alpha^+)/2, \) \( \epsilon < 1/n \) for \( n \) fixed. Since \( f_n \leq f \) and (2), we have that

\[
y(2\delta - 1) \int_{\Omega} (u_n + \epsilon)^{2\delta-2} \, dx \leq \int_{\Omega} f(u_n + \epsilon)^{2\delta-1-\alpha^-} \, dx + \int_{\Omega} f(u_n + \epsilon)^{2\delta-1-\alpha^+} \, dx.
\]

(52)

Using Sobolev Embedding Theorem \((H^1_0(\Omega) \hookrightarrow L^{2\gamma}(\Omega))\) on the left hand side, it follows that

\[
\frac{Sy}{\delta^2} \left( \int_{\Omega} (u_n + \epsilon)^{2\delta-2} \, dx \right)^{2/\gamma} \leq \int_{\Omega} f(u_n + \epsilon)^{2\delta-1-\alpha^-} \, dx + \int_{\Omega} f(u_n + \epsilon)^{2\delta-1-\alpha^+} \, dx.
\]

(53)

where \( S \) is the constant of the Sobolev Embedding Theorem. Using Hölder's inequality and Lemma 5 on the right hand side, we get

\[
\frac{Sy}{\delta^2} \left( \int_{\Omega} (u_n + \epsilon)^{2\delta-2} \, dx \right)^{2/\gamma} \leq \int_{\Omega} f(u_n + \epsilon)^{2\delta-1-\alpha^-} \, dx + \int_{\Omega} \frac{f}{(u_n + \epsilon)^{2\delta+\alpha^-+1}} \, dx
\]

\[
\leq \int_{\Omega} f(u_n + \epsilon)^{2\delta-1-\alpha^-} \, dx + \int_{\Omega} \frac{f}{u_n^{2\delta+\alpha^-+1}} \, dx
\]

\[
\leq \|f\|_{L^m(\Omega)} \left( \int_{\Omega} (u_n + \epsilon)^{2\delta-1-\alpha^-} \, dx \right)^{1/m'} + \|f\|_{L^m(\Omega)} \left( \int_{\Omega} (u_n + \epsilon)^{2\delta-1-\alpha^+} \, dx \right)^{1/m'} + C\|f\|_{L^m(\Omega)}.
\]

(54)

Letting \( \epsilon \to 0 \), we obtain that

\[
\left( \int_{\Omega} u_n^{2\gamma} \, dx \right)^{2/\gamma} \leq \frac{\delta^2}{S_y(2\delta - 1)} \|f\|_{L^m(\Omega)} + C
\]

(55)

where \( \delta \) is chosen in such a way that \( 2^\gamma \delta = (2\delta - 1 - \alpha^-)m' \); that is

\[
\delta = \frac{(1 + \alpha^-)(N - 2) m}{2(N - 2m)}.
\]

(56)

If \( 1 < m < (1 + \alpha^+)N/((1 + \alpha^-)(N-2) + 2(1 + \alpha^+)) \), it is clear that the inequality on \( m \) holds true if and only if \( (1 + \alpha^-)/2 < \delta < (1 + \alpha^+)/2 \), and arguing as to the case \( m = (1 + \alpha^+)N/((1 + \alpha^-)(N-2) + 2(1 + \alpha^+)) \) in the proof of Theorem 9, we also obtain that \( u_n \) is bounded in \( L^s(\Omega) \), with \( s = Nm(1 + \alpha^-)/(N - 2m) \). Since \( \delta < 1 \),

\[
\int_{\Omega} \frac{\|u_n\|^2}{(u_n + \epsilon)^{2\delta-2}} \, dx = \int_{\Omega} \|u_n\|^2 (u_n + \epsilon)^{2\delta-2} \, dx \leq C.
\]

(57)

If \( q = Nm(1 + \alpha^-)/(N - m(1 - \alpha^-)) < 2 \), similarly to the proof of Theorem 9, we have by Hölder's inequality that

\[
\int_{\Omega} \|u_n\|^q \, dx \leq C \left( \int_{\Omega} (u_n + \epsilon)^{2(1-\delta)(1/2-q)} \, dx \right)^{1-q/2}.
\]

(58)

Since the choice of \( \delta \) and the value of \( q \), the right hand side of the above inequality is bounded with respect to \( n \) and \( \epsilon \). Hence, \( u_n \) is bounded in \( W^{1,q}_0(\Omega) \).

**4. The Case** \( 1 < \alpha^- \leq \alpha(x) \leq \alpha^+ \)

The case \( 1 < \alpha^- \leq \alpha(x) \leq \alpha^+ \) has many analogies with the case \( 0 < \alpha^- < \alpha^+ < 1 \). In this case, we can also prove that \( u_n \) is bounded in \( H^1_0(\Omega) \) only if \( f \) is more regular than \( L^2(\Omega) \) and \( \alpha^- \) and \( \alpha^+ \) are close to 1.

**Lemma 11.** Suppose that \( f \in L^m(\Omega) \) \( (m > 1) \), and let \( u_n \) be the solution of (7) with \( 1 < \alpha^- < \alpha^+ < 2 - 1/m \). Then \( u_n \) is bounded in \( H^1_0(\Omega) \).
Proof. Taking $u_n$ as a test function in (7), by (2), we obtain that
\[
\gamma \int_{\Omega} |\nabla u_n|^2 \, dx \leq \int_{\Omega} \frac{f}{u_n^{\gamma(x)-1}} \, dx.
\]  
(59)

Using Lemmas 2 and 3, we know that $u_n \geq u_1$ and there exists a constant $M > 0$ s.t. $u_1 \leq M$. Hence $(M/u_1)^{\alpha(x)-1} \leq (M/u_1)^{\alpha-1}$, and we have
\[
\gamma \int_{\Omega} |\nabla u_n|^2 \, dx \leq \int_{\Omega} \frac{f}{u_n^{\alpha-1}} \, dx \leq \int_{\Omega} \frac{f}{u_1^{\alpha-1}} \, dx \leq \left(1 + M^{\alpha-\alpha} \right) \int_{\Omega} \frac{f}{u_1^{\alpha-1}} \, dx.
\]  
(60)

Using Hölder’s inequality on the right hand side, and Lemma 5, we obtain
\[
\gamma \int_{\Omega} |\nabla u_n|^2 \, dx \leq \left(1 + M^{\alpha-\alpha} \right) \int_{\Omega} \frac{f}{u_1^{\alpha-1}} \, dx \leq \left(1 + M^{\alpha-\alpha} \right) \|f\|_{L^m(\Omega)} \left( \int_{\Omega} u_1^{\alpha(x) - 1} \, dx \right)^{1/m'} \leq C \left(1 + M^{\alpha-\alpha} \right) \|f\|_{L^m(\Omega)}.
\]  
(61)

Therefore, $u_n$ is bounded in $H^1_0(\Omega)$. \hfill \Box

Once we have the boundedness of $u_n$, we can prove the following existence theorem along the lines of Theorem 7.

**Theorem 12.** Suppose that $f \in L^m(\Omega)$ $(m > 1)$ and $1 < \alpha^- < \alpha^+ < 2 - 1/m$. Then problem (1) has a solution $u$ in $H^1_0(\Omega)$.

The summability of $u$ can be proved along the lines of Lemma 8 with little changes.

**Lemma 13.** Suppose that $f \in L^m(\Omega)$ $(m > 1)$ and $1 < \alpha^- < \alpha^+ < 2 - 1/m$. Then the solution $u$ of (1) given by Theorem 12 is such that

(i) if $m > N/2$, then $u \in L^\infty(\Omega)$;

(ii) if $N(1 + \alpha^+)/(1 + \alpha^-)(N - 2) + 2(1 + \alpha^+) < m < N/2$, then $u \in L^1(\Omega)$, $s = Nm(1 + \alpha^-)/(N - 2m)$.

**Proof.** The proof of (i) is similar to the proof of Lemma 8(i); we omit the details here.

To prove (ii) we choose $u_n^{\delta - 1}$ as a test function with $\delta \geq (1 + \alpha^+)/2$ in (7); similarly to the proof of Lemma 8, we obtain that
\[
\frac{S\gamma (2\delta - 1)}{\delta^2} \left( \int_{\Omega} u_n^{2\delta} \, dx \right)^{2/\delta^2} \leq \int_{\Omega} f u_n^{2\delta - 1 - \alpha^+} \, dx + \int_{\Omega} f u_n^{\delta - 1 + \alpha^-} \, dx.
\]  
(62)

If $m = N(1 + \alpha^+)/((1 + \alpha^-)(N - 2) + 2(1 + \alpha^+))$, choosing $\delta = (1 + \alpha^+)/2$ in (62), by Hölder’s inequality, we get
\[
\frac{S\gamma (2\delta - 1)}{\delta^2} \left( \int_{\Omega} u_n^{2\delta} \, dx \right)^{2/\delta^2} \leq \|f\|_{L^m(\Omega)} \left( \int_{\Omega} u_n^{(2\delta - 1 - \alpha^+)/m'} \, dx \right)^{1/m'} + |\Omega|^{1/m} \|f\|_{L^m(\Omega)}.
\]  
(63)

We choose $\delta$ in such a way that $2\delta = (2\delta - 1 - \alpha^+)/m'$; that is, $\delta = (1 + \alpha^-)m/(N - 2)/2(1 + \alpha^-)$. Since $m = N(1 + \alpha^+)/((1 + \alpha^-)(N - 2) + 2(1 + \alpha^-))$ being $2/2^+ > 1/m'$, it follows that $u_\delta$ is bounded in $L^{(1 + \alpha^-)}/(N - 2, (\Omega))$.

If $N(1 + \alpha^+)/((1 + \alpha^-)(N - 2) + 2(1 + \alpha^-)) < m < N/2$, starting from inequality (62) and Hölder’s inequality, we have that
\[
\frac{S\gamma (2\delta - 1)}{\delta^2} \left( \int_{\Omega} u_n^{2\delta} \, dx \right)^{2/\delta^2} \leq \|f\|_{L^m(\Omega)} \cdot \left( \int_{\Omega} u_n^{(2\delta - 1 - \alpha^-)/m'} \, dx \right)^{1/m'} \cdot \left( \int_{\Omega} u_n^{(2\delta - 1 - \alpha^-)/m'} \, dx \right)^{(2\delta - 1 - \alpha^-)/(2\delta - 1 - \alpha^-)}
\]  
(64)

We also choose $\delta$ in such a way that $2\delta = (2\delta - 1 - \alpha^-)/m'$, which yields that $\delta > (1 + \alpha^-)/2$, if and only if $m > N(1 + \alpha^+)/((1 + \alpha^-)(N - 2) + 2(1 + \alpha^-))$, and that $2\delta = s$. So, since $2/2^+ > 1/m'$ being $m < N/2$, we have the boundedness of $u_\delta$ in $L^1(\Omega)$, and so does $u \in L^1(\Omega)$. \hfill \Box

Moreover, we can only prove that a positive power of $u_n$ is bounded in $H^1_0(\Omega)$ only if $f$ is more regular than $L^1(\Omega)$ and $\alpha^+$ is close to $\alpha^-$ and we only have the boundedness of $u_n$ in $H^1_{loc}(\Omega)$.

**Lemma 14.** Suppose that $f \in L^m(\Omega)$ $(m > 1)$, and let $u_n$ be the solution of (7) with $1 < \alpha^- < \alpha(x) < \alpha^+ < 1 + 1/m$. Then $u_n^{(1+m)/2}$ is bounded in $H^1_{loc}(\Omega)$, and $u_n$ is bounded in $H^1_{loc}(\Omega)$ and in $L^1(\Omega)$, with $s = N(1 + \alpha^-)/(N - 2)$.

**Proof.** Taking $u_n^{-1}$ as a test function in (7), since $u_n^{\alpha^-}/(u_n + 1/n)^{\alpha^-} \leq 1$ and $f_n \leq f$, by Lemma 5 and (2), we get that
\[
\gamma \int_{\Omega} |\nabla u_n|^{2\alpha^-} \, dx \leq \int_{\Omega} \frac{f u_n^{\alpha^-}}{(u_n + 1/n)^{\alpha^-}} \, dx + \int_{\Omega} f u_n^{\alpha^-} \, dx \leq |\Omega|^{1/m} \|f\|_{L^m(\Omega)} + \|f\|_{L^m(\Omega)} \int_{\Omega} \frac{1}{u_n^{\alpha^-} \, dx} \leq |\Omega|^{1/m} \|f\|_{L^m(\Omega)} + C \|f\|_{L^m(\Omega)}.
\]  
(65)
Since
\[ \int_{\Omega} |\nabla u_n|^2 u_n^{\alpha-1} dx = \frac{4}{(1 + \alpha)^2} \int_{\Omega} |\nabla u_n^{(1+\alpha)/2}|^2 dx, \] (66)
we have that
\[ \frac{4\alpha - \gamma}{(1 + \alpha)^2} \int_{\Omega} |\nabla u_n^{(1+\alpha)/2}|^2 dx \leq (C + |\Omega|^{1-1/m}) \|\nabla u_n\|_{L^m(\Omega)}. \] (67)

Thus, we have that \( u_n^{(1+\alpha)/2} \) is bounded in \( H_0^1(\Omega) \).

Applying Sobolev Embedding Theorem to \( u_n^{(1+\alpha)/2} (H_0^1(\Omega) \hookrightarrow L^2(\Omega)) \), we have that
\[ S \left( \int_{\Omega} |u_n^{(1+\alpha)/2}|^2 dx \right)^{2/2'} \leq \int_{\Omega} |\nabla u_n^{(1+\alpha)/2}|^2 dx, \] (68)
where \( S \) is the best constant of the Sobolev embedding. Since the boundedness of \( u_n^{(1+\alpha)/2} \) in \( H_0^1(\Omega) \), we thus have the boundedness of \( u_n \) in \( L^{2'}(\Omega) \).

To prove the boundedness of \( u_n \) in \( H_0^1(\Omega) \), we choose \( u_n\varphi^2 \) as a test function in (7), where \( \varphi \in C_0^1(\Omega), \Omega' = \{ x \in \Omega, \varphi \neq 0 \} \). By (2) and (12), we have that
\[ \frac{\gamma}{2} \int_{\Omega} |\nabla u_n| ^2 \varphi^2 dx + 2 \int_{\Omega} (M(x) \nabla u_n) \cdot \nabla u_n \varphi dx \leq \frac{1}{\min \left\{ C_\Omega^\alpha, C_\Omega^{\alpha - 1} \right\}} \int_{\Omega} f \varphi^2 dx \] (69)

By Young's inequality, we get that
\[ 2\beta \int_{\Omega} \nabla u_n \cdot \nabla u_n \varphi \varphi dx \leq \frac{\gamma}{2} \int_{\Omega} |\nabla u_n|^2 \varphi^2 dx + \frac{2\beta^2}{\gamma} \int_{\Omega} |\nabla \varphi|^2 u_n^2 dx. \] (70)

Since \( u_n \) is bounded in \( L'(\Omega) \) (where \( s \geq 2 \)), by Hölder inequality, we obtain that
\[ \frac{\gamma}{2} \int_{\Omega} |\nabla u_n| ^2 \varphi^2 dx \leq \frac{1}{\min \left\{ C_\Omega^\alpha, C_\Omega^{\alpha - 1} \right\}} \int_{\Omega} f \varphi^2 dx \] (71)
\[ \int_{\Omega} f \varphi^2 dx \leq \frac{2\beta^2}{\gamma} \int_{\Omega} |\nabla \varphi|^2 u_n^2 dx \]
and hence \( u_n \) is bounded in \( H_0^1(\Omega) \).

Once we have the boundedness of \( u_n \), we can prove the following existence theorem along the lines of Theorem 7.

Theorem 15. Suppose that \( f \) is a nonnegative function in \( L^m(\Omega) \) (\( m > 1 \)), \( f \neq 0 \), \( 1 < \alpha^- \leq \alpha(x) \leq \alpha^+ \) and \( \alpha^- - \alpha^+ < 1 - 1/m \). Then problem (1) has a solution \( u \) in \( H_0^1(\Omega) \).

Furthermore, \( u^{(1+\alpha)/2} \) belongs to \( H_0^1(\Omega) \).

The summability of \( u \) can be proved along the lines of Lemma 8 with little changes.

Lemma 16. Suppose that \( f \in L^m(\Omega) \), \( 1 < \alpha^- \leq \alpha(x) \leq \alpha^+ \), and \( \alpha^- - \alpha^+ < 1 - 1/m \). Then the solution \( u \) of (1) given by Theorem 15 is such that

(i) if \( m > N/2 \), then \( u \in L^{\infty}(\Omega) \);

(ii) if \( N(1 + \alpha^-)/(1 + \alpha^+)(N - 2) + 2(1 + \alpha^+) \leq m < N/2 \), then \( u \in L^s(\Omega), s = Nm(1 + \alpha^+)/((N - 2)m) \).

Proof. The proof of (i) is similar to the proof of Lemma 8(i); we omit the details here.

To prove (ii), we choose \( u_n^{(1+\alpha)/2} (\delta \geq (1 + \alpha^+)/2) \) as a test function in (7); applying (2) and Sobolev Embedding Theorem, we have that
\[ \frac{2\beta^2}{\gamma} \int_{\Omega} |\nabla \varphi|^2 u_n^2 dx \leq \int_{\Omega} f u_n^{2\alpha - 1} dx + \int_{\Omega} f u_n^{2\alpha - 1} \varphi^2 dx. \] (72)
If $m = N(1 + \alpha^+)/((1 + \alpha^-)(N - 2) + 2(1 + \alpha^+))$, we choose $\delta = (1 + \alpha^+)/2$; using Hölder inequality, we have that
\[
\frac{S y (2\delta - 1)}{\delta^2} \left( \int_\Omega u_n^{2\delta-1} dx \right)^{2/2'} \leq \|f\|_{L^m(\Omega)} \left( \int_\Omega u_n^{(2\delta-1-2\alpha^-)m'} dx \right)^{1/m'} + |\Omega|^{1-1/m} \|f\|_{L^m(\Omega)}.
\] (73)

We also choose $\delta$ in such a way that $2\delta = (2\delta - 1 - \alpha^-)m'$, since $m = N(1+\alpha^+)/((1+\alpha^-)(N-2)+2(1+\alpha^+))$ being $2/2' > 1/m'$, and we have the boundedness of $u_n$ in $L^{N(1+\alpha^+)/((N-2)+2(1+\alpha^+))}(\Omega)$ which is the value of $s$ for $m = N(1+\alpha^+)/((1+\alpha^-)(N-2)+2(1+\alpha^+))$. If $N(1+\alpha^+)/((1+\alpha^-)(N-2)+2(1+\alpha^+)) < m < N/2$, starting from (72) using Hölder inequality, we get that
\[
\frac{S y (2\delta - 1)}{\delta^2} \left( \int_\Omega u_n^{2\delta-1} dx \right)^{2/2'} \leq \|f\|_{L^m(\Omega)} \left( \int_\Omega u_n^{(2\delta-1-2\alpha^-)m'} dx \right)^{1/m'} + \|f\|_{L^m(\Omega)} (\int_\Omega u_n^{(2\delta-1-2\alpha^-)m'} dx)^{1/m'}.
\] (74)

We also choose $\delta$ in such a way that $2\delta = (2\delta - 1 - \alpha^-)m'$, which yields that $\delta > (1 + \alpha^+)/2$, if and only if $m > N(1 + \alpha^+)/((1 + \alpha^-)(N - 2) + 2(1 + \alpha^+))$, and we have the boundedness of $u_n$ in $L^{1}(\Omega)$, and so does $u \in L^{1}(\Omega)$. \(\square\)

5. The Case $0 < \alpha^- < 1 < \alpha^+$

If $\alpha^- < 1 < \alpha^+$, the boundedness of $u_n$ in $H_0^1(\Omega)$ can also be obtained only if $f$ is more regular than $L^1(\Omega)$, and the proof has many analogies with the case $0 < \alpha^- < \alpha^+ < 1$. We have the following results.

Lemma 17. Suppose that $f \in L^m(\Omega)$, with $m = 2N/(N + 2 + (N - 2)\alpha^-)$, and let $u_n$ be the solution of (7) with $0 < \alpha^- < 1 < \alpha^+ < 2 - 1/m$. Then the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$.

Proof. We choose $u_n$ as a test function in (7), by Hölder’s inequality, (2), and Lemma 5, since $f_n \leq f$, we have that
\[
\int_\Omega |\nabla u_n|^2 dx \leq \|f\|_{L^m(\Omega)} \left( \int_\Omega u_n^{1-\alpha^-m'} dx \right)^{1/m'} + \|f\|_{L^m(\Omega)} \left( \int_\Omega u_n^{1-\alpha^-m'} dx \right)^{1/m'}.
\] (80)

So the boundedness of $u_n$ in $L^2(\Omega)$ is obtained; using the estimate and (75) again, we have the estimate of $u_n$ in $H_0^1(\Omega)$. \(\square\)

Once the boundedness of $u_n$ in $H_0^1(\Omega)$ is obtained, we can prove the following existence theorem.

Theorem 18. Suppose that $f \in L^m(\Omega)$ with $m = 2N/(N + 2 + (N - 2)\alpha^-)$, and $0 < \alpha^- < 1 < \alpha^+ < 2 - 1/m$. Then problem (1) has a solution $u$ in $H_0^1(\Omega)$.

Lemma 19. Suppose that $f \in L^m(\Omega)$ with $m \geq 2N/(N + 2 + (N - 2)\alpha^-)$, and $\alpha^- < 1 < \alpha^+ < 2 - 1/m$. Then the solution $u$ of (1) given by Theorem 18 is such that
(i) if $m > N/2$, then $u \in L^\infty(\Omega)$;
(ii) if $2N/(N+2+(N-2)\alpha^-) \leq m < N/2$, then $u \in L^s(\Omega),
\quad s = Nm(1+\alpha^-)/(N-2m).

Proof. The proof of (i) is similar to that for Lemma 8(i), and we also omit the details here.
To prove (ii), if $N(1+\alpha^+)/(1+\alpha^-)(N-2)+2(1+\alpha^+) < m < N/2$, the proof is identical to that for Lemma 13, and we also omit it here.
If $m = 2N/(N+2+(N-2)\alpha^-)$, we can prove the results by Sobolev Embedding Theorem.
If $2N/(N+2+(N-2)\alpha^-) < m < N(1+\alpha^+)/(1+\alpha^-)(N-2)+2(1+\alpha^+)$, we choose $1 < \delta < (1+\alpha^+)/2$, and we use once again $u_n^{(\delta-1)}$ as a test function in (7). Using $\delta > 1 > (1+\alpha^-)/2$, as well as Hölder’s inequality, Sobolev Embedding Theorem, Lemma 5, and (2), we get
\[
\frac{\delta^2}{2} \left( \int_\Omega ^{2/\delta} \left( \int_\Omega ^{2\delta-1} \| f \|_{L^m(\Omega)}^{1/m} + C \left\| f \|_{L^m(\Omega)} \right. \right)
\]
\[
\leq \left\| f \right\|_{L^m(\Omega)}^{1/m} \left( \int_\Omega ^{2\delta-1} \| u_n \|_{L^\infty(\Omega)} dx \right) \quad (81)
\]
The choice of $\delta$ in such a way that $2^\delta \delta = (2\delta - 1 - \alpha^-)m$ yields that $1 < \delta < (1+\alpha^+)/2$, if and only if $2N/(N+2+(N-2)\alpha^-) < m < N(1+\alpha^+)/(1+\alpha^-)(N-2)+2(1+\alpha^+)$, and that $2^\delta \delta = s$. The choice of $m < N/2$ implies that $2^\delta \delta > 1/m^\delta$. Thus we have the boundedness of $u_n \in L^\delta(\Omega)$, and do so limit it in $L^\delta(\Omega)$.

\[\square\]

**Competing Interests**
The authors declare that they have no competing interests.

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