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Research Article

Existence of Solutions to a Class of Semilinear Elliptic Problem with Nonlinear Singular Terms and Variable Exponent

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The authors of this paper prove the existence and regularity results for the homogeneous Dirichlet boundary value problem to the equation $-\operatorname{div}(M(x)\nabla u_n) = f(x)/u^{\alpha(x)}$ with $f \in L^m(\Omega)$ $(m \geqslant 1)$ and $\alpha(x) > 0$. The results show the dependence of the summability of f in some Lebesgue spaces and on the values of $\alpha(x)$.

1. Introduction

In this paper, we study the existence of solutions for the following semilinear elliptic problem with nonlinear singular terms and variable exponent:

$$\begin{gathered} -\operatorname{div}(M(x)\nabla u) = \frac{f(x)}{u^{\alpha(x)}}, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1}$$

where Ω is a bounded domain in $R^N (N \geqslant 2)$ with smooth boundary $\partial\Omega$, $\alpha(x)$ is a continuous function on $\overline{\Omega}$, $\alpha(x) > 0$, $\alpha^+ = \sup_{x\in\overline{\Omega}} \alpha(x)$, $\alpha^- = \inf_{x\in\overline{\Omega}} \alpha(x)$, f is a nonnegative function belonging to the Lebesgue space $L^m(\Omega)$, for some suitable $m \geqslant 1$, and M is a bounded positive definite matrix; that is, there exist $0 < \gamma \leqslant \beta$ such that

$$\begin{gathered} \gamma |\eta|^2 \leqslant (M(x)\eta)\cdot\eta, \\ |M(x)| \leqslant \beta, \end{gathered} \tag{2}$$

for every η in R^N, for almost every x in Ω.

Problem (1) has been widely applied in many areas, such as the contexts of chemical heterogeneous catalysts, non-Newtonian fluids, and also the theory of heat conduction in electrically conducting materials; see [1–4] for detailed discussion.

Problem (1) has been extensively studied in the past. In [5], Lazer and Mckenna dealt with model (1) with f, a continuous function; they proved that the solution was in $H_0^1(\Omega)$ if and only if $\alpha < 3$, while it was not in $C^1(\overline{\Omega})$ if $\alpha > 1$.

Later, Lair and Shaker in [6] studied the existence of solutions to the elliptic equation

$$\begin{gathered} -\Delta u = f(x) u^{-\alpha}, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{gathered} \tag{3}$$

They proved that problem (3) with $0 < \alpha < 1$ has a unique weak positive solution in $H_0^1(\Omega)$ if $f(x)$ is a nonnegative nontrivial function in $L^2(\Omega)$.

Moreover, the results of Lair and Shaker were generalized by Shi and Yao (see [7]); they studied the following problem:

$$\begin{gathered} -\Delta u = f(x,u), \quad x \in \Omega, \\ u = \varphi, \quad x \in \partial\Omega, \end{gathered} \tag{4}$$

where Ω is a bounded domain in R^N, $N \geqslant 2$, $\varphi \geqslant 0$ may take the value 0 on $\partial\Omega$, and $f(x,s)$ is possibly singular near $s = 0$. They proved the existence and the uniqueness of positive solutions without assuming monotonicity or strict positivity on $f(x,s)$.

Recently, Boccardo and Orsina in [8] studied the existence, regularity, and nonexistence of solutions for the following problem:

$$-\operatorname{div}(M(x)\nabla u) = \frac{f(x)}{u^{\alpha}}, \quad x \in \Omega, \qquad u = 0, \quad x \in \partial\Omega. \tag{5}$$

They discussed the dependence of the results on the summability of f and the values of α. For the other results of singular elliptic equations, see [9, 10]. In this paper, we generalize the results in [8] to the case when α is a variable exponent by applying the method of regularization, Schauder fixed point theorem, the integrability of solution to the approximate problem with $n = 1$, and a necessary compactness argument to overcome some difficulties arising from the singular terms with variable exponent.

2. Preliminaries

Firstly, we give the definition of weak solutions to problem (1).

Definition 1. A function $u \in H_0^1(\Omega)$ is called a weak solution of problem (1), if the following identity holds:

$$\int_\Omega M(x)\nabla u \nabla\varphi\, dx = \int_\Omega \frac{f}{u^{\alpha(x)}}\varphi\, dx, \quad \forall \varphi \in C_0^1(\Omega). \tag{6}$$

In order to prove our results, we will consider the following approximation problem:

$$-\operatorname{div}(M(x)\nabla u_n) = \frac{f_n}{(u_n + 1/n)^{\alpha(x)}}, \quad x \in \Omega, \qquad u_n = 0, \quad x \in \partial\Omega, \tag{7}$$

where $n \in N$, $f_n(x) = \min\{f(x), n\}$.

Lemma 2. *Problem (7) has a nonnegative solution u_n in $H_0^1(\Omega)\bigcap L^\infty(\Omega)$.*

Proof. Let $n \in N$ be fixed and ω a function in $L^2(\Omega)$. It is not difficult to prove that the following problem has a unique solution $v \in H_0^1(\Omega)\bigcap L^\infty(\Omega)$ (see [11, 12]):

$$-\operatorname{div}(M(x)\nabla v) = \frac{f_n}{(|\omega| + 1/n)^{\alpha(x)}}, \quad x \in \Omega, \qquad v = 0, \quad x \in \partial\Omega. \tag{8}$$

So, for any $\omega \in L^2(\Omega)$, we define the mapping $\Gamma : L^2(\Omega) \to H_0^1(\Omega)\bigcap L^\infty(\Omega)$ as $\Gamma(\omega) = v$. Taking v as a test function, we have, using (2),

$$\begin{aligned}\gamma\int_\Omega |\nabla v|^2\, dx &\leqslant \int_\Omega (M(x)\nabla v)\cdot\nabla v\, dx \\ &= \int_\Omega \frac{f_n v}{(|\omega| + 1/n)^{\alpha(x)}}\, dx \\ &\leqslant n^{1+\alpha^+}\int_\Omega |v|\, dx.\end{aligned} \tag{9}$$

By Poincaré inequality (on the left hand side) and Hölder's inequality (on the right hand side), we get that

$$\int_\Omega |v|^2\, dx \leqslant Cn^{1+\alpha^+}\left(\int_\Omega |v|^2\, dx\right)^{1/2}, \tag{10}$$

for some constant C independent of ω. This implies that

$$\|v\|_{L^2(\Omega)} \leqslant Cn^{1+\alpha^+}. \tag{11}$$

Therefore, the ball of $L^2(\Omega)$ of radius $Cn^{1+\alpha^+}$ is invariant under the mapping Γ. Since the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, we obtain that Γ is a compact operator and $\|v\|_{L^2(\Omega)} \leqslant C$. It is also easy to prove that Γ is continuous on $L^2(\Omega)$, so by Schauder's fixed point theorem, we get that there exists a function $u_n \in H_0^1(\Omega)$, for every fixed $n \in N$, such that $u_n = S(u_n)$; that is, problem (7) has a solution. Since $f_n/(u_n + 1/n)^{\alpha(x)} \geqslant 0$, the maximum principle implies that $u_n \geqslant 0$. Since the right hand side of (7) belongs to $L^\infty(\Omega)$, the result of Theorem 4.2 in [13] implies that $u_n \in L^\infty(\Omega)$. □

Lemma 3. *The sequence u_n is increasing with respect to n, $u_n > 0$ in Ω, and for every $\Omega' \subset\subset \Omega$ there exists $C_{\Omega'} > 0$ (independent of n) such that*

$$u_n(x) \geqslant C_{\Omega'} > 0 \quad \text{for every } x \in \Omega', \text{ for every } n \in N. \tag{12}$$

Proof. Due to $0 \leqslant f_n \leqslant f_{n+1}$ and $\alpha(x) > 0$, we have that

$$\begin{aligned}-\operatorname{div}(M(x)\nabla u_n) &= \frac{f_n}{(u_n + 1/n)^{\alpha(x)}} \\ &\leqslant \frac{f_{n+1}}{(u_n + 1/(n+1))^{\alpha(x)}}, \\ -\operatorname{div}(M(x)\nabla u_{n+1}) &= \frac{f_{n+1}}{(u_{n+1} + 1/(n+1))^{\alpha(x)}},\end{aligned} \tag{13}$$

so that

$$\begin{aligned}&-\operatorname{div}(M(x)\nabla(u_n - u_{n+1})) \\ &\leqslant f_{n+1}\left(\frac{1}{(u_n + 1/(n+1))^{\alpha(x)}} - \frac{1}{(u_{n+1} + 1/(n+1))^{\alpha(x)}}\right) = f_{n+1} \\ &\cdot \frac{(u_{n+1} + 1/(n+1))^{\alpha(x)} - (u_n + 1/(n+1))^{\alpha(x)}}{(u_n + 1/(n+1))^{\alpha(x)}(u_{n+1} + 1/(n+1))^{\alpha(x)}}.\end{aligned} \tag{14}$$

Choosing $(u_n - u_{n+1})_+ = \max\{u_n - u_{n+1}, 0\}$ as a test function, observing that

$$\left(\left(u_{n+1} + \frac{1}{n+1}\right)^{\alpha(x)} - \left(u_n + \frac{1}{n+1}\right)^{\alpha(x)}\right) \cdot (u_n - u_{n+1})_+ \leqslant 0, \tag{15}$$

and applying (2), we get that

$$0 \leqslant \gamma \int_{\Omega} \left|\nabla \left(u_n - u_{n+1}\right)_+\right|^2 dx \leqslant 0. \tag{16}$$

This implies that $(u_n - u_{n+1})_+ = 0$ a.e. in Ω; that is, $u_n \leqslant u_{n+1}$ for every $n \in N$. Since the sequence $\{u_n\}$ is increasing with respect to n, we only need to prove that (12) holds for u_1. Applying Lemma 2, we know that $u_1 \in L^{\infty}(\Omega)$; that is, there exists a constant C (depending only on Ω and N) such that

$$\|u_1\|_{L^{\infty}(\Omega)} \leqslant C \|f_1\|_{L^{\infty}(\Omega)} \leqslant C, \tag{17}$$

and then

$$-\operatorname{div}\left(M(x)\nabla u_1\right) = \frac{f_1}{\left(u_1+1\right)^{\alpha(x)}} \geqslant \frac{f_1}{(C+1)^{\alpha(x)}}. \tag{18}$$

Due to $f_1/(C+1)^{\alpha(x)} \geqslant 0$, $f_1/(C+1)^{\alpha(x)} \not\equiv 0$, the strong maximum principle implies that $u_1 > 0$ in Ω and (12) holds for u_1. The monotonicity of u_n implies that (12) holds for u_n. □

Remark 4. If u_n and v_n are two solutions of (7), following the lines of the proof of the first part in Lemma 3 we may show that $u_n \leqslant v_n$. By symmetry, this implies that the solution of (7) is unique.

Lemma 5. *The solution u_1 to problem (7) with $n = 1$ satisfies*

$$\int_{\Omega} u_1^{-r} dx < \infty, \quad \forall r < 1. \tag{19}$$

Proof. By $\min\{f(x), 1\}/(u_1+1)^{\alpha(x)} \leqslant 1$ and Lemma 2.2 in [14], we know that there exists $0 < \beta < 1$ such that $u_1 \in C^{1,\beta}(\overline{\Omega})$ and $\|u_1\|_{C^{1,\beta}} \leqslant C$, which implies that the gradient of u_1 exists everywhere; then Hopf Lemma in [15] shows that $\partial u_1(x)/\partial\nu > 0$ in $\overline{\Omega}$, where ν is the outward unit normal vector of $\partial\Omega$ at x. Moreover, following the lines of the proof of Lemma in [5], we get that

$$\int_{\Omega} u_1^{r} dx < \infty, \quad \text{iff } r > -1. \tag{20}$$

□

We know clearly that the estimates on u_n depend on f and $\alpha(x)$, and we will discuss it in different cases.

3. The Case $0 < \alpha^- \leqslant \alpha(x) \leqslant \alpha^+ < 1$

In this case, we obtain a priori estimates on u_n in $H_0^1(\Omega)$ only if f is more regular than $L^1(\Omega)$, and we have the following results.

Lemma 6. *Let u_n be the solution of (7) with $0 < \alpha^- \leqslant \alpha(x) \leqslant \alpha^+ < 1$, and suppose that $f \in L^m(\Omega)$ with $m = 2N/(N+2+(N-2)\alpha^-) = (2^*/(1-\alpha^-))'$. Then the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$.*

Proof. Choosing u_n as a test function in (7), by Hölder's inequality, (2), and the fact that $f_n \leqslant f$, we get

$$\begin{aligned}
\gamma \int_{\Omega} |\nabla u_n|^2 dx &\leqslant \int_{\Omega} \frac{f_n u_n}{\left(u_n + 1/n\right)^{\alpha(x)}} dx \\
&\leqslant \int_{\Omega} f u_n^{1-\alpha(x)} dx \leqslant \int_{\Omega} f u_n^{1-\alpha^-} dx + \int_{\Omega} f u_n^{1-\alpha^+} dx \\
&\leqslant \|f\|_{L^m(\Omega)} \left(\int_{\Omega} u_n^{(1-\alpha^-)m'} dx\right)^{1/m'} + \|f\|_{L^m(\Omega)} \\
&\quad \cdot \left(\int_{\Omega} u_n^{(1-\alpha^+)m'} dx\right)^{1/m'} \leqslant \|f\|_{L^m(\Omega)} \\
&\quad \cdot \left(\int_{\Omega} u_n^{(1-\alpha^-)m'} dx\right)^{1/m'} + \|f\|_{L^m(\Omega)} \\
&\quad \cdot \left(\left(\int_{\Omega} 1^{(1-\alpha^-)/(\alpha^+-\alpha^-)} dx\right)^{(\alpha^+-\alpha^-)/(1-\alpha^-)}\right. \\
&\quad \left. \cdot \left(\int_{\Omega} u_n^{(1-\alpha^-)m'} dx\right)^{(1-\alpha^+)/(1-\alpha^-)}\right)^{1/m'} = \|f\|_{L^m(\Omega)} \\
&\quad \cdot \left(\int_{\Omega} u_n^{(1-\alpha^-)m'} dx\right)^{1/m'} + |\Omega|^{(\alpha^+-\alpha^-)/(1-\alpha^-)m'} \\
&\quad \cdot \|f\|_{L^m(\Omega)} \left(\int_{\Omega} u_n^{(1-\alpha^-)m'} dx\right)^{(1-\alpha^+)/(1-\alpha^-)m'}.
\end{aligned} \tag{21}$$

By the assumption of m, we have $(1-\alpha^-)m' = 2^*$, and using Sobolev Embedding Theorem (on the left hand side), we have that

$$\begin{aligned}
\gamma S \left(\int_{\Omega} u_n^{2^*} dx\right)^{2/2^*} &\leqslant \gamma \int_{\Omega} |\nabla u_n|^2 dx \leqslant \|f\|_{L^m(\Omega)} \\
&\quad \cdot \left(\int_{\Omega} u_n^{2^*} dx\right)^{1/m'} + |\Omega|^{(\alpha^+-\alpha^-)/(1-\alpha^-)m'} \|f\|_{L^m(\Omega)} \\
&\quad \cdot \left(\int_{\Omega} u_n^{2^*} dx\right)^{(1-\alpha^+)/(1-\alpha^-)m'};
\end{aligned} \tag{22}$$

that is,

$$\begin{aligned}
\gamma S \left(\int_{\Omega} u_n^{2^*} dx\right)^{2/2^*} &\leqslant \|f\|_{L^m(\Omega)} \left(\int_{\Omega} u_n^{2^*} dx\right)^{(1-\alpha^-)/2^*} \\
&\quad + |\Omega|^{(\alpha^+-\alpha^-)/(1-\alpha^-)m'} \|f\|_{L^m(\Omega)} \left(\int_{\Omega} u_n^{2^*} dx\right)^{(1-\alpha^+)/2^*}.
\end{aligned} \tag{23}$$

Since $1-\alpha^+ < 1-\alpha^- < 2$, (22) yields the boundedness of u_n in $L^{2^*}(\Omega)$. By this estimate and (22), the conclusion follows. □

Once we have the boundedness of u_n, we can prove an existence result for (1).

Theorem 7. *Suppose that f is a nonnegative function in $L^m(\Omega)(f \not\equiv 0)$, with $m = 2N/(N+2+(N-2)\alpha^-) = (2^*/(1-\alpha^-))'$, and let $0 < \alpha^- \leqslant \alpha(x) \leqslant \alpha^+ < 1$. Then problem (1) has a solution $u \in H_0^1(\Omega)$ satisfying (6).*

Proof. Since u_n is bounded in $H_0^1(\Omega)$ by Lemma 6 and u_n converges to u pointwise in Ω (by Lemma 3), then we know that there exists $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \\ &\qquad \text{weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega), \\ &u_n \longrightarrow u \quad \text{a.e. in } \Omega, \\ &\nabla u_n \rightharpoonup \nabla u \quad \text{weakly in } L^2(\Omega). \end{aligned} \tag{24}$$

So we have that

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega (M(x)\nabla u_n)\cdot\nabla\varphi\, dx \\ &\quad = \int_\Omega (M(x)\nabla u)\cdot\nabla\varphi\, dx, \quad \forall\varphi \in C_0^1(\Omega). \end{aligned} \tag{25}$$

Since u_n satisfies (12), we get that

$$0 \leqslant \left|\frac{f_n\varphi}{(u_n+1/n)^{\alpha(x)}}\right| \leqslant \frac{\|\varphi\|_{L^\infty(\Omega)} f}{\min\{C_{\Omega'}^{\alpha^-}, C_{\Omega'}^{\alpha^+}\}}, \tag{26}$$

where $\Omega' = \{x : \varphi \neq 0\}$. Then by Lebesgue Dominated Convergence Theorem, we have that

$$\lim_{n\to\infty} \int_\Omega \frac{f_n\varphi}{(u_n+1/n)^{\alpha(x)}} dx = \int_\Omega \frac{f\varphi}{u^{\alpha(x)}} dx. \tag{27}$$

Since u_n is a solution of (7), this implies that

$$\begin{aligned} \int_\Omega (M(x)\nabla u_n)\nabla\varphi\, dx = \int_\Omega \frac{f_n\varphi}{(u_n+1/n)^{\alpha(x)}} dx, \\ \forall\varphi \in C_0^1(\Omega). \end{aligned} \tag{28}$$

Letting $n \to \infty$, combining (25) with (27), we get that

$$\int_\Omega (M(x)\nabla u)\cdot\nabla\varphi\, dx = \int_\Omega \frac{f\varphi}{u^{\alpha(x)}} dx, \tag{29}$$

which proves that (1) has a solution u in $H_0^1(\Omega)$. □

The summability of u depends on the summability of f, which is proved in the next Lemma.

Lemma 8. *Suppose that $f \in L^m(\Omega)$, $m \geqslant 2N/(N+2+(N-2)\alpha^-)$, and let $0 < \alpha^- \leqslant \alpha(x) \leqslant \alpha^+ < 1$. Then the solution u of (1) given by Theorem 7 is such that*

(i) *if $m > N/2$, then $u \in L^\infty(\Omega)$;*

(ii) *if $2N/(N+2+(N-2)\alpha^-) \leqslant m < N/2$, then $u \in L^s(\Omega)$, $s = Nm(1+\alpha^-)/(N-2m)$.*

Proof. To prove (i), let $k > 1$ and define $G_k(s) = (s-k)_+$. Taking $G_k(u_n)$ as a test function in (7), using (2), we get

$$\begin{aligned} &\gamma \int_\Omega |\nabla G_k(u_n)|^2 dx \\ &\quad \leqslant \int_\Omega (M(x)\nabla G_k(u_n))\cdot\nabla G_k(u_n)\, dx \\ &\quad = \int_\Omega \frac{f_n G_k(u_n)}{(u_n+1/n)^{\alpha(x)}} dx. \end{aligned} \tag{30}$$

Since $G_k(u_n) \neq 0$, it follows that

$$\gamma \int_\Omega |\nabla G_k(u_n)|^2 dx \leqslant \int_\Omega f G_k(u_n)\, dx. \tag{31}$$

Starting from inequality (31), Theorem 4.2 in [13] shows that there exists a constant C (independent of n), such that

$$\|u_n\|_{L^\infty(\Omega)} \leqslant C\|f\|_{L^m(\Omega)}, \tag{32}$$

which implies that u belongs to $L^\infty(\Omega)$.

To prove (ii), noting that if $m = 2N/(N+2+(N-2)\alpha^-)$, $s = 2N/(N-2) = 2^*$, since $u \in H_0^1(\Omega)$, the result when $m = 2N/(N+2+(N-2)\alpha^-)$ is true by Sobolev Embedding Theorem. If $2N/(N+2+(N-2)\alpha^-) < m < N/2$, letting $\delta > 1$ and choosing $u_n^{2\delta-1}$ as a test function in (7), using Hölder's inequality, we get that

$$\begin{aligned} &\gamma(2\delta-1)\int_\Omega |\nabla u_n|^2 u_n^{2\delta-2} dx \leqslant \int_\Omega \frac{f u_n^{2\delta-1}}{(u_n+1/n)^{\alpha(x)}} dx \\ &\quad \leqslant \int_{\{x\in\Omega, u_n\geqslant 1\}} \frac{f u_n^{2\delta-1}}{u_n^{\alpha^-}} dx + \int_{\{x\in\Omega, u_n<1\}} \frac{f u_n^{2\delta-1}}{u_n^{\alpha^+}} dx \\ &\quad \leqslant \int_\Omega f u_n^{2\delta-1-\alpha^-} dx + \int_\Omega f u_n^{2\delta-1-\alpha^+} dx \leqslant \|f\|_{L^m(\Omega)} \\ &\quad \cdot \left(\int_\Omega u_n^{(2\delta-1-\alpha^-)m'} dx\right)^{1/m'} + \|f\|_{L^m(\Omega)} \\ &\quad \cdot \left(\int_\Omega u_n^{(2\delta-1-\alpha^+)m'} dx\right)^{1/m'} = \|f\|_{L^m(\Omega)} \\ &\quad \cdot \left(\left(\int_\Omega u_n^{(2\delta-1-\alpha^-)m'} dx\right)^{1/m'} \right. \\ &\quad + |\Omega|^{(\alpha^+-\alpha^-)/(2\delta-1-\alpha^-)m'} \\ &\quad \left. \cdot \left(\int_\Omega u_n^{(2\delta-1-\alpha^-)m'} dx\right)^{(2\delta-1-\alpha^+)/(2\delta-1-\alpha^-)m'}\right). \end{aligned} \tag{33}$$

By Sobolev inequality (on the left hand side), we have that

$$\begin{aligned} \int_\Omega |\nabla u_n|^2 u_n^{2\delta-2} dx &= \frac{1}{\delta^2}\int_\Omega |\nabla u_n^\delta|^2 dx \\ &\geqslant \frac{S}{\delta^2}\left(\int_\Omega u_n^{2^*\delta} dx\right)^{2/2^*}, \end{aligned} \tag{34}$$

where S is the constant of the Sobolev embedding; combining with (33) and (34), we have that

$$\begin{aligned}\frac{S\gamma(2\delta-1)}{\delta^2}\left(\int_\Omega u_n^{2^*\delta}dx\right)^{2/2^*} &\leqslant \|f\|_{L^m(\Omega)}\\ &\cdot\left(\left(\int_\Omega u_n^{(2\delta-1-\alpha^-)m'}dx\right)^{1/m'}\right.\\ &+|\Omega|^{(\alpha^+-\alpha^-)/(2\delta-1-\alpha^-)m'}\\ &\left.\cdot\left(\int_\Omega u_n^{(2\delta-1-\alpha^-)m'}dx\right)^{(2\delta-1-\alpha^+)/(2\delta-1-\alpha^-)m'}\right).\end{aligned} \tag{35}$$

We choose δ in such a way that $2^*\delta=(2\delta-1-\alpha^-)m'$; that is,

$$\delta=\frac{(1+\alpha^-)m(N-2)}{2(N-2m)}, \tag{36}$$

which yields that $\delta>1$ if and only if $2N/(N+2+(N-2)\alpha^-)<m<N/2$, and that $2^*\delta=Nm(1+\alpha^-)/(N-2m)=s$. Therefore, (35) becomes

$$\begin{aligned}\left(\int_\Omega u_n^s dx\right)^{2/2^*} &\leqslant \frac{\delta^2}{S\gamma(2\delta-1)}\|f\|_{L^m(\Omega)}\\ &\cdot\left(\left(\int_\Omega u_n^s dx\right)^{1/m'}+|\Omega|^{(\alpha^+-\alpha^-)/(2\delta-1-\alpha^-)m'}\right.\\ &\left.\cdot\left(\int_\Omega u_n^s dx\right)^{(2\delta-1-\alpha^+)/(2\delta-1-\alpha^-)m'}\right),\end{aligned} \tag{37}$$

which implies that

$$\begin{aligned}\left(\int_\Omega u_n^s dx\right)^{(1+\alpha^+)/2^*\delta} &\leqslant \frac{\delta^2}{S\gamma(2\delta-1)}\|f\|_{L^m(\Omega)}\\ &\cdot\left(\left(\int_\Omega u_n^s dx\right)^{(\alpha^+-\alpha^-)/2^*\delta}+|\Omega|^{(\alpha^+-\alpha^-)/2^*\delta}\right).\end{aligned} \tag{38}$$

Using Young's inequality on the right hand side in (38), we have that

$$\begin{aligned}\left(\int_\Omega u_n^s dx\right)^{(1+\alpha^+)/2^*\delta} &\leqslant \frac{\delta^2}{S\gamma(2\delta-1)}\|f\|_{L^m(\Omega)}\\ &\cdot\left(\varepsilon\left(\int_\Omega u_n^s dx\right)^{(1+\alpha^+)/2^*\delta}+\varepsilon^{-(\alpha^+-\alpha^-)/(1+\alpha^-)}\right.\\ &\left.+|\Omega|^{(\alpha^+-\alpha^-)/2^*\delta}\right),\end{aligned} \tag{39}$$

where $\varepsilon=S\gamma(2\delta-1)/2\delta^2\|f\|_{L^m(\Omega)}$. Thus, we get that

$$\begin{aligned}&\left(\int_\Omega u_n^s dx\right)^{(1+\alpha^+)/2^*\delta}\\ &\leqslant \frac{2\delta^2\|f\|_{L^m(\Omega)}}{S\gamma(2\delta-1)}\left(\left(\frac{2\delta^2\|f\|_{L^m(\Omega)}}{S\gamma(2\delta-1)}\right)^{(\alpha^+-\alpha^-)/(1+\alpha^-)}\right.\\ &\left.+|\Omega|^{(\alpha^+-\alpha^-)/2^*\delta}\right).\end{aligned} \tag{40}$$

Therefore, we know that u_n is bounded in $L^s(\Omega)$, and so does $u\in L^s(\Omega)$. □

Theorem 9. *Suppose that $f\in L^m(\Omega)$, $(1+\alpha^+)N/((1+\alpha^-)(N-2)+2(1+\alpha^+))\leqslant m<2N/(N+2+(N-2)\alpha^-)$, and $0<\alpha^-\leqslant\alpha(x)\leqslant\alpha^+<1$. Then problem (1) has a solution u in $W_0^{1,q}(\Omega)$, $q=Nm(1+\alpha^-)/(N-m(1-\alpha^-))$.*

Proof. The lines of our proof are that if we can prove that u_n is bounded in $W_0^{1,q}(\Omega)$ (with q as in the statement), the existence of a solution u in $W_0^{1,q}(\Omega)$ of (1) will be proved by passing to the limit in (7) as in the proof of Theorem 7. To prove that u_n is bounded in $W_0^{1,q}(\Omega)$, we begin by proving that it is bounded in $L^s(\Omega)$, with $s=Nm(1+\alpha^-)/(N-2m)$. To attain this goal, we choose $u_n^{2\delta-1}$ as a test function in (7) as in the statement of Lemma 8, where $(1+\alpha^+)/2\leqslant\delta<1$; however $\nabla u_n^{2\delta-1}$ will be singular at $u_n=0$, and therefore, we choose $(u_n+\varepsilon)^{2\delta-1}-\varepsilon^{2\delta-1}$ as a test function in (7), where $\varepsilon<1/n$ for n fixed; by (2) and $f_n\leqslant f$, we have that

$$\begin{aligned}&\gamma(2\delta-1)\int_\Omega|\nabla u_n|^2(u_n+\varepsilon)^{2\delta-2}dx\\ &\leqslant\int_\Omega f(u_n+\varepsilon)^{2\delta-1-\alpha^-}dx\\ &+\int_\Omega f(u_n+\varepsilon)^{2\delta-1-\alpha^+}dx.\end{aligned} \tag{41}$$

By Sobolev Embedding Theorem ($H_0^1(\Omega)\hookrightarrow L^{2^*}(\Omega)$) on the left hand side, it follows that

$$\begin{aligned}&\int_\Omega|\nabla u_n|^2(u_n+\varepsilon)^{2\delta-2}dx\\ &=\int_\Omega\frac{\left|\nabla\left((u_n+\varepsilon)^\delta-\varepsilon^\delta\right)\right|^2}{\delta^2}dx\\ &\geqslant\frac{S}{\delta^2}\left(\int_\Omega\left((u_n+\varepsilon)^\delta-\varepsilon^\delta\right)^{2^*}dx\right)^{2/2^*},\end{aligned} \tag{42}$$

where S is the best constant of the Sobolev Embedding Theorem. Combining (41) with (42), we have that

$$\begin{aligned}&\frac{S\gamma(2\delta-1)}{\delta^2}\left(\int_\Omega\left((u_n+\varepsilon)^\delta-\varepsilon^\delta\right)^{2^*}dx\right)^{2/2^*}\\&\leqslant\int_\Omega f(u_n+\varepsilon)^{2\delta-1-\alpha^-}dx\\&\quad+\int_\Omega f(u_n+\varepsilon)^{2\delta-1-\alpha^+}dx.\end{aligned}\tag{43}$$

Using Hölder's inequality on the right hand side, we get

$$\begin{aligned}&\frac{S\gamma(2\delta-1)}{\delta^2}\left(\int_\Omega\left((u_n+\varepsilon)^\delta-\varepsilon^\delta\right)^{2^*}dx\right)^{2/2^*}\\&\leqslant\|f\|_{L^m(\Omega)}\left(\int_\Omega(u_n+\varepsilon)^{(2\delta-1-\alpha^-)m'}dx\right)^{1/m'}\\&+|\Omega|^{(\alpha^+-\alpha^-)/(2\delta-1-\alpha^-)m'}\|f\|_{L^m(\Omega)}\\&\cdot\left(\int_\Omega(u_n+\varepsilon)^{(2\delta-1-\alpha^-)m'}dx\right)^{(2\delta-1-\alpha^+)/(2\delta-1-\alpha^-)m'}.\end{aligned}\tag{44}$$

Letting $\varepsilon\to0$, we get (35); that is,

$$\begin{aligned}&\left(\int_\Omega u_n^{2^*\delta}dx\right)^{2/2^*}\leqslant\frac{\delta^2}{S\gamma(2\delta-1)}\|f\|_{L^m(\Omega)}\\&\cdot\left(\left(\int_\Omega u_n^{(2\delta-1-\alpha^-)m'}dx\right)^{1/m'}\right.\\&+|\Omega|^{(\alpha^+-\alpha^-)/(2\delta-1-\alpha^-)m'}\\&\left.\cdot\left(\int_\Omega u_n^{(2\delta-1-\alpha^-)m'}dx\right)^{(2\delta-1-\alpha^+)/(2\delta-1-\alpha^-)m'}\right),\end{aligned}\tag{45}$$

where δ is chosen in such a way that $2^*\delta=(2\delta-1-\alpha^-)m'$; that is,

$$\delta=\frac{(1+\alpha^-)(N-2)m}{2(N-2m)}.\tag{46}$$

If $m=(1+\alpha^+)N/((1+\alpha^-)(N-2)+2(1+\alpha^+))$, choosing $\delta=(1+\alpha^+)/2$ in (43), and letting $\varepsilon\to0$, we have that

$$\begin{aligned}&\left(\int_\Omega u_n^{2^*\delta}dx\right)^{2/2^*}\\&\leqslant\frac{\delta^2}{S\gamma(2\delta-1)}\left(\int_\Omega fu_n^{2\delta-1-\alpha^-}dx+\int_\Omega f\,dx\right).\end{aligned}\tag{47}$$

Using Hölder's inequality and Young's inequality, we get

$$\begin{aligned}&\left(\int_\Omega u_n^{2^*\delta}dx\right)^{2/2^*}\leqslant\frac{\delta^2}{S\gamma(2\delta-1)}\left(\|f\|_{L^m(\Omega)}\right.\\&\left.\cdot\left(\int_\Omega u_n^{(2\delta-1-\alpha^-)m'}dx\right)^{1/m'}+\int_\Omega f\,dx\right)\\&\leqslant\frac{\delta^2}{S\gamma(2\delta-1)}\left(\|f\|_{L^m(\Omega)}\right.\\&\left.\cdot\left(\varepsilon\left(\int_\Omega u_n^{2^*\delta}dx\right)^{2/2^*}+\varepsilon^{-2^*/(2m'-2^*)}\right)+\int_\Omega f\,dx\right),\end{aligned}\tag{48}$$

where $\varepsilon=S\gamma(2\delta-1)/2\delta^2\|f\|_{L^m(\Omega)}$. Thus we have that

$$\begin{aligned}&\left(\int_\Omega u_n^{2^*\delta}dx\right)^{2/2^*}\leqslant\frac{2\delta^2}{S\gamma(2\delta-1)}\left(\|f\|_{L^m(\Omega)}\right.\\&\left.\cdot\left(\frac{2\delta^2\|f\|_{L^m(\Omega)}}{S\gamma(2\delta-1)}\right)^{2^*/(2m'-2^*)}+\int_\Omega f\,dx\right).\end{aligned}\tag{49}$$

Therefore we obtain that u_n is bounded in $L^{N(1+\alpha^+)/(N-2)}(\Omega)$, where $N(1+\alpha^+)/(N-2)$ is the value of s for $m=(1+\alpha^+)N/((1+\alpha^-)(N-2)+2(1+\alpha^+))$.

If $(1+\alpha^+)N/((1+\alpha^-)(N-2)+2(1+\alpha^+))<m<2N/(N+2+(N-2)\alpha^-)$, it is clear that the inequality on m holds true if and only if $(1+\alpha^+)/2<\delta<1$, starting from (35) and arguing as in the proof of Lemma 8, we also get that u_n is bounded in $L^s(\Omega)$ with $s=Nm(1+\alpha^-)/(N-2m)$.

The right hand side of (41) is bounded with respect to n (and ε, which we take smaller than 1) by using the estimate on u_n in $L^s(\Omega)$ and the choice of δ.

Since $\delta<1$,

$$\int_\Omega\frac{|\nabla u_n|^2}{(u_n+\varepsilon)^{2-2\delta}}dx=\int_\Omega|\nabla u_n|^2(u_n+\varepsilon)^{2\delta-2}dx\leqslant C.\tag{50}$$

If $q=Nm(1+\alpha^-)/(N-m(1-\alpha^-))<2$, by Hölder's inequality, we have that

$$\begin{aligned}\int_\Omega|\nabla u_n|^q dx&=\int_\Omega\frac{|\nabla u_n|^q}{(u_n+\varepsilon)^{(1-\delta)q}}(u_n+\varepsilon)^{(1-\delta)q}dx\\&\leqslant\left(\int_\Omega\left(\frac{|\nabla u_n|^q}{(u_n+\varepsilon)^{(1-\delta)q}}\right)^{2/q}dx\right)^{q/2}\\&\cdot\left(\int_\Omega(u_n+\varepsilon)^{2(1-\delta)q/(2-q)}dx\right)^{1-q/2}\end{aligned}$$

$$= \left(\int_\Omega \frac{|\nabla u_n|^2}{(u_n+\varepsilon)^{2(1-\delta)}} dx \right)^{q/2} \cdot \left(\int_\Omega (u_n+\varepsilon)^{2(1-\delta)q/(2-q)} dx \right)^{1-q/2} \leqslant C \left(\int_\Omega (u_n+\varepsilon)^{2(1-\delta)q/(2-q)} dx \right)^{1-q/2}. \tag{51}$$

The choice of δ and the value of q are such that $2(1-\delta)q/(2-q) = s$, so that the right hand side of (51) is bounded with respect to n and ε. Hence, u_n is bounded in $W_0^{1,q}(\Omega)$. □

Theorem 10. *Suppose that $f \in L^m(\Omega)$, $1/(2\delta - \alpha^+) < m < ((1+\alpha^+)N/((1+\alpha^-)(N-2)+2(1+\alpha^+)))$ with $(1+\alpha^-)/2 < \delta < (1+\alpha^+)/2$, and $0 < \alpha^- \leqslant \alpha(x) \leqslant \alpha^+ < 1$. Then problem (1) has a solution u in $W_0^{1,q}(\Omega)$, $q = Nm(1+\alpha^-)/(N-m(1-\alpha^-))$.*

Proof. The lines of our proof are similar to that in the proof of Theorem 9. We also begin by proving that u_n is bounded in $L^s(\Omega)$, with $s = Nm(1+\alpha^-)/(N-2m)$. To this aim, we also choose $(u_n+\varepsilon)^{2\delta-1} - \varepsilon^{2\delta-1}$ as a test function in (7), where $(1+\alpha^-)/2 < \delta < (1+\alpha^+)/2$, $\varepsilon < 1/n$ for n fixed. Since $f_n \leqslant f$ and (2), we have that

$$\gamma(2\delta-1)\int_\Omega |\nabla u_n|^2 (u_n+\varepsilon)^{2\delta-2} dx \leqslant \int_\Omega f(u_n+\varepsilon)^{2\delta-1-\alpha^-} dx + \int_\Omega f(u_n+\varepsilon)^{2\delta-1-\alpha^+} dx. \tag{52}$$

Using Sobolev Embedding Theorem ($H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$) on the left hand side, it follows that

$$\frac{S\gamma(2\delta-1)}{\delta^2}\left(\int_\Omega \left((u_n+\varepsilon)^\delta - \varepsilon^\delta\right)^{2^*} dx\right)^{2/2^*} \leqslant \int_\Omega f(u_n+\varepsilon)^{2\delta-1-\alpha^-} dx + \int_\Omega f(u_n+\varepsilon)^{2\delta-1-\alpha^+} dx, \tag{53}$$

where S is the constant of the Sobolev Embedding Theorem.

Using Hölder's inequality and Lemma 5 on the right hand side, we get

$$\begin{aligned}\frac{S\gamma(2\delta-1)}{\delta^2}\left(\int_\Omega \left((u_n+\varepsilon)^\delta - \varepsilon^\delta\right)^{2^*} dx\right)^{2/2^*} &\leqslant \int_\Omega f(u_n+\varepsilon)^{2\delta-1-\alpha^-} dx + \int_\Omega \frac{f}{(u_n+\varepsilon)^{-2\delta+\alpha^++1}} dx \\ &\leqslant \int_\Omega f(u_n+\varepsilon)^{2\delta-1-\alpha^-} dx + \int_\Omega \frac{f}{u_1^{-2\delta+\alpha^++1}} dx \\ &\leqslant \|f\|_{L^m(\Omega)} \left(\int_\Omega (u_n+\varepsilon)^{(2\delta-1-\alpha^-)m'} dx\right)^{1/m'} + \|f\|_{L^m(\Omega)} \left\|u_1^{2\delta-1-\alpha^+}\right\|_{L^{m'}(\Omega)} \\ &\leqslant \|f\|_{L^m(\Omega)} \left(\int_\Omega (u_n+\varepsilon)^{(2\delta-1-\alpha^-)m'} dx\right)^{1/m'} + C\|f\|_{L^m(\Omega)}.\end{aligned} \tag{54}$$

Letting $\varepsilon \to 0$, we obtain that

$$\left(\int_\Omega u_n^{2^*\delta} dx\right)^{2/2^*} \leqslant \frac{\delta^2}{S\gamma(2\delta-1)}\|f\|_{L^m(\Omega)} \cdot \left(\left(\int_\Omega u_n^{(2\delta-1-\alpha^-)m'} dx\right)^{1/m'} + C\right), \tag{55}$$

where δ is chosen in such a way that $2^*\delta = (2\delta-1-\alpha^-)m'$; that is

$$\delta = \frac{(1+\alpha^-)(N-2)m}{2(N-2m)}. \tag{56}$$

If $1 < m < (1+\alpha^+)N/((1+\alpha^-)(N-2)+2(1+\alpha^+))$, it is clear that the inequality on m holds true if and only if $(1+\alpha^-)/2 < \delta < (1+\alpha^+)/2$, and arguing as to the case $m = (1+\alpha^+)N/((1+\alpha^-)(N-2)+2(1+\alpha^+))$ in the proof of Theorem 9, we also obtain that u_n is bounded in $L^s(\Omega)$, with $s = Nm(1+\alpha^-)/(N-2m)$.

Since $\delta < 1$,

$$\int_\Omega \frac{|\nabla u_n|^2}{(u_n+\varepsilon)^{2-2\delta}} dx = \int_\Omega |\nabla u_n|^2 (u_n+\varepsilon)^{2\delta-2} dx \leqslant C. \tag{57}$$

If $q = Nm(1+\alpha^-)/(N-m(1-\alpha^-)) < 2$, similarly to the proof of Theorem 9, we have by Hölder's inequality that

$$\int_\Omega |\nabla u_n|^q dx \leqslant C\left(\int_\Omega (u_n+\varepsilon)^{2(1-\delta)q/(2-q)} dx\right)^{1-q/2}. \tag{58}$$

Since the choice of δ and the value of q, the right hand side of the above inequality is bounded with respect to n and ε. Hence, u_n is bounded in $W_0^{1,q}(\Omega)$. □

4. The Case $1 < \alpha^- \leqslant \alpha(x) \leqslant \alpha^+$

The case $1 < \alpha^- \leqslant \alpha(x) \leqslant \alpha^+$ has many analogies with the case $0 < \alpha^- < \alpha^+ < 1$. In this case, we can also prove that u_n is bounded in $H_0^1(\Omega)$ only if f is more regular than $L^1(\Omega)$ and α^+ and α^- are close to 1.

Lemma 11. *Suppose that $f \in L^m(\Omega)$ $(m > 1)$, and let u_n be the solution of (7) with $1 < \alpha^- < \alpha^+ < 2 - 1/m$. Then u_n is bounded in $H_0^1(\Omega)$.*

Proof. Taking u_n as a test function in (7), by (2), we obtain that

$$\gamma \int_\Omega |\nabla u_n|^2 dx \leqslant \int_\Omega \frac{f}{u_n^{\alpha(x)-1}} dx. \tag{59}$$

Using Lemmas 2 and 3, we know that $u_n \geqslant u_1$ and there exists a constant $M > 0$ s.t. $u_1 \leqslant M$. Hence $(M/u_1)^{\alpha(x)-1} \leqslant (M/u_1)^{\alpha^+-1}$, and we have

$$\begin{aligned}\gamma \int_\Omega |\nabla u_n|^2 dx &\leqslant \int_\Omega \frac{f}{u_n^{\alpha(x)-1}} dx \leqslant \int_\Omega \frac{f}{u_1^{\alpha(x)-1}} dx \\ &\leqslant \left(1 + M^{\alpha^+-\alpha^-}\right) \int_\Omega \frac{f}{u_1^{\alpha^+-1}} dx.\end{aligned} \tag{60}$$

Using Hölder's inequality on the right hand side, and Lemma 5, we obtain

$$\begin{aligned}\gamma \int_\Omega |\nabla u_n|^2 dx &\leqslant \left(1 + M^{\alpha^+-\alpha^-}\right) \int_\Omega \frac{f}{u_1^{\alpha^+-1}} dx \\ &\leqslant \left(1 + M^{\alpha^+-\alpha^-}\right) \|f\|_{L^m(\Omega)} \left(\int_\Omega \frac{1}{u_1^{(\alpha^+-1)m'}} dx\right)^{1/m'} \\ &\leqslant C\left(1 + M^{\alpha^+-\alpha^-}\right) \|f\|_{L^m(\Omega)}.\end{aligned} \tag{61}$$

Therefore, u_n is bounded in $H_0^1(\Omega)$. □

Once we have the boundedness of u_n, we can prove the following existence theorem along the lines of Theorem 7.

Theorem 12. *Suppose that $f \in L^m(\Omega)$ $(m > 1)$ and $1 < \alpha^- < \alpha^+ < 2 - 1/m$. Then problem (1) has a solution u in $H_0^1(\Omega)$.*

The summability of u can be proved along the lines of Lemma 8 with little changes.

Lemma 13. *Suppose that $f \in L^m(\Omega)$ $(m > 1)$ and $1 < \alpha^- < \alpha^+ < 2 - 1/m$. Then the solution u of (1) given by Theorem 12 is such that*

(i) *if $m > N/2$, then $u \in L^\infty(\Omega)$;*

(ii) *if $N(1+\alpha^+)/((1+\alpha^-)(N-2)+2(1+\alpha^+)) \leqslant m < N/2$, then $u \in L^s(\Omega)$, $s = Nm(1+\alpha^-)/(N-2m)$.*

Proof. The proof of (i) is similar to the proof of Lemma 8(i); we omit the details here.

To prove (ii) we choose $u_n^{2\delta-1}$ as a test function with $\delta \geqslant (1+\alpha^+)/2$ in (7); similarly to the proof of Lemma 8, we obtain that

$$\begin{aligned}&\frac{S\gamma(2\delta-1)}{\delta^2}\left(\int_\Omega u_n^{2^*\delta} dx\right)^{2/2^*} \\ &\quad\leqslant \int_\Omega f u_n^{2\delta-1-\alpha^-} dx + \int_\Omega f u_n^{2\delta-1-\alpha^+} dx.\end{aligned} \tag{62}$$

If $m = N(1+\alpha^+)/((1+\alpha^-)(N-2)+2(1+\alpha^+))$, choosing $\delta = (1+\alpha^+)/2$ in (62), by Hölder's inequality, we get

$$\begin{aligned}&\frac{S\gamma(2\delta-1)}{\delta^2}\left(\int_\Omega u_n^{2^*\delta} dx\right)^{2/2^*} \\ &\quad\leqslant \|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{(2\delta-1-\alpha^-)m'} dx\right)^{1/m'} \\ &\qquad + |\Omega|^{1-1/m}\|f\|_{L^m(\Omega)}.\end{aligned} \tag{63}$$

We choose δ in such a way that $2^*\delta = (2\delta-1-\alpha^-)m'$; that is, $\delta = (1+\alpha^-)m(N-2)/2(N-2m)$. Since $m = N(1+\alpha^+)/((1+\alpha^-)(N-2)+2(1+\alpha^+))$ being $2/2^* > 1/m'$, it follows that u_n is bounded in $L^{N(1+\alpha^+)/(N-2)}(\Omega)$.

If $N(1+\alpha^+)/((1+\alpha^-)(N-2)+2(1+\alpha^+)) < m < N/2$, starting from inequality (62) and Hölder's inequality, we have that

$$\begin{aligned}&\frac{S\gamma(2\delta-1)}{\delta^2}\left(\int_\Omega u_n^{2^*\delta} dx\right)^{2/2^*} \leqslant \|f\|_{L^m(\Omega)} \\ &\quad\cdot\left(\left(\int_\Omega u_n^{(2\delta-1-\alpha^-)m'} dx\right)^{1/m'}\right. \\ &\quad+ |\Omega|^{(\alpha^+-\alpha^-)/(2\delta-1-\alpha^-)m'} \\ &\quad\left.\cdot\left(\int_\Omega u_n^{(2\delta-1-\alpha^-)m'} dx\right)^{(2\delta-1-\alpha^+)/(2\delta-1-\alpha^-)m'}\right).\end{aligned} \tag{64}$$

We also choose δ in such a way that $2^*\delta = (2\delta-1-\alpha^-)m'$, which yields that $\delta > (1+\alpha^+)/2$, if and only if $m > N(1+\alpha^+)/((1+\alpha^-)(N-2)+2(1+\alpha^+))$, and that $2^*\delta = s$. So, since $2/2^* > 1/m'$ being $m < N/2$, we have the boundedness of u_n in $L^s(\Omega)$, and so does $u \in L^s(\Omega)$. □

Moreover, we can only prove that a positive power of u_n is bounded in $H_0^1(\Omega)$ only if f is more regular than $L^1(\Omega)$ and α^+ is close to α^- and we only have the boundedness of u_n in $H_{\mathrm{loc}}^1(\Omega)$.

Lemma 14. *Suppose that $f \in L^m(\Omega)$ $(m > 1)$, and let u_n be the solution of (7) with $1 < \alpha^- \leqslant \alpha(x) \leqslant \alpha^+$ and $\alpha^+ - \alpha^- < 1 - 1/m$. Then $u_n^{(1+\alpha^-)/2}$ is bounded in $H_0^1(\Omega)$, and u_n is bounded in $H_{\mathrm{loc}}^1(\Omega)$ and in $L^s(\Omega)$, with $s = N(1+\alpha^-)/(N-2)$.*

Proof. Taking $u_n^{\alpha^-}$ as a test function in (7), since $u_n^{\alpha^-}/(u_n + 1/n)^{\alpha^-} \leqslant 1$ and $f_n \leqslant f$, by Lemma 5 and (2), we get that

$$\begin{aligned}&\alpha^-\gamma \int_\Omega |\nabla u_n|^2 u_n^{\alpha^--1} dx \\ &\quad\leqslant \int_\Omega \frac{f u_n^{\alpha^-}}{(u_n + 1/n)^{\alpha^-}} dx + \int_\Omega \frac{f u_n^{\alpha^-}}{u_n^{\alpha^+}} dx \\ &\quad\leqslant |\Omega|^{1-1/m}\|f\|_{L^m(\Omega)} + \|f\|_{L^m(\Omega)} \int_\Omega \frac{1}{u_n^{(\alpha^+-\alpha^-)m'}} dx \\ &\quad\leqslant |\Omega|^{1-1/m}\|f\|_{L^m(\Omega)} + C\|f\|_{L^m(\Omega)}.\end{aligned} \tag{65}$$

Since

$$\int_\Omega |\nabla u_n|^2 u_n^{\alpha^- -1} dx = \frac{4}{(1+\alpha^-)^2} \int_\Omega \left|\nabla u_n^{(1+\alpha^-)/2}\right|^2 dx, \quad (66)$$

we have that

$$\begin{aligned} &\frac{4\alpha^-\gamma}{(1+\alpha^-)^2} \int_\Omega \left|\nabla u_n^{(1+\alpha^-)/2}\right|^2 dx \\ &\quad \leqslant \left(C + |\Omega|^{1-1/m}\right) \|f\|_{L^m(\Omega)}. \end{aligned} \quad (67)$$

Thus, we have that $u_n^{(1+\alpha^-)/2}$ is bounded in $H_0^1(\Omega)$.

Applying Sobolev Embedding Theorem to $u_n^{(1+\alpha^-)/2}(H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega))$, we have that

$$S\left(\int_\Omega \left|u_n^{(1+\alpha^-)/2}\right|^{2^*} dx\right)^{2/2^*} \leqslant \int_\Omega \left|\nabla u_n^{(1+\alpha^-)/2}\right|^2 dx, \quad (68)$$

where S is the best constant of the Sobolev embedding. Since the boundedness of $u_n^{(1+\alpha^-)/2}$ in $H_0^1(\Omega)$, we thus have the boundedness of u_n in $L^s(\Omega)$.

To prove the boundedness of u_n in $H^1_{\mathrm{loc}}(\Omega)$, we choose $u_n\varphi^2$ as a test function in (7), where $\varphi \in C_0^1(\Omega)$, $\Omega' = \{x \in \Omega, \varphi \neq 0\}$. By (2) and (12), we have that

$$\begin{aligned} &\gamma \int_\Omega |\nabla u_n|^2 \varphi^2 dx + 2\int_\Omega (M(x)\nabla u_n)\cdot \nabla\varphi u_n \varphi\, dx \\ &\quad \leqslant \int_\Omega (M(x)\nabla u_n)\nabla\left(u_n\varphi^2\right) dx \\ &\quad = \int_\Omega \frac{f_n u_n \varphi^2}{(u_n+1/n)^{\alpha(x)}} dx \leqslant \int_\Omega \frac{f_n\varphi^2}{u_n^{\alpha(x)-1}} dx \\ &\quad \leqslant \int_\Omega \frac{f_n\varphi^2}{C_{\Omega'}^{\alpha(x)-1}} dx \\ &\quad \leqslant \frac{1}{\min\left\{C_{\Omega'}^{\alpha^+-1}, C_{\Omega'}^{\alpha^--1}\right\}} \int_\Omega f_n\varphi^2 dx. \end{aligned} \quad (69)$$

By Young's inequality, we get that

$$\begin{aligned} &2\beta \int_\Omega \nabla u_n \cdot \nabla\varphi u_n \varphi\, dx \\ &\quad \leqslant \frac{\gamma}{2}\int_\Omega |\nabla u_n|^2 \varphi^2 dx + \frac{2\beta^2}{\gamma}\int_\Omega |\nabla\varphi|^2 u_n^2 dx. \end{aligned} \quad (70)$$

Since u_n is bounded in $L^s(\Omega)$ (where $s \geqslant 2$), by Hölder inequality, we obtain that

$$\begin{aligned} &\frac{\gamma}{2}\int_\Omega |\nabla u_n|^2 \varphi^2 dx \\ &\quad \leqslant \frac{1}{\min\left\{C_{\Omega'}^{\alpha^+-1}, C_{\Omega'}^{\alpha^--1}\right\}} \int_\Omega f_n\varphi^2 dx \\ &\qquad + \frac{2\beta^2}{\gamma}\int_\Omega |\nabla\varphi|^2 u_n^2 dx \\ &\quad \leqslant \frac{\|\varphi\|^2_{L^\infty(\Omega)}}{\min\left\{C_{\Omega'}^{\alpha^+-1}, C_{\Omega'}^{\alpha^--1}\right\}} \int_\Omega f\, dx \\ &\qquad + \frac{2\beta^2}{\gamma}\|\nabla\varphi\|^2_{L^\infty(\Omega)} \int_\Omega u_n^2 dx \\ &\quad \leqslant \frac{|\Omega|^{1-1/m}\|\varphi\|^2_{L^\infty(\Omega)}}{\min\left\{C_{\Omega'}^{\alpha^+-1}, C_{\Omega'}^{\alpha^--1}\right\}} \|f\|_{L^m(\Omega)} \\ &\qquad + \frac{2\beta^2}{\gamma}\|\nabla\varphi\|^2_{L^\infty(\Omega)} |\Omega|^{1-2/s} \left(\int_\Omega u_n^s dx\right)^{2/s} \\ &\quad \leqslant C(f,\varphi), \end{aligned} \quad (71)$$

and hence u_n is bounded in $H^1_{\mathrm{loc}}(\Omega)$. □

Once we have the boundedness of u_n, we can prove the following existence theorem along the lines of Theorem 7.

Theorem 15. *Suppose that f is a nonnegative function in $L^m(\Omega)$ $(m > 1)$, $(f \not\equiv 0)$, $1 < \alpha^- \leqslant \alpha(x) \leqslant \alpha^+$ and $\alpha^+ - \alpha^- < 1 - 1/m$. Then problem (1) has a solution u in $H^1_{\mathrm{loc}}(\Omega)$. Furthermore, $u^{(1+\alpha^-)/2}$ belongs to $H_0^1(\Omega)$.*

The summability of u can be proved along the lines of Lemma 8 with little changes.

Lemma 16. *Suppose that $f \in L^m(\Omega)$, $1 < \alpha^- \leqslant \alpha(x) \leqslant \alpha^+$, and $\alpha^+ - \alpha^- < 1 - 1/m$. Then the solution u of (1) given by Theorem 15 is such that*

(i) *if $m > N/2$, then $u \in L^\infty(\Omega)$;*

(ii) *if $N(1+\alpha^+)/((1+\alpha^-)(N-2)+2(1+\alpha^+)) \leqslant m < N/2$, then $u \in L^s(\Omega)$, $s = Nm(1+\alpha^-)/(N-2m)$.*

Proof. The proof of (i) is similar to the proof of Lemma 8(i); we omit the details here.

To prove (ii), we choose $u_n^{2\delta-1}$ $(\delta \geqslant (1+\alpha^+)/2)$ as a test function in (7); applying (2) and Sobolev Embedding Theorem, we have that

$$\begin{aligned} &\frac{S\gamma(2\delta-1)}{\delta^2}\left(\int_\Omega u_n^{2^*\delta} dx\right)^{2/2^*} \\ &\quad \leqslant \int_\Omega f u_n^{2\delta-1-\alpha^-} dx + \int_\Omega f u_n^{2\delta-1-\alpha^+} dx. \end{aligned} \quad (72)$$

If $m = N(1+\alpha^+)/((1+\alpha^-)(N-2)+2(1+\alpha^+))$, we choose $\delta = (1+\alpha^+)/2$; using Hölder inequality, we have that

$$\begin{aligned}\frac{S\gamma(2\delta-1)}{\delta^2}\left(\int_\Omega u_n^{2^*\delta}dx\right)^{2/2^*} &\leqslant \|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{(2\delta-1-\alpha^-)m'}dx\right)^{1/m'} \\ &\quad + |\Omega|^{1-1/m}\|f\|_{L^m(\Omega)}.\end{aligned} \tag{73}$$

We also choose δ in such a way that $2^*\delta = (2\delta-1-\alpha^-)m'$, since $m = N(1+\alpha^+)/((1+\alpha^-)(N-2)+2(1+\alpha^+))$ being $2/2^* > 1/m'$, and we have the boundedness of u_n in $L^{N(1+\alpha^+)/(N-2)}(\Omega)$ which is the value of s for $m = N(1+\alpha^+)/((1+\alpha^-)(N-2)+2(1+\alpha^+))$.

If $N(1+\alpha^+)/((1+\alpha^-)(N-2)+2(1+\alpha^+)) < m < N/2$, starting from (72) using Hölder inequality, we get that

$$\begin{aligned}\frac{S\gamma(2\delta-1)}{\delta^2}\left(\int_\Omega u_n^{2^*\delta}dx\right)^{2/2^*} &\leqslant \|f\|_{L^m(\Omega)} \\ &\quad\cdot\left(\int_\Omega u_n^{(2\delta-1-\alpha^-)m'}dx\right)^{1/m'} + \|f\|_{L^m(\Omega)} \\ &\quad\cdot|\Omega|^{(\alpha^+-\alpha^-)/(2\delta-1-\alpha^-)m'} \\ &\quad\cdot\left(\int_\Omega u_n^{(2\delta-1-\alpha^-)m'}dx\right)^{(2\delta-1-\alpha^+)/(2\delta-1-\alpha^-)m'}.\end{aligned} \tag{74}$$

We also choose δ in such a way that $2^*\delta = (2\delta-1-\alpha^-)m'$, which yields that $\delta > (1+\alpha^+)/2$, if and only if $m > N(1+\alpha^+)/((1+\alpha^-)(N-2)+2(1+\alpha^+))$, and that $2^*\delta = s$. Since $2/2^* > 1/m'$ being $m < N/2$, we have the boundedness of u_n in $L^s(\Omega)$, and so does $u \in L^s(\Omega)$. □

5. The Case $0 < \alpha^- < 1 < \alpha^+$

If $\alpha^- < 1 < \alpha^+$, the boundedness of u_n in $H_0^1(\Omega)$ can also be obtained only if f is more regular than $L^1(\Omega)$, and the proof has many analogies with the case $0 < \alpha^- < \alpha^+ < 1$. We have the following results.

Lemma 17. *Suppose that $f \in L^m(\Omega)$, with $m = 2N/(N+2+(N-2)\alpha^-)$, and let u_n be the solution of (7) with $0 < \alpha^- < 1 < \alpha^+ < 2-1/m$. Then the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$.*

Proof. We choose u_n as a test function in (7), by Hölder's inequality, (2), and Lemma 5, since $f_n \leqslant f$, we have that

$$\begin{aligned}\gamma\int_\Omega|\nabla u_n|^2dx &\leqslant \|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{(1-\alpha^-)m'}dx\right)^{1/m'} \\ &\quad + \|f\|_{L^m(\Omega)}\left(\int_\Omega \frac{1}{u_1^{(\alpha^+-1)m'}}dx\right)^{1/m'} \\ &\leqslant \|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{(1-\alpha^-)m'}dx\right)^{1/m'} \\ &\quad + C\|f\|_{L^m(\Omega)}.\end{aligned} \tag{75}$$

Applying Sobolev Embedding Theorem on the left hand side, we get

$$\begin{aligned}S\gamma\left(\int_\Omega u_n^{2^*}dx\right)^{2/2^*} &\leqslant \|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{(1-\alpha^-)m'}dx\right)^{1/m'} \\ &\quad + C\|f\|_{L^m(\Omega)}.\end{aligned} \tag{76}$$

Let $2^* = (1-\alpha^-)m'$, and it follows that

$$\begin{aligned}S\gamma\left(\int_\Omega u_n^{2^*}dx\right)^{2/2^*} &\leqslant \|f\|_{L^m(\Omega)}\left(\int_\Omega u_n^{2^*}dx\right)^{1/m'} \\ &\quad + C\|f\|_{L^m(\Omega)}.\end{aligned} \tag{77}$$

By Young's inequality, we get that

$$\begin{aligned}&S\gamma\left(\int_\Omega u_n^{2^*}dx\right)^{2/2^*} \\ &\quad\leqslant \|f\|_{L^m(\Omega)}\left(\varepsilon\left(\int_\Omega u_n^{2^*}dx\right)^{2/2^*} + \varepsilon^{-2^*/(2m'-2^*)}\right) \\ &\quad\quad + C\|f\|_{L^m(\Omega)}.\end{aligned} \tag{78}$$

Thus, we have that

$$\begin{aligned}&\left(\int_\Omega u_n^{2^*}dx\right)^{2/2^*} \\ &\quad\leqslant \frac{\|f\|_{L^m(\Omega)}}{S\gamma - \|f\|_{L^m(\Omega)}\varepsilon}\left(\varepsilon^{-2^*/(2m'-2^*)} + C\right).\end{aligned} \tag{79}$$

We choose $\varepsilon = S\gamma/2\|f\|_{L^m(\Omega)}$ to get

$$\begin{aligned}&\left(\int_\Omega u_n^{2^*}dx\right)^{2/2^*} \\ &\quad\leqslant \frac{2\|f\|_{L^m(\Omega)}}{S\gamma}\left(\left(\frac{2\|f\|_{L^m(\Omega)}}{S\gamma}\right)^{2^*/(2m'-2^*)} + C\right).\end{aligned} \tag{80}$$

So the boundedness of u_n in $L^{2^*}(\Omega)$ is obtained; using the estimate and (75) again, we have the estimate of u_n in $H_0^1(\Omega)$. □

Once the boundedness of u_n in $H_0^1(\Omega)$ is obtained, we can prove the following existence theorem.

Theorem 18. *Suppose that $f \in L^m(\Omega)$ with $m = 2N/(N+2+(N-2)\alpha^-)$, and $0 < \alpha^- < 1 < \alpha^+ < 2-1/m$. Then problem (1) has a solution u in $H_0^1(\Omega)$.*

Lemma 19. *Suppose that $f \in L^m(\Omega)$ with $m \geqslant 2N/(N+2+(N-2)\alpha^-)$, and $\alpha^- < 1 < \alpha^+ < 2-1/m$. Then the solution u of (1) given by Theorem 18 is such that*

(i) *if* $m > N/2$, *then* $u \in L^{\infty}(\Omega)$;

(ii) *if* $2N/(N+2+(N-2)\alpha^{-}) \leqslant m < N/2$, *then* $u \in L^{s}(\Omega)$, $s = Nm(1+\alpha^{-})/(N-2m)$.

Proof. The proof of (i) is similar to that for Lemma 8(i), and we omit the details here.

To prove (ii), if $N(1+\alpha^{+})/((1+\alpha^{-})(N-2)+2(1+\alpha^{+})) \leqslant m < N/2$, the proof is identical to that for Lemma 13, and we also omit it here.

If $m = 2N/(N+2+(N-2)\alpha^{-})$, we can prove the results by Sobolev Embedding Theorem.

If $2N/(N+2+(N-2)\alpha^{-}) < m < N(1+\alpha^{+})/((1+\alpha^{-})(N-2)+2(1+\alpha^{+}))$, we choose $1 < \delta < (1+\alpha^{+})/2$, and we use once again $u_n^{2\delta-1}$ as a test function in (7). Using $\delta > 1 > (1+\alpha^{-})/2$, as well as Hölder's inequality, Sobolev Embedding Theorem, Lemma 5, and (2), we get

$$\begin{aligned} &\frac{S\gamma(2\delta-1)}{\delta^2}\left(\int_{\Omega} u_n^{2^*\delta}dx\right)^{2/2^*} \\ &\quad \leqslant \|f\|_{L^m(\Omega)}\left(\int_{\Omega} u_n^{(2\delta-1-\alpha^{-})m'}dx\right)^{1/m'} + C\|f\|_{L^m(\Omega)}. \end{aligned} \tag{81}$$

The choice of δ in such a way that $2^*\delta = (2\delta-1-\alpha^{-})m'$ yields that $1 < \delta < (1+\alpha^{+})/2$, if and only if $2N/(N+2+(N-2)\alpha^{-}) < m < N(1+\alpha^{+})/((1+\alpha^{-})(N-2)+2(1+\alpha^{+}))$, and that $2^*\delta = s$. The choice of $m < N/2$ implies that $2/2^* > 1/m'$. Thus we have the boundedness of u_n in $L^s(\Omega)$, and so does the limit u in $L^s(\Omega)$. □

Competing Interests

The authors declare that they have no competing interests.

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