Hindawi
Journal of Function Spaces
Volume 2017, Article ID 1014250, 6 pages
https://doi.org/10.1155/2017/1014250

Research Article

Global Structure of Positive Solutions for Some Second-Order Multipoint Boundary Value Problems

Hongyu Li and Junting Zhang

College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, Shandong 266590, China

Correspondence should be addressed to Hongyu Li; sdlhy1978@163.com

Received 11 May 2017; Accepted 29 August 2017; Published 15 October 2017

Academic Editor: Lishan Liu

We investigate in this paper the following second-order multipoint boundary value problem: $-(L\varphi)(t) = \lambda f(t,\varphi(t))$, $0 \le t \le 1$, $\varphi'(0) = 0$, $\varphi(1) = \sum_{i=1}^{m-2}\beta_i\varphi(\eta_i)$. Under some conditions, we obtain global structure of positive solution set of this boundary value problem and the behavior of positive solutions with respect to parameter λ by using global bifurcation method. We also obtain the infinite interval of parameter λ about the existence of positive solution.

1. Introduction

In this paper, we shall study the following second-order multipoint boundary value problem:

$$\begin{aligned} &-(L\varphi)(t) = \lambda f(t,\varphi(t)), \quad 0 \le t \le 1, \\ &\varphi'(0) = 0, \\ &\varphi(1) = \sum_{i=1}^{m-2}\beta_i\varphi(\eta_i), \end{aligned} \tag{1}$$

where $(L\varphi)(t) = (p(t)\varphi'(t))' + q(t)\varphi(t)$, $\eta_i \in (0,1)$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $\beta_i \in [0+\infty)$, and λ is a positive parameter.

The multipoint boundary value problems for ordinary differential equations play an important role in physics and applied mathematics, and so on. The existence and multiplicity of nontrivial solutions for multipoint boundary value problems have been extensively considered (including positive solutions, negative solutions, or sign-changing solutions) by using the fixed point theorem with lattice, fixed point index theory, coincidence degree theory, Leray-Schauder continuation theorems, upper and lower solution method, and so on (see [1–25] and references therein). On the other hand, some scholars have studied the global structure of nontrivial solutions for second-order multipoint boundary value problems (see [26–32] and references therein).

There are few papers about the global structure of nontrivial solutions for the boundary value problem (1). Motivated by [1, 26–32], we shall investigate the global structure of positive solutions of the boundary value problem (1). In [1], the authors only have studied the existence of positive solutions, but in this paper, we prove that the set of nontrivial positive solutions of the boundary value problem (1) possesses an unbounded connected component.

This paper is arranged as follows. In Section 2, some notation and lemmas are presented. In Section 3, we prove the main results of the boundary value problem (1). Finally, in Section 4, two examples are given to illustrate the main results obtained in Section 3.

2. Preliminaries

Let E be a Banach space, $P \subset E$ be a cone, and $A : P \to P$ be a completely continuous operator.

Definition 1 (see [33]). Let $\Omega \subset P$ be an open set, $A : \Omega \to P$, and $\lambda_0 \in (0, +\infty)$. If, for any $\epsilon > 0$, there exists the solution $(\lambda, x) \in R^+ \times \Omega$ of the equation $x = \lambda A x$, satisfying

$$\begin{gathered} \left|\lambda - \lambda_0\right| < \epsilon, \\ 0 < \|x\| < \epsilon, \end{gathered} \tag{2}$$

then λ_0 is called a bifurcation point of the cone operator A.

Definition 2 (see [33]). Let $\Omega \subset P$ be an open set, $A : \Omega \to P$, and $\lambda_0 \in (0, +\infty)$. If, for any $\epsilon > 0$, there exists the solution $(\lambda, x) \in R^+ \times \Omega$ of the equation $x = \lambda A x$, satisfying

$$\begin{gathered} \left|\lambda - \lambda_0\right| < \epsilon, \\ \|x\| > \frac{1}{\epsilon}, \end{gathered} \tag{3}$$

then λ_0 is called an asymptotic bifurcation point of the cone operator A.

Definition 3 (see [34]). Let $T : E \to E$ be a linear operator and T map P into P. The linear operator T is u_0-positive if there exists $u_0 \in P \setminus \{\theta\}$ such that, for any $x \in P \setminus \{\theta\}$, we can find an integer n and real numbers $\alpha_0 > 0$, $\beta_0 > 0$ such that $\alpha_0 u_0 \le T^n x \le \beta_0 u_0$.

Lemma 4 (see [34]). *Let $\Omega(P)$ be an open set of P. Assume that the operator A has no fixed points on $\partial\Omega(P)$. If there exists a linear operator B and $u^* \in P \setminus \{\theta\}$ such that*

(i) $Ax \ge Bx$, $x \in \partial\Omega(P)$;

(ii) *for some* n, $B^n u^* \ge u^*$,

then $i(A, \Omega(P), P) = 0$.

Lemma 5 (see [34]). *Let $A : P \to P$ be completely continuous and T be a completely continuous u_0-bounded linear operator. If, for any $x \in P$, $Ax \ge Tx$, $\lambda A x = x$, then $\lambda \le 1/r(T)$, where $1/r(T)$ is unique eigenvalue of T corresponding to positive eigenfunction.*

Lemma 6 (see [34]). *Let M be a compact metric space and A and B be disjoint, compact subsets of M. If there does not exist connected subset C of M such that $C \cap A \ne \emptyset$ and $C \cap B \ne \emptyset$, then there exist disjoint compact subsets M_A and M_B such that $A \subset M_A$, $B \subset M_B$ and $M = M_A \cup M_B$.*

3. Main Results

Let $E = C[0,1]$ with the norm $\|\varphi(t)\| = \max_{t\in[0,1]}|\varphi(t)|$; then E is a Banach space. Let $P = \{\varphi \in E \mid \varphi(t) \ge 0,\ t \in [0,1]\}$. Obviously, P is a normal cone of E.

In this paper, we always assume that

(H_1) $p(t) \in C^1[0,1]$, $p(t) > 0$, $q(t) \in C[0,1]$, $q(t) \le 0$.

Lemma 7 (see [1]). *Suppose that* (H_1) *holds. Let Φ_1 and Φ_2 be the solutions of*

$$\begin{gathered} (L\varphi)(t) = 0, \quad 0 < t < 1, \\ \varphi'(0) = 0, \\ \varphi(1) = 1, \end{gathered} \tag{4}$$

$$\begin{gathered} (L\varphi)(t) = 0, \quad 0 < t < 1, \\ \varphi(0) = 1, \\ \varphi(1) = 0, \end{gathered} \tag{5}$$

respectively. Then

(i) Φ_1 *is increasing on* $[0,1]$ *and* $\Phi_1 > 0$, $t \in [0,1]$;

(ii) Φ_2 *is decreasing on* $[0,1]$ *and* $\Phi_2 > 0$, $t \in [0,1)$.

Let

$$G(t,s) = \begin{cases} \dfrac{1}{\rho}\Phi_1(t)\Phi_2(s), & 0 \le t \le s \le 1, \\ \dfrac{1}{\rho}\Phi_1(s)\Phi_2(t), & 0 \le s \le t \le 1, \end{cases} \tag{6}$$

where $\rho = -\Phi_1(0)\Phi_2'(0) > 0$ *by [1]. Let*

$$K(t,s) = G(t,s) + D^{-1}\Phi_1(t)\sum_{i=1}^{m-2}\beta_i G(\eta_i, s), \qquad 0 \le t, s \le 1, \tag{7}$$

where $D = 1 - \sum_{i=1}^{m-2}\beta_i\Phi_1(\eta_i)$.

Define the operators A, B, and F:

$$(A\varphi)(t) = \int_0^1 K(t,s)\,\widetilde{p}(s)\,f(s,\varphi(s))\,ds, \tag{8}$$

$$(B\varphi)(t) = \int_0^1 K(t,s)\,\widetilde{p}(s)\,\varphi(s)\,ds, \tag{9}$$

$$(F\varphi)(t) = f(t,\varphi(t)), \tag{10}$$

where $\widetilde{p}(s) = (1/p(s))\exp(\int_0^s (p'(x)/p(x))dx)$ *and* $K(t,s)$ *is defined by (7).*

Obviously, $A = BF$. *It is easy to know that the solutions of the boundary value problem (1) are equivalent to the solutions of the equation*

$$\varphi = \lambda A\varphi. \tag{11}$$

Let $L = \overline{\{(\lambda,\varphi) \in (0,+\infty) \times P \mid \varphi = \lambda A\varphi,\ \varphi \ne \theta\}}$ *be the closure of nontrivial positive solution set of (11). Then L is also the closure of the nontrivial positive solution set of the boundary value problem (1).*

We give the following assumptions:

(H_2) $\sum_{i=1}^{m-2}\beta_i\Phi_1(\eta_i) < 1$, *where* $\Phi_1(t)$ *is the solution of (4).*

(H_3) $f : [0,1] \times R^+ \to R^+$ *is continuous,* $f(t,0) = 0$, uniformly on $t \in [0,1]$.

(H_4) $\liminf_{u\to 0^+}(f(t,u)/u) \geq \alpha > 0$, uniformly on $t \in [0,1]$.

(H_5) $\limsup_{u\to+\infty}(f(t,u)/u) = 0$, uniformly on $t \in [0,1]$.

Lemma 8 (see [1]). *Suppose that* (H_1)–(H_3) *are satisfied. Then, for the operator B defined by (9), the spectral radius* $r(B) \neq 0$ *and B has a positive eigenfunction corresponding to its first eigenvalue* $\lambda_1 = (r(B))^{-1}$.

Theorem 9. *Suppose that* (H_1)–(H_4) *are satisfied. Then*

(i) *the operator A defined by (8) has at least a bifurcation point* $\lambda^* \in [0, \lambda_1/\alpha]$ *corresponding to positive solution; the operator A has no bifurcation points in* $(\lambda_1/\alpha, +\infty)$ *corresponding to positive solution, where* λ_1 *is defined by Lemma 8;*

(ii) *L possesses an unbounded connected component* $C \subset (0,+\infty) \times P$ *passing through* (λ^*, θ), *and* $C \cap ((\lambda_1/\alpha, +\infty) \times \{\theta\}) = \emptyset$, *where* λ_1 *is defined by Lemma 8.*

Proof. By (H_1)–(H_3), it is easy to know that $A : P \to P$ is completely continuous and $B : P \to P$ is completely continuous; and $A\theta = \theta$. By Lemma 8, we have $r(B) = 1/\lambda_1$.

By (H_3), for any $\epsilon > 0$, there exists $r_\epsilon > 0$ such that

$$\frac{f(t,u)}{u} \geq \alpha - \epsilon, \quad \forall t \in [0,1], \ 0 \leq u \leq r_\epsilon, \tag{12}$$

that is,

$$f(t,u) \geq (\alpha - \epsilon) u, \quad \forall t \in [0,1], \ 0 \leq u \leq r_\epsilon. \tag{13}$$

Let $N_{r_\epsilon}(P) = \{\varphi \in P \mid \|\varphi\| < r_\epsilon\}$. From (8) and (13), for any $\varphi \in \overline{N}_{r_\epsilon}(P)$, we have

$$\begin{aligned}(A\varphi)(t) &\geq (\alpha-\epsilon)\int_0^1 K(t,s)\,\widetilde{p}(s)\,\varphi(s)\,ds \\ &= (\alpha-\epsilon)(B\varphi)(t) = (T\varphi)(t),\end{aligned} \tag{14}$$

where $T = (\alpha - \epsilon)B$. Clearly, $T : P \to P$ is completely continuous and $r(T) = (\alpha-\epsilon) r(B) = (\alpha - \epsilon)/\lambda_1$.

By Lemma 7 and (6) and (7), it follows that

$$\begin{aligned}&\Phi_1(t)\left(D^{-1}\sum_{i=1}^{m-2}\beta_i G(\eta_i,s)\right) \leq K(t,s) \\ &\quad \leq \left(\frac{1}{\rho}\Phi_2(s) + D^{-1}\sum_{i=1}^{m-2}\beta_i G(\eta_i,s)\right)\Phi_1(t), \\ &\qquad \forall t,s \in [0,1].\end{aligned} \tag{15}$$

For any $\varphi \in P$, by (14) and (15), we have

$$\begin{aligned}(T\varphi)(t) &\geq (\alpha-\epsilon)\,\Phi_1(t) \\ &\cdot \int_0^1\left(D^{-1}\sum_{i=1}^{m-2}\beta_i G(\eta_i,s)\right)\widetilde{p}(s)\,\varphi(s)\,ds, \\ (T\varphi)(t) &\leq (\alpha-\epsilon)\,\Phi_1(t) \\ &\cdot \int_0^1\left(\frac{1}{\rho}\Phi_2(s) + D^{-1}\sum_{i=1}^{m-2}\beta_i G(\eta_i,s)\right)\widetilde{p}(s)\,\varphi(s)\,ds.\end{aligned} \tag{16}$$

Let $u_0 = \Phi_1(t)$. It follows from (16) that T is a u_0-bounded operator by Definition 3. By Krein-Rutman theorem, there exists $\varphi^* \in P \setminus \{\theta\}$ such that

$$T\varphi^* = r(T)\,\varphi^*. \tag{17}$$

By (14) and (17), we have

$$\begin{aligned}&\lambda(A\varphi) \geq \lambda(T\varphi), \quad \forall \varphi \in \partial N_{r_\epsilon}(P), \ \lambda \geq \frac{\lambda_1}{\alpha-\epsilon}, \\ &\lambda(T\varphi^*) = \lambda r(T)\,\varphi^* \geq \varphi^*, \quad \forall \lambda \geq \frac{\lambda_1}{\alpha-\epsilon}.\end{aligned} \tag{18}$$

So, by (18) and Lemma 4, we have

$$i\left(\lambda A, N_{r_\epsilon}(P), P\right) = 0, \quad \forall \lambda \geq \frac{\lambda_1}{\alpha-\epsilon}. \tag{19}$$

In the following, we prove that the operator A has at least one bifurcation point on $[0, \lambda_1/(\alpha-\epsilon)]$ and has no bifurcation points on $(\lambda_1/(\alpha-\epsilon), +\infty)$.

We shall prove that, for any $\overline{\epsilon} \in (0, r_\epsilon)$, there must exist $\lambda_{\overline{\epsilon}} \in [0, \lambda_1/(\alpha-\epsilon)]$ and $\varphi_{\overline{\epsilon}} \in \partial N_{\overline{\epsilon}}(P)$ such that

$$\varphi_{\overline{\epsilon}} = \lambda_{\overline{\epsilon}} A \varphi_{\overline{\epsilon}}, \tag{20}$$

where $N_{\overline{\epsilon}}(P) = \{\varphi \in P \mid \|\varphi\| < \overline{\epsilon}\}$.

Without loss of generality, we may assume that the equation $(\lambda_1/(\alpha-\epsilon))A\varphi = \varphi$ has no solutions on $\partial N_{\overline{\epsilon}}(P)$. By (19), we get

$$i\left(\frac{\lambda_1}{\alpha-\epsilon}A, N_{\overline{\epsilon}}(P), P\right) = 0. \tag{21}$$

Obviously,

$$i\left(0, N_{\overline{\epsilon}}(P), P\right) = 1. \tag{22}$$

Set

$$H(t,\varphi) = \varphi - t\frac{\lambda_1}{\alpha-\epsilon}A\varphi, \quad t \in [0,1]. \tag{23}$$

By (21) and (22) and the homotopy invariance of fixed point index, there exists $t^* \in [0,1]$ such that $H(t^*, \varphi) = \theta$ has a solution $\varphi^*_{\overline{\epsilon}} \in \partial N_{\overline{\epsilon}}(P)$. Namely, $\varphi^*_{\overline{\epsilon}} = \lambda^*_{\overline{\epsilon}} A \varphi^*_{\overline{\epsilon}}$, where $\lambda^*_{\overline{\epsilon}} = \lambda_1 t^*/(\alpha-\epsilon) \in [0, \lambda_1/(\alpha-\epsilon)]$. □

Choose $1/n < r_\epsilon$. Then there exist $\lambda_n \in [0, \lambda_1/(\alpha-\epsilon)]$ and $\varphi_n \in P$ with $\|\varphi_n\| = 1/n$ such that $\varphi_n = \lambda_n A\varphi_n$. And $\varphi_n \neq \theta$, $\varphi_n \to \theta (n \to \infty)$. Assume that $\lambda_n \to \lambda^* (n \to \infty)$. Then $\lambda^* \in [0, \lambda_1/(\alpha-\epsilon)]$ is a bifurcation point of the cone operator A.

By (14) and Lemma 5, for any $0 < r < r_\epsilon$, the equation $\varphi = \lambda A\varphi$ has no solutions in $(\lambda_1/(\alpha-\epsilon), +\infty) \times \partial N_r(P)$, where $N_r(P) = \{\varphi \in P \mid \|\varphi\| < r\}$. Hence, A has no bifurcation points in $(\lambda_1/(\alpha-\epsilon), +\infty)$ corresponding to positive solution, and $L_P \cap ((\lambda_1/(\alpha-\epsilon), +\infty) \times \{\theta\}) = \emptyset$.

Let $G = \{(\lambda, \theta) \mid \lambda \in [0, \lambda_1/(\alpha-\epsilon)], \lambda$ is a bifurcation point of the cone operator $A\}$. By the above proof, we know that $G \neq \emptyset$. If, for any $\lambda \in [0, \lambda_1/(\alpha-\epsilon)]$, the connected component C_λ of L_P is bounded, which passes through (λ, θ), then C_λ is a compact set.

Let Q_λ be an open neighborhood of C_λ in $[0, +\infty) \times P$. If $\partial Q_\lambda \cap L_P \neq \emptyset$, then $Z = \overline{Q}_\lambda \cap L_p$ is a compact metric space, and $\partial Q_\lambda \cap L_P$ and C_λ are disjoint, compact subsets of Z. Since C_λ has the property of maximal connectivity, there exists no connected subset $\widetilde{C}$ of Z such that $\widetilde{C} \cap C_\lambda = \emptyset$ and $\widetilde{C} \cap (\partial Q_\lambda \cap L_P) = \emptyset$. By Lemma 6, we know that there exist two compact subsets Z_1, Z_2 of Z such that

$$\begin{gathered} Z = Z_1 \cup Z_2, \\ Z_1 \cap Z_2 = \emptyset, \\ C_\lambda \subset Z_1, \\ \partial Q_\lambda \cap L_P \subset Z_2. \end{gathered} \tag{24}$$

Obviously, the distance $\rho = d(Z_1, Z_2) > 0$. Let $Q'_\lambda = \{u \in [0, +\infty) \times P \mid d(u, Z_1) < \rho/3\}$. Then Q'_λ is an open neighborhood of Z_1. Let $Q''_\lambda = Q_\lambda \cap Q'_\lambda$. Then $\partial Q''_\lambda \cap L_P = \emptyset$. Let

$$Q^*_\lambda = \begin{cases} Q_\lambda, & \text{if } \partial Q_\lambda \cap L_P = \emptyset, \\ Q''_\lambda, & \text{if } \partial Q_\lambda \cap L_P \neq \emptyset. \end{cases} \tag{25}$$

Clearly, Q^*_λ is a bounded open set of $[0, +\infty) \times P$, and $\partial Q^*_\lambda \cap L_P = \emptyset$. Hence $\{Q^*_\lambda \mid (\lambda, \theta) \in G\}$ is an open covering of G. Since G is compact, there exist $(\lambda_i, \theta) \in G$ $(i = 1, 2, \ldots, n)$ such that $\{Q^*_{\lambda_i} \mid i = 1, 2, \ldots, n\}$ is also an open covering of G. Let $Q^* = \bigcup_{i=1}^n Q^*_{\lambda_i}$. Then Q^* is a bounded open set of $[0, +\infty) \times P$, and $G \subset Q^*, \partial Q^* \cap L_P = \emptyset$.

Take sufficiently large $\widetilde{\lambda} > \lambda_1/(\alpha-\epsilon)$ such that $\overline{Q}^* \subset [0, \widetilde{\lambda}] \times P$. For $0 < r < r_\epsilon$, let $U_r = [0, \widetilde{\lambda}] \times N_r(P)$, where $N_r(P) = \{\varphi \in P \mid \|\varphi\| < r\}$. Evidently, U_r is an open set of $[0, \widetilde{\lambda}] \times P$, and $\partial U_r = [0, \widetilde{\lambda}] \times \partial N_r(P)$. And $\lambda A\varphi = \varphi$ has no nontrivial solutions on $\partial U_r \setminus Q^*$ when r is sufficiently small.

Let $X = Q^* \cup U_r$. Then $\partial X \subset \partial Q^* \cup \partial(U_r \setminus Q^*)$. Since $[0, \widetilde{\lambda}] \times \{\theta\} \subset X$ and $\partial Q^* \cap L_P = \emptyset$, we know that $\varphi = \lambda A\varphi$ has no solutions on ∂X. By the general homotopy invariance of topological degree, we get

$$i\left(\widetilde{\lambda} A, X\left(\widetilde{\lambda}\right), P\right) = i\left(0A, X(0), P\right) = 1. \tag{26}$$

Since $Q^* \subset [0, \widetilde{\lambda}] \times P$, $Q^*(\widetilde{\lambda}) = (\{\widetilde{\lambda}\} \times P) \cap Q^* = \emptyset$, so

$$\begin{aligned} X\left(\widetilde{\lambda}\right) &= \left(\left\{\widetilde{\lambda}\right\} \times P\right) \cap U_r = U_r\left(\widetilde{\lambda}\right) \\ &= \left(\left\{\widetilde{\lambda}\right\} \times P\right) \cap N_r(P) = N_r\left(\widetilde{\lambda}, P\right). \end{aligned} \tag{27}$$

Therefore, by (20), we have

$$\begin{aligned} i\left(\widetilde{\lambda} A, X\left(\widetilde{\lambda}\right), P\right) &= i\left(\widetilde{\lambda} A, N_r\left(\widetilde{\lambda}, P\right), P\right) \\ &= i\left(\widetilde{\lambda} A, N_r(P), P\right) = 0, \end{aligned} \tag{28}$$

which contradicts (26).

Hence, L_P possesses an unbounded connected component $C_\lambda \subset (0, +\infty) \times P$ passing through (λ, θ).

By the above proof and the arbitrariness of ϵ, we obtain that (i) the cone operator A has at least a bifurcation point $\lambda^* \in [0, \lambda_1/\alpha]$ (the cone operator A has no bifurcation point in $(\lambda_1/\alpha, +\infty)$) and (ii) L_P possesses an unbounded connected component $C \subset (0, +\infty) \times P$ passing through (λ^*, θ), and $C \cap ((\lambda_1/\alpha, +\infty) \times \{\theta\}) = \emptyset$.

Theorem 10. *Suppose that* (H_1)–(H_3) *and* (H_5) *are satisfied. Then the operator A has no asymptotic bifurcation points in* $[0, +\infty)$.

Proof. For any $\lambda_0 \in [0, +\infty)$, there exists sufficiently small $\epsilon_0 > 0$ such that

$$\lambda_0 \epsilon_0 < \lambda_1, \tag{29}$$

where λ_1 is defined by Lemma 8.

By (H_5), for the above $\epsilon_0 > 0$, there exists $R > 0$ such that

$$\frac{f(t,u)}{u} \le \epsilon_0, \quad \forall t \in [0,1], \ u \ge R, \tag{30}$$

that is,

$$f(t,u) \le u\epsilon_0, \quad \forall t \in [0,1], \ u \ge R. \tag{31}$$

Set $M = \max_{t\in[0,1], 0\le u\le R} f(t,u)$; then

$$f(t,u) \le \epsilon_0 u + M, \quad \forall t \in [0,1], \ u \ge 0. \tag{32}$$

Let $G(\lambda_0) = \{\varphi \in P \mid \varphi = \lambda A\varphi, \ 0 \le \lambda \le \lambda_0\}$. For any $\overline{\varphi} \in G(\lambda_0)$, there exists $\overline{\lambda} \in [0, \lambda_0]$ such that $\overline{\varphi} = \overline{\lambda} A\overline{\varphi}$. By (32), we have

$$\begin{aligned} \overline{\varphi}(t) &= \overline{\lambda} A\overline{\varphi}(t) = \overline{\lambda} \int_0^1 K(t,s)\, \widetilde{p}(s)\, f(s, \overline{\varphi}(s))\, ds \\ &\le \lambda_0 \epsilon_0 \int_0^1 K(t,s)\, \widetilde{p}(s)\, \overline{\varphi}(s)\, ds \\ &\quad + M \int_0^1 K(t,s)\, \widetilde{p}(s)\, ds = \overline{T}(\overline{\varphi})(t) + v_0, \end{aligned} \tag{33}$$

where $\overline{T} = \lambda_0 \epsilon_0 B$, $v_0 = M \int_0^1 K(t,s)\widetilde{p}(s) ds$. It follows from (33) that $r(\overline{T}) = \lambda_0 \epsilon_0 r(B) < 1$.

By (33), we get that $\overline{\varphi}(t) \le (I - \overline{T})^{-1} v_0$. So $\|\overline{\varphi}\| \le \|(I - \overline{T})^{-1}\| \|v_0\|$. Therefore, $G(\lambda_0)$ is bounded. By the arbitrariness of λ_0, the operator A has no asymptotic bifurcation point in $(0, +\infty)$. □

By Theorems 9 and 10, we have the following theorem.

Theorem 11. *Suppose that* (H_1)–(H_5) *are satisfied. Then, for any* $\lambda \in (\lambda_1/\alpha, +\infty)$, *the boundary value problem (1) has at least one positive solution.*

Furthermore, we take $\alpha = +\infty$ *in* (H_4), *that is, the following condition* (H_4').

(H_4') $\liminf_{u \to 0^+}(f(t,u)/u) = +\infty$, uniformly on $t \in [0, 1]$.

Theorem 12. *Suppose that* (H_1)–(H_3) (H_5) *and* (H_4') *are satisfied. Then*

(i) *the operator* A *has no asymptotic bifurcation points in* $[0, +\infty)$;

(ii) L_P *possesses an unbounded connected component* $C \subset (0, +\infty) \times P$ *passing through* $(0, \theta)$, *and* $C \cap ((0, +\infty) \times \{\theta\}) = \emptyset$.

Proof. Since (H_1)–(H_3) and (H_5) are satisfied, it follows from Theorem 10 that (i) holds.

By (H_4'), for sufficiently large $M > 0$, there exists $r_M > 0$ such that

$$\frac{f(t,u)}{u} \geq M, \quad \forall t \in [0,1], \; 0 \leq u \leq r_M, \tag{34}$$

that is,

$$f(t,u) \geq Mu, \quad \forall t \in [0,1], \; 0 \leq u \leq r_M. \tag{35}$$

Let $\widetilde{N}(P) = \{\varphi \in P \mid \|\varphi\| < r_M\}$. From (7) and (35), for any $\varphi \in \widetilde{N}(P)$, we have

$$\begin{aligned} (A\varphi)(t) &\geq M \int_0^1 K(t,s)\,\widetilde{p}(s)\,\varphi(s)\,ds = M(B\varphi)(t) \\ &= (\widetilde{T}\varphi)(t), \end{aligned} \tag{36}$$

where $\widetilde{T} = MB$. Clearly, $\widetilde{T} : P \to P$ is completely continuous and $r(\widetilde{T}) = Mr(B) = M/\lambda_1$.

Similar to the proof of Theorem 9, we obtain that the operator A has a bifurcation point $\lambda_* \in [0, \lambda_1/M]$ corresponding to positive solution and L_P possesses an unbounded connected component $C \subset (0, +\infty) \times P$ passing through (λ_*, θ), and $C \cap ((\lambda_1/M, +\infty) \times \{\theta\}) = \emptyset$. Since M can take sufficiently large value, we know that (i) and (ii) hold. The proof is completed. □

It follows from Theorem 12 that we have the following theorem.

Theorem 13. *Suppose that* (H_1)–(H_3) (H_4') *and* (H_5) *are satisfied. Then, for any* $\lambda \in (0, +\infty)$, *the boundary value problem (1) has at least one positive solution.*

4. Applications

In this section, two examples are given to illustrate our main results.

Example 14. Consider the following boundary value problem:

$$\begin{aligned} -\varphi''(t) &= \lambda f(t, \varphi(t)), \quad 0 \leq t \leq 1, \\ \varphi'(0) &= 0, \\ \varphi(1) &= \frac{1}{2}\varphi\left(\frac{1}{2}\right), \end{aligned} \tag{37}$$

where

$$f(t,u) = \begin{cases} 2u + u^2 t, & t \in [0,1], \; u \in [0,10), \\ 100t + \sqrt{10u} + 10, & t \in [0,1], \; u \in [10, +\infty). \end{cases} \tag{38}$$

By simple calculations, $\lambda_1 \approx 6.9497$. The nonlinear term f satisfies the conditions of Theorem 11. Thus, for any $\lambda > 3.4749$, the boundary value problem (37) has at least one positive solution by Theorem 11.

Example 15. Consider the following boundary value problem:

$$\begin{aligned} -\varphi''(t) &= \lambda f(t, \varphi(t)), \quad 0 \leq t \leq 1, \\ \varphi'(0) &= 0, \\ \varphi(1) &= \frac{1}{2}\varphi\left(\frac{1}{2}\right), \end{aligned} \tag{39}$$

where

$$f(t,u) = \begin{cases} t\sqrt{u} + u^{1/3}, & t \in [0,1], \; u \in [0,1], \\ t + \sqrt{u}, & t \in [0,1], \; u \in [1, +\infty). \end{cases} \tag{40}$$

The nonlinear term f satisfies the conditions of Theorem 13. Thus, for any $\lambda > 0$, the boundary value problem (39) has at least one positive solution by Theorem 13.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

Acknowledgments

The project is supported by the National Science Foundation of China (11571207).

References

[1] G. Zhang and J. Sun, "Existence of positive solutions for singular second-order m-point boundary value problems," *Acta Mathematicae Applicatae Sinica. English Series*, vol. 20, no. 4, pp. 655–664, 2004.

[2] Y. Sun, L. Liu, J. Zhang, and R. P. Agarwal, "Positive solutions of singular three-point boundary value problems for second-order differential equations," *Journal of Computational and Applied Mathematics*, vol. 230, no. 2, pp. 738–750, 2009.

[3] Y. Cui, F. Wang, and Y. Zou, “Computation for the fixed point index and its applications,” *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal*, vol. 71, no. 1-2, pp. 219–226, 2009.

[4] K. Zhang and X. Xie, “Existence of sign-changing solutions for some asymptotically linear three-point boundary value problems,” *Nonlinear Analysis*, vol. 70, no. 7, pp. 2796–2805, 2009.

[5] L. Liu, B. Liu, and Y. Wu, “Nontrivial solutions for higher-order m-point boundary value problem with a sign-changing nonlinear term,” *Applied Mathematics and Computation*, vol. 217, no. 8, pp. 3792–3800, 2010.

[6] F. Wang and Y. Cui, “On the existence of solutions for singular boundary value problem of third-order differential equations,” *Mathematica Slovaca*, vol. 60, no. 4, pp. 485–494, 2010.

[7] Z. Bai, “On solutions of some fractional m-point boundary value problems at resonance,” *Electronic Journal of Qualitative Theory of Differential Equations*, pp. 1–15, 2010.

[8] L. Liu, X. Hao, and Y. Wu, “Unbounded solutions of second-order multipoint boundary value problem on the half-line,” *Boundary Value Problems*, vol. 2010, Article ID 236560, 2010.

[9] Z. Bai and Y. Zhang, “The existence of solutions for a fractional multi-point boundary value problem,” *Computers & Mathematics with Applications. An International Journal*, vol. 60, no. 8, pp. 2364–2372, 2010.

[10] X. Hao, L. Liu, and Y. Wu, “On positive solutions of an m-point nonhomogeneous singular boundary value problem,” *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal*, vol. 73, no. 8, pp. 2532–2540, 2010.

[11] F. Wang, Y. Cui, and F. Zhang, “Existence of nonnegative solutions for second order m-point boundary value problems at resonance,” *Applied Mathematics and Computation*, vol. 217, no. 10, pp. 4849–4855, 2011.

[12] Z. Bai, “Solvability for a class of fractional m-point boundary value problem at resonance,” *Computers & Mathematics with Applications. An International Journal*, vol. 62, no. 3, pp. 1292–1302, 2011.

[13] J. Jiang, L. Liu, and Y. Wu, “Symmetric positive solutions to singular system with multi-point coupled boundary conditions,” *Applied Mathematics and Computation*, vol. 220, pp. 536–548, 2013.

[14] Y. Cui, “Computation of topological degree in ordered banach spaces with lattice structure and applications,” *Applications of Mathematics*, vol. 58, no. 6, pp. 689–702, 2013.

[15] H. Li, “Existence of nontrivial solutions for unilaterally asymptotically linear three-point boundary value problems,” *Abstract and Applied Analysis*, Article ID 263042, Art. ID 263042, 7 pages, 2014.

[16] Y. Cui and Y. Zou, “An existence and uniqueness theorem for a second order nonlinear system with coupled integral boundary value conditions,” *Applied Mathematics and Computation*, vol. 256, pp. 438–444, 2015.

[17] L. Guo, L. Liu, and Y. Wu, “Existence of positive solutions for singular higher-order fractional differential equations with infinite-point boundary conditions,” *Boundary Value Problems*, Paper No. 114, 22 pages, 2016.

[18] L. Liu, C. Liu, F. Wang, and Y. Wu, “Strong convergence of a general iterative algorithm for asymptotically nonexpansive semigroups in Banach spaces,” *Journal of Nonlinear Science and its Applications. JNSA*, vol. 9, no. 10, pp. 5695–5711, 2016.

[19] Z. Bai, S. Zhang, S. Sun, and C. Yin, “Monotone iterative method for fractional differential equations,” *Electronic Journal of Differential Equations*, vol. 2016, article 6, 2016.

[20] X. Zhang, L. Liu, and Y. Wu, “Fixed point theorems for the sum of three classes of mixed monotone operators and applications,” *Fixed Point Theory and Applications*, Paper No. 49, 22 pages, 2016.

[21] Y. Cui, “Existence of solutions for coupled integral boundary value problem at resonance,” *Publicationes Mathematicae Debrecen*, vol. 89, no. 1-2, pp. 73–88, 2016.

[22] L. Guo, L. Liu, and Y. Wu, “Existence of positive solutions for singular fractional differential equations with infinite-point boundary conditions,” *Lithuanian Association of Nonlinear Analysts (LANA). Nonlinear Analysis. Modelling and Control*, vol. 21, no. 5, pp. 635–650, 2016.

[23] Y. Cui and Y. Zou, “Existence of solutions for second-order integral boundary value problems,” *Lithuanian Association of Nonlinear Analysts (LANA). Nonlinear Analysis. Modelling and Control*, vol. 21, no. 6, pp. 828–838, 2016.

[24] Z. Bai, X. Dong, and C. Yin, “Existence results for impulsive nonlinear fractional differential equation with mixed boundary conditions,” *Boundary Value Problems*, Paper No. 63, 11 pages, 2016.

[25] Y. Cui, “Uniqueness of solution for boundary value problems for fractional differential equations,” *Applied Mathematics Letters. An International Journal of Rapid Publication*, vol. 51, pp. 48–54, 2016.

[26] L. Liu, F. Sun, X. Zhang, and Y. Wu, “Bifurcation analysis for a singular differential system with two parameters via to topological degree theory,” *Lithuanian Association of Nonlinear Analysts (LANA). Nonlinear Analysis. Modelling and Control*, vol. 22, no. 1, pp. 31–50, 2017.

[27] Y. An, “Global structure of nodal solutions for second-order m -point boundary value problems with superlinear nonlinearities,” *Boundary Value Problems*, vol. 2011, Article ID 715836, 2011.

[28] X. Zhang, “Global structure of positive solutions for superlinear singular m-point boundary-value problems,” *Electronic Journal of Differential Equations*, No. 13, 12 pages, 2008.

[29] J. Sun, X. Xian, and D. O’Regan, “Nodal solutions for m-point boundary value problems using bifurcation methods,” *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal*, vol. 68, no. 10, pp. 3034–3046, 2008.

[30] X. Liu and J. Sun, “Bifurcation points and global structure of positive eigenvalues for cone mappings with no differentiability at the zero point,” *Journal of Systems Science and Mathematical Sciences*, vol. 30, no. 8, pp. 1111–1118, 2010.

[31] Y. Cui, J. Sun, and Y. Zou, “Global bifurcation and multiple results for Sturm-Liouville problems,” *Journal of Computational and Applied Mathematics*, vol. 235, no. 8, pp. 2185–2192, 2011.

[32] Z. Bai, “Eigenvalue intervals for a class of fractional boundary value problem,” *Computers & Mathematics with Applications. An International Journal*, vol. 64, no. 10, pp. 3253–3257, 2012.

[33] D. Guo, *Nonlinear Functional Analysis*, Shandong Sci. and Tech. Press, Jinan, China, 2nd edition, 2001.

[34] J. Sun, *Nonlinear Functional Analysis And Applications*, Science Press, Beijing, China, 2008.