Research Article

**Essential Norms of Volterra Type Operators between Zygmund Type Spaces**

Shanli Ye¹ and Caishu Lin²

¹School of Science, Zhejiang University of Science and Technology, Hangzhou 310023, China
²Department of Mathematics, Fujian Normal University, Fuzhou 350007, China

Correspondence should be addressed to Shanli Ye; ye_shanli@aliyun.com

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We investigate the boundedness of some Volterra type operators between Zygmund type spaces. Then, we give the essential norms of such operators in terms of \( g, \varphi \), their derivatives, and the \( n \)th power \( \varphi^n \) of \( \varphi \).

1. Introduction

Let \( D = \{ z : |z| < 1 \} \) be the open unit disk in the complex plane \( \mathbb{C} \) and let \( \partial D = \{ z : |z| = 1 \} \) be its boundary, and \( H(D) \) denote the set of all analytic functions on \( D \).

For every \( 0 < \alpha < \infty \), we denote by \( B^\alpha \) the Bloch type space of all functions \( f \in H(D) \) satisfying

\[
\|f\|_{B^\alpha} = |f(0)| + b^\alpha(f) = |f(0)| + \sup_{z \in D} \left( 1 - |z|^2 \right)^\alpha |f'(z)| < \infty
\]

(1)

endowed with the norm \( \|f\|_{B^\alpha} = |f(0)| + b^\alpha(f) \). The little Bloch type space \( B^1_0 \) consists of all \( f \in B^\alpha \) satisfying

\[
\lim_{|z| \to 1} (1 - |z|^2)^\alpha |f'(z)| = 0,
\]

and is obviously the closed subspace of \( B^\alpha \). When \( \alpha = 1 \), we get the classical Bloch space \( B^1 = B \) and little Bloch space \( B^1_0 = B_0 \). It is well known that, for \( 0 < \alpha < 1 \), \( B^\alpha \) is a subspace of \( H^\infty \), the Banach space of bounded analytic functions on \( D \). Some sources for results and references about the Bloch type functions are the papers of Yoneda [1], Stevic [2, 3], and the first author [4–7].

For \( 0 < \alpha < \infty \) we denote by \( \mathcal{L}^\alpha \), the Zygmund type space of those functions \( f \in H(D) \) satisfying

\[
\sup_{z \in D} \left( 1 - |z|^2 \right)^\alpha |f''(z)| < \infty,
\]

(2)

and the little Zygmund type space \( \mathcal{L}^\alpha_0 \) consists of all \( f \in \mathcal{L}^\alpha \) satisfying

\[
\lim_{|z| \to 1} (1 - |z|^2)^\alpha |f''(z)| = 0,
\]

and that \( \mathcal{L}^\alpha_0 \) is a closed subspace of \( \mathcal{L}^\alpha \). It can easily be proved that \( \mathcal{L}^\alpha \) is a Banach space under the norm

\[
\|f\|_\mathcal{L}^\alpha = |f(0)| + |f'(0)| + \sup_{z \in D} \left( 1 - |z|^2 \right)^\alpha |f''(z)|
\]

(3)

and that \( \mathcal{L}^\alpha_0 \) is a closed subspace of \( \mathcal{L}^\alpha \). When \( \alpha = 1 \), we get the classical Zygmund space \( \mathcal{L}^1 = \mathcal{L} \) and the little Zygmund space \( \mathcal{L}^1_0 = \mathcal{L}_0 \). It is clear that \( f \in \mathcal{L} \) if and only if \( f' \in B^1 \).

We consider the weighted Banach spaces of analytic functions

\[
H^{\infty}_\nu = \left\{ f \in H(D) : \|f\|_\nu = \sup_{z \in D} \nu(z) |f(z)| < \infty \right\}
\]

(4)

endowed with norm \( \| \cdot \|_\nu \), where the weight \( \nu : \mathbb{D} \to \mathbb{R}_+ \) is a continuous, strictly positive, and bounded function. The weight \( \nu \) is called radial, if \( \nu(z) = \nu(|z|) \) for all \( z \in \mathbb{D} \). For a weight \( \nu \) the associated weight \( \bar{\nu} \) is defined by

\[
\bar{\nu}(z) = \left( \sup_{f \in H^{\infty}_\nu} \|f\|_\nu \right)^{-1}, \quad z \in \mathbb{D}.
\]

(5)
We notice the standard weights \( v_\alpha(z) = (1 - |z|^2)^\alpha \), where \( 0 < \alpha < \infty \), and it is well known that \( \overline{v_\alpha} = v_\alpha \). We also consider the logarithmic weight

\[
\overline{v}_\log = \left( \log \frac{2}{1 - |z|^2} \right)^{-1}, \quad z \in D. \tag{6}
\]

It is straightforward to show that \( \overline{v}_\log = v_\log \).

For an analytic self-map \( \varphi \) of \( D \) and a function \( u \in H(D) \), we define the weighted composition operator as \( uC_\varphi f = u \cdot (f \circ \varphi) \) for \( f \in H(D) \). Weighted composition operators have been extensively studied recently. It is interesting to provide a function theoretic characterization when \( \varphi \) and \( u \) induce a bounded or compact composition operator on various function spaces. Some results on the boundedness and compactness of concrete operators between some spaces of analytic functions one of which is of Zygmund type can be found, for example, in [8–19].

Suppose that \( g : D \to \mathbb{C} \) is an analytic map. Let \( T_g \) and \( I_g \) denote the Volterra type operators with the analytic symbol \( g \) on \( D \), respectively:

\[
T_g f(z) = \int_0^z f(\xi) g'(\xi) d\xi, \quad z \in D, \tag{7}
\]

\[
I_g f(z) = \int_0^z f'(\xi) g(\xi) d\xi, \quad z \in D.
\]

If \( g(z) = z \), then \( T_g \) is an integral operator. While \( g(z) = \ln(1/(1-z)) \), then \( T_g \) is Cesàro operator. Pommerenke introduced the Volterra type operator \( T_g \) and characterized the boundedness of \( T_g \) between \( H^2 \) spaces in [20]. More recently, boundedness and compactness of Volterra type operators between several spaces of analytic functions have been studied by many authors; one may see [21, 22].

In this paper, we consider the following integral type operators, which were introduced by Li and Stević (see, e.g., [10, 23]): they can be defined by

\[
(C_g T_g f)(z) = \int_0^z f(\xi) g'(\xi) d\xi,
\]

\[
(C_g I_g f)(z) = \int_0^z f'(\xi) g(\xi) d\xi,
\]

\[
(T_g C_g f)(z) = \int_0^z f(\varphi(\xi)) g'(\xi) d\xi,
\]

\[
(I_g C_g f)(z) = \int_0^z f'(\varphi(\xi)) g(\xi) d\xi.
\]

We will characterize the boundedness of those integral type operators between Zygmund type spaces and also estimate their essential norms. The boundedness and compactness of these operators on the logarithmic Bloch space have been characterized in [22].

Recall that essential norm \( \|T\|_{c,X \to Y} \) of a bounded linear operator \( T : X \to Y \) is defined as the distance from \( T \) to \( \mathcal{H}(X,Y) \), the space of compact operators from \( X \) to \( Y \), namely,

\[
\|T\|_{c,X \to Y} = \inf \{\|T + K\|_{X \to Y} : K : X \to Y \text{ is compact} \}.
\]

It provides a measure of noncompactness of \( T \). Clearly, \( T \) is compact if and only if \( \|T\|_{c,X \to Y} = 0 \).

Throughout this paper, constants are denoted by \( C \), they are positive and may differ from one occurrence to the other. The notation \( a \asymp b \) means that there are positive constants \( C_1, C_2 \) such that \( C_1 a \leq b \leq C_2 a \).

2. Boundedness

In order to prove the main results of this paper. We need some auxiliary results.

Lemma 1 (see [8, 13]). For \( 0 < \alpha < 2 \) and \( \{f_n\} \) be a bounded sequence in \( \mathcal{L}^\alpha \) which converges to 0 uniformly on compact subsets of \( D \). Then \( \lim_{n \to \infty} \sup_{x \in D} |f_n(z)| = 0 \).

Lemma 2 (see [8, 13]). For every \( f \in \mathcal{L}^\alpha \), where \( \alpha > 0 \), one has

(i) \[ |f'(z)| \leq \frac{2}{(1 - \alpha)} \|f\|_{\mathcal{L}^\alpha} \quad \text{and} \quad |f(z)| \leq \frac{2}{(1 - \alpha)} \|f\|_{\mathcal{L}^\alpha}, \quad \text{for} \quad 0 < \alpha < 1; \]

(ii) \[ |f'(z)| \leq 2 \log(2/(1 - |z|)) \|f\|_{\mathcal{L}^\alpha} \quad \text{and} \quad |f(z)| \leq \|f\|_{\mathcal{L}^\alpha}, \quad \text{for} \quad \alpha = 1; \]

(iii) \[ |f'(z)| \leq \frac{2}{(1 - \alpha)} \|f\|_{\mathcal{L}^\alpha/(1 - |z|)^{\alpha-1}}, \quad \text{for} \quad 0 < \alpha < 1; \]

(iv) \[ |f(z)| \leq \frac{2}{(\alpha - 1)(2 - \alpha)} \|f\|_{\mathcal{L}^\alpha}, \quad \text{for} \quad 1 < \alpha < 2; \]

(v) \[ |f(z)| \leq 2 \log(2/(1 - |z|)) \|f\|_{\mathcal{L}^\alpha}, \quad \text{for} \quad \alpha = 2; \]

(vi) \[ |f(z)| \leq \frac{2}{(2 - \alpha)(1 - \alpha - 2)} \|f\|_{\mathcal{L}^\alpha/(1 - |z|)^{\alpha-2}}, \quad \text{for} \quad \alpha > 2. \]

Lemma 3 (see [8]). Let \( 0 < \alpha < \infty \) and \( v \) a radial, non-increasing weight tending to 0 at boundary of \( D \), and let the weighted composition operator \( uC_\varphi : \mathcal{L}^\alpha \to H^v \) be bounded.

(i) If \( 0 < \alpha < 2 \), then \( uC_\varphi \) is a compact operator.

(ii) \[
\|uC_\varphi\|_{c,\mathcal{L}^\alpha \to H^v} = \limsup_{n \to \infty} (\log n) \|u\varphi^n\|_v = \limsup_{\|\varphi(z)\| \to 1} v(\varphi(z)) \log \frac{2}{1 - |\varphi(z)|^2}. \tag{10}
\]

(iii) If \( \alpha > 2 \), then

\[
\|uC_\varphi\|_{c,\mathcal{L}^\alpha \to H^v} \asymp \limsup_{n \to \infty} \left( n + 1 \right)^{\alpha-2} \|u\varphi^n\|_v \asymp \limsup_{\|\varphi(z)\| \to 1} v(\varphi(z)) \|u(z)\| \left( 1 - |\varphi(z)|^2 \right)^{\alpha-2}. \tag{11}
\]

The following lemma is due to [24, 25].
Theorem 6. Let \( \nu \) and \( \omega \) be radial, nonincreasing weights tending to zero at the boundary of \( D \). Then

(i) The weighted composition operator \( uC_\varphi \) maps \( H_\nu^\infty \) into \( H_\omega^\infty \) if and only if

\[
\sup_{n \geq 0} \frac{\|u\varphi^n\|_\omega}{\|u\varphi^n\|_\nu} = \sup_{z \in D} \frac{\varphi'(z)|u'(z)|}{\varphi(z)} < \infty, 
\]

(12)

with norm comparable to the above supremum.

(ii)

\[
\|uC_\varphi\|_{c:H_\nu^\infty \to H_\omega^\infty} = \lim_{n \to \infty} \frac{\|u\varphi^n\|_\omega}{\|u\varphi^n\|_\nu} = \lim_{n \to \infty} \frac{\varphi(z)}{|\varphi'(z)|} |u'(z)|.
\]

(13)

Lemma 4. Let \( \nu \) and \( \omega \) be radial, nonincreasing weights tending to zero at the boundary of \( D \). Then

(i) The weighted composition operator \( uC_\varphi \) maps \( H_\nu^\infty \) into \( H_\omega^\infty \) if and only if

\[
\sup_{n \geq 0} \frac{\|u\varphi^n\|_\omega}{\|u\varphi^n\|_\nu} = \sup_{z \in D} \frac{\varphi'(z)|u'(z)|}{\varphi(z)} < \infty,
\]

(12)

with norm comparable to the above supremum.

(ii)

\[
\|uC_\varphi\|_{c:H_\nu^\infty \to H_\omega^\infty} = \lim_{n \to \infty} \frac{\|u\varphi^n\|_\omega}{\|u\varphi^n\|_\nu} = \lim_{n \to \infty} \frac{\varphi(z)}{|\varphi'(z)|} |u'(z)|.
\]

(13)

Lemma 5 (see [26]). For every \( 0 < \alpha < \infty \), one has

\[
\lim_{n \to \infty} \frac{\log(n+1)}{\log n} \|e^n\|_{\psi^\alpha} = 1.
\]

(15)

Theorem 6. Let \( \varphi \) be an analytic self-map of \( D \) and \( g \in H(D) \).

(i) If \( 0 < \alpha < 1 \), then \( I_{g\varphi} : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta \) is a bounded operator if and only if \( g' \in H_\psi^\infty \) and

\[
\sup_{n \geq 0} \|g(\varphi^n)\|_\psi = \sup_{z \in D} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)\alpha} |g(z)| |\varphi'(z)| < \infty.
\]

(16)

(ii) If \( \alpha = 1 \), then \( I_{g\varphi} : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta \) is a bounded operator if and only if

\[
\sup_{n \geq 0} \|g(\varphi^n)\|_\psi = \sup_{z \in D} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)\alpha} |g(z)| |\varphi'(z)| < \infty.
\]

(16)

Proof. Suppose that \( I_{g\varphi} \) is bounded from \( \mathcal{Z}^\alpha \) to \( \mathcal{Z}^\beta \). Using the test functions \( f(z) = z \) and \( f(z) = z^2 \), we have

\[
(1-|z|^2)^\beta \left| (I_{g\varphi}z)^\alpha \right| = (1-|z|^2)^\beta \left| g'(z) \right| < \infty,
\]

(19)

\[
(1-|z|^2)^\beta \left| (I_{g\varphi}z^2)^\alpha \right| = (1-|z|^2)^\beta \left| 2g'(z) g(z) + 2\varphi(z) g'(z) \right| < \infty.
\]

(22)

Since \( \varphi \) is a self-map, we get that \( g' \in H_\psi^\infty, \varphi' g \in H_\psi^\infty \).

For every \( 0 < \alpha < \infty \) and given nonzero \( a \in D \), we take the test functions

\[
f_n(z) = \frac{1}{a^2} \left[ \frac{(1-|a|^2)^2}{(1-\overline{a}z)^\alpha} - \frac{1-|a|^2}{(1-\overline{a}z)^\alpha+1} \right],
\]

(23)

\[
h_n(z) = \frac{1}{a^2} \left[ \frac{1-|a|^2}{(1-\overline{a}w)^\alpha} \right] dw,
\]

(24)

\[
g_n(z) = f_n(z) - h_n(z),
\]

(25)

for every \( z \in D \). One can show that \( f_n, h_n, \) and \( g_n \) are in \( \mathcal{Z}^\alpha \), \( \sup_{1/2 < a < 1} \|f_n\|_{\mathcal{Z}^\alpha} < \infty \), \( \sup_{1/2 < a < 1} \|h_n\|_{\mathcal{Z}^\alpha} < \infty \). Since \( g_0'(a) = 0, g_0''(a) = \alpha/(1-|a|^2)^\alpha \), it follows that for all \( z \in D \) with \( |\varphi(z)| > 1/2 \), we have

\[
+\infty > C \|g_n\|_{\mathcal{Z}^\alpha} \geq \|I_{g\varphi} (g_n(z))\|_{\mathcal{Z}^\beta}
\]

\[
\geq \left| (1-|z|^2)^\beta \left| \varphi'(z) g(z) \right| g_n''(z) (\varphi(z)) \right| - \left| (1-|z|^2)^\beta \left| g'(z) \right| g_n'(z) (\varphi(z)) \right|
\]

(26)
Then
\[
\sup_{z \in D} |f'(z)| g(z) \left( 1 - |z|^2 \right)^a \left( 1 - |\varphi(z)|^2 \right)^a \\
\lesssim \sup_{|\varphi(z)| \leq 1/2} \left| f'(z) \right| g(z) \left( 1 - |z|^2 \right)^\beta \left( 1 - |\varphi(z)|^2 \right)^\alpha \\
\quad + \sup_{|\varphi(z)| > 1/2} \left| f'(z) \right| g(z) \left( 1 - |z|^2 \right)^\beta \left( 1 - |\varphi(z)|^2 \right)^\alpha \\
\lesssim \left( \frac{4}{3} \right)^\alpha \left\| \varphi' g \right\|_{\mathcal{Y}_g} + C \left\| g_a \right\|_{\mathcal{X}^s} < \infty.
\]

Now we use (14) and Lemma 4 to conclude that
\[
\sup_{n \leq 0} \left( n + 1 \right)^\alpha \| g(\varphi') \varphi' \|_{\mathcal{Y}_g} \\
= \sup_{z \in D} \left( 1 - |z|^2 \right)^\beta \left| g(z) \varphi'(z) \right| < \infty,
\]
which shows that (16) is necessary for all cases.

Conversely, suppose that \( g' \in H^\infty_{\mathcal{Y}_g} \) and (16) holds. Assume that \( f \in \mathcal{X}^s \). From Lemma 2, it follows that
\[
(1 - |z|^2)^\beta \left| (I_{g'} C_\varphi)''(z) \right| = (1 - |z|^2)^\beta \\
\quad \cdot \left| f'(z) g(z) f''(\varphi(z)) + g'(z) f'(\varphi(z)) \right| \\
\leq (1 - |z|^2)^\beta \left| f'(z) g(z) f''(\varphi(z)) \right| + (1 - |z|^2)^\beta \\
\quad \cdot \left| g'(z) f'(\varphi(z)) \right| \\
\leq \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^\beta \left| \varphi'(z) g(z) \right| \\
\quad \cdot \left\| f \right\|_{\mathcal{X}^s} + C \left( 1 - |z|^2 \right)^\beta \left| g'(z) \right| \left\| f \right\|_{\mathcal{X}^s} \leq C \left\| f \right\|_{\mathcal{X}^s},
\]
\[
|I_{g'} C_\varphi (f) (0)| = 0,
\]
\[
\left| (I_{g'} C_\varphi) (f)' (0) \right| = \left| f'(\varphi(0)) g(0) \right| \leq \left\| f \right\|_{\mathcal{X}^s} \\
\quad \cdot \left| g(\varphi(0)) \right|,
\]
which implies that \( I_{g'} C_\varphi \) is bounded. This completes the proof of (i).

Next we will prove (ii). The necessity in condition (17) has been proved above. Fixing \( a \in D \) with \( |a| > 1/2 \), we take the function
\[
k_a (z) = \frac{D(2z)}{a} \left( \log \frac{1}{1 - |a|} \right)^{-1},
\]
for \( z \in D \), where
\[
p(z) = (z - 1) \left( 1 + \log \frac{1}{1 - z} \right)^2 + 1.
\]

Then we have \( \sup_{|z| > 1/2} \left| k_a \right| \leq C \) by [11]. Let \( a = \varphi(z) \). It follows that
\[
\begin{aligned}
\left\| I_{g'} C_\varphi \left( k_{\varphi(z)} \right) \right\|_{\mathcal{X}^s} &\leq \left( 1 - |z|^2 \right)^\beta \left| g'(z) \right| \left| k_{\varphi(z)}' \left( \varphi(z) \right) \right| \\
&\quad - \left( 1 - |z|^2 \right)^\beta \left| f'(z) g(z) \right| \left| k_{\varphi(z)}'' \left( \varphi(z) \right) \right| \\
&= \left( 1 - |z|^2 \right)^\beta \left| g'(z) \right| \log \frac{1}{1 - |\varphi(z)|^2} \\
&\quad - \left( 1 - |z|^2 \right)^\beta \left| f'(z) g(z) \right| \left( 1 - |z|^2 \right)^\alpha \left( 1 - |\varphi(z)|^2 \right)^\alpha.
\end{aligned}
\]

Since (17) holds and \( I_{g'} C_\varphi \) is bounded, we obtain that
\[
\begin{aligned}
\sup_{|z| > 1/2} \left( 1 - |z|^2 \right)^\beta \log \frac{1}{1 - |\varphi(z)|^2} \left| g'(z) \right| \\
&\leq \sup_{|z| > 1/2} \left( 1 - |z|^2 \right)^\beta \left| f'(z) g(z) \right| \left( 1 - |z|^2 \right)^\alpha \left( 1 - |\varphi(z)|^2 \right)^\alpha \\
&\quad + \sup_{|z| > 1/2} \left\| I_{g'} C_\varphi \left( k_{\varphi(z)} \right) \right\|_{\mathcal{X}^s} < \infty.
\end{aligned}
\]

Noting \( g' \in H^\infty_{\mathcal{Y}_g} \) and together with (15) and Lemma 4, we conclude that (18) holds.

The converse implication can be shown as in the proof of (i).

Finally we will prove (iii). We have proved that (19) holds above. To prove (20), we take function \( f_{\varphi(z)} \) defined in (23) for every \( z \in D \) with \( |\varphi(z)| > 1/2 \) and obtain that
\[
\begin{aligned}
\left\| I_{g'} C_\varphi \left( f_{\varphi(z)} \right) \right\|_{\mathcal{X}^s} &\geq \left( 1 - |z|^2 \right)^\beta \left| g'(z) \right| \left| f_{\varphi(z)}' \left( \varphi(z) \right) \right| \\
&\quad - \left( 1 - |z|^2 \right)^\beta \left| f'(z) g(z) \right| \left| f_{\varphi(z)}'' \left( \varphi(z) \right) \right| \\
&= \left( 1 - |z|^2 \right)^\beta \left| g'(z) \right| \frac{1}{|\varphi(z)|} \left( 1 - |\varphi(z)|^2 \right)^{\beta - 1} \\
&\quad - \left( 1 - |z|^2 \right)^\beta \left| f'(z) g(z) \right| \left( 2 \varphi(z) \right) \left( 1 - |\varphi(z)|^2 \right)^{\alpha}.
\end{aligned}
\]

Since \( I_{g'} C_\varphi \) is bounded and (19) holds, we obtain that
\[
\begin{aligned}
\sup_{|z| > 1/2} \left( 1 - |z|^2 \right)^\beta \left| g'(z) \right| &\leq 2 \sup_{|z| > 1/2} \left( 1 - |z|^2 \right)^\beta \left| f'(z) g(z) \right| \left( 1 - |\varphi(z)|^2 \right)^{\alpha - 1} \\
&\quad + 2 \sup_{|z| > 1/2} \left\| I_{g'} C_\varphi \left( f_{\varphi(z)} \right) \right\|_{\mathcal{X}^s} < \infty;
\end{aligned}
\]
therefore, we deduce that (20) holds by (14) and Lemma 4.
The converse implication can be shown as in the proof of (i).

**Theorem 7.** Let $\varphi$ be an analytic self-map of $D$ and $g \in H(D)$.

(i) If $0 < \alpha < 1$, then $C_{g}I_{g} : \mathcal{L}^{\alpha} \to \mathcal{L}^{\beta}$ is a bounded operator if and only if $((g \circ \varphi)(\varphi''') + (g' \circ \varphi)(\varphi')^{2}) \in H^{\infty}_{\varphi}$ and

$$
\sup_{n \geq 0} (n + 1)^{\alpha} \left\| (g \circ \varphi) (\varphi')^{2} \varphi'' \right\|_{\varphi} 
\approx \sup_{z \in D} \frac{(1 - |z|^{2})^{\beta}}{(1 - |\varphi(z)|^{2})} \left| g \left( \varphi(z) \right) \left( \varphi'(z) \right)^{2} \right| < \infty.
$$

(ii) If $\alpha = 1$, then $C_{g}I_{g} : \mathcal{L}^{\alpha} \to \mathcal{L}^{\beta}$ is a bounded operator if and only if

$$
\sup_{n \geq 0} (n + 1)(g \circ \varphi) (\varphi')^{2} \varphi'' \approx \sup_{z \in D} \frac{(1 - |z|^{2})^{\beta}}{1 - |\varphi(z)|^{2}} \left| \varphi'(z) \right|^{2} \left| \varphi''(z) \right| < \infty,
$$

$$
\sup_{n \geq 0} (\log n) \left\| (g' \circ \varphi)(\varphi'')^{2} + (g \circ \varphi)(\varphi')^{2} \right\|_{\varphi} 
\approx \sup_{z \in D} \frac{(1 - |z|^{2})^{\beta}}{1 - |\varphi(z)|^{2}} \left| \varphi'(z) \right|^{2} \left| \varphi''(z) \right| < \infty.
$$

(iii) If $\alpha > 1$, then $C_{g}I_{g} : \mathcal{L}^{\alpha} \to \mathcal{L}^{\beta}$ is a bounded operator if and only if (36) holds and

$$
\sup_{n \geq 0} (n + 1)^{\alpha - 1} \left\| (g' \circ \varphi)(\varphi')^{2} \varphi'' \right\|_{\varphi} 
\approx \sup_{z \in D} \frac{(1 - |z|^{2})^{\beta}}{(1 - |\varphi(z)|^{2})^{\alpha - 1}} \left| \varphi'(z) \right|^{2} \left| \varphi''(z) \right| < \infty.
$$

The proof is similar to that of Theorem 6, and the details are omitted.

**Theorem 8.** Let $\varphi$ be an analytic self-map of $D$ and $g \in H(D)$.

(i) If $0 < \alpha < 1$, then $C_{g}T_{g} : \mathcal{L}^{\alpha} \to \mathcal{L}^{\beta}$ is a bounded operator if and only if $((g' \circ \varphi)(\varphi')^{2} \varphi'' + (g'' \circ \varphi)(\varphi')^{2}) \in H^{\infty}_{\varphi}$ and $((g' \circ \varphi)(\varphi')^{2} \varphi'' + (g'' \circ \varphi)(\varphi')^{2}) \in H^{\infty}_{\varphi}$.

(ii) If $\alpha = 1$, then $C_{g}T_{g} : \mathcal{L}^{\alpha} \to \mathcal{L}^{\beta}$ is a bounded operator if and only if $((g' \circ \varphi)(\varphi')^{2} \varphi'' + (g'' \circ \varphi)(\varphi')^{2}) \in H^{\infty}_{\varphi}$ and

$$
\sup_{n \geq 0} (n + 1) \left\| (g' \circ \varphi)(\varphi')^{2} \varphi'' \right\|_{\varphi} 
\approx \sup_{z \in D} \frac{(1 - |z|^{2})^{\beta}}{(1 - |\varphi(z)|^{2})^{\alpha - 1}} \left| \varphi'(z) \right|^{2} \left| \varphi''(z) \right| < \infty.
$$

(iii) If $\alpha > 1$, then $C_{g}T_{g} : \mathcal{L}^{\alpha} \to \mathcal{L}^{\beta}$ is a bounded operator if and only if (40) holds and

$$
\sup_{n \geq 0} (n + 1)^{\alpha - 2} \left\| (g' \circ \varphi)(\varphi')^{2} \varphi'' + (g'' \circ \varphi)(\varphi')^{2} \right\|_{\varphi} 
\approx \sup_{z \in D} \frac{(1 - |z|^{2})^{\beta}}{(1 - |\varphi(z)|^{2})^{\alpha - 2}} \left| \varphi'(z) \right|^{2} \left| \varphi''(z) \right| < \infty.
$$
Proof. Suppose that $C_T g$ is bounded from $\mathcal{L}^\alpha$ to $\mathcal{L}^\beta$ space.

(i) Case $0 < \alpha < 1$. Using functions $f = 1 \in \mathcal{L}^\alpha$ and $f = z \in \mathcal{L}^\alpha$, we obtain

$$
\sup_{z \in D} (1 - |z|^2) \beta
\cdot |g''(\varphi(z)) (\varphi'(z))^2 + g'(\varphi(z)) \varphi''(z)| < \infty, \quad (44)
$$

Then we obtain that $(g' \circ \varphi)(\varphi'') + (g'' \circ \varphi)(\varphi')^2 \in H^\infty_\varphi$ and $(g' \circ \varphi)(\varphi')^2 \in H^\infty_\varphi$ are necessary for all case.

For the converse implication, suppose that $(g' \circ \varphi)(\varphi'') + (g'' \circ \varphi)(\varphi')^2 \in H^\infty_\varphi$ and $(g' \circ \varphi)(\varphi')^2 \in H^\infty_\varphi$. For $f \in \mathcal{L}^\alpha$, it follows from Lemma 2 that

$$
(1 - |z|^2) \beta \left| (C_T g, f)'(z) \right| = (1 - |z|^2) \beta \left| (\varphi'(z))^2 \right|
\cdot g'(\varphi(z)) f'(\varphi(z))
\cdot \left( g''(\varphi(z)) (\varphi'(z))^2 + g'(\varphi(z)) \varphi''(z) \right)
\cdot f(\varphi(z))
\cdot f'(\varphi(z)) + (1 - |z|^2) \beta
\cdot \left( g''(\varphi(z)) (\varphi'(z))^2 + g'(\varphi(z)) \varphi''(z) \right)
\cdot f(\varphi(z))
\cdot \|f\|_{\mathcal{L}^\alpha} + (1 - |z|^2) \beta
\cdot g'(\varphi(z)) (\varphi'(z))^2
\cdot \|f\|_{\mathcal{L}^\alpha} \leq C \|f\|_{\mathcal{L}^\alpha},
$$

$$
|C_T g, f\rangle(0) = \left[ f(0) \int_0^\varphi(0) f(\zeta) \varphi'(\zeta) \, d\zeta \right]
\leq \max_{K \in \{\varphi(0)\}} |f(\zeta)| \max_{K \in \{\varphi(0)\}} |g'(\zeta)| \leq \frac{2}{1 - \alpha} \|f\|_{\mathcal{L}^\alpha}
\cdot \max_{K \in \{\varphi(0)\}} |g'(\zeta)|,
$$

Then $C_T g$ is bounded. This completes the proof of (i).

(ii) Case $\alpha = 1$. We consider the test function $k_{\varphi(z)}(z)$ defined in (30) for every $z \in D$ with $|\varphi(z)| > 1/2$. It follows that

$$
\| C_T g \|_{\mathcal{L}^\beta} \geq \log \left( \frac{1}{1 - |\varphi(z)|} \right) (1 - |z|^2) \beta
\cdot \left( (\varphi'(z))^2 g'\varphi(z) \right) - (1 - |z|^2) \beta
\cdot \left( \varphi''(z) g'(\varphi(z)) + g''(\varphi(z)) (\varphi(z))^2 \right)
\cdot \left| k_{\varphi(z)}(\varphi(z)) \right|.
$$

Since $(g' \circ \varphi)(\varphi'') + (g'' \circ \varphi)(\varphi')^2 \in H^\infty_\varphi$ and $sup_{|\varphi(z)| > 1/2} \| C_T g \|_{\mathcal{L}^\beta} \leq C$, we get

$$
\sup_{|\varphi(z)| > 1/2} \log \left( \frac{1}{1 - |\varphi(z)|} \right) (1 - |z|^2) \beta \left| (\varphi'(z))^2 \right|
\cdot g'(\varphi(z)) \leq \sup_{|\varphi(z)| > 1/2} (1 - |z|^2) \beta
\cdot \left| \varphi''(z) g'(\varphi(z)) + g''(\varphi(z)) (\varphi(z))^2 \right|
\cdot \left| k_{\varphi(z)}(\varphi(z)) \right| + \| C_T g \|_{\mathcal{L}^\beta} < \infty.
$$

Then we have

$$
\sup_{z \in D} \log \left( \frac{1}{1 - |\varphi(z)|} \right) (1 - |z|^2) \beta
\cdot \left| (\varphi'(z))^2 g'(\varphi(z)) \right|
\leq \sup_{|\varphi(z)| > 1/2} \log \left( \frac{1}{1 - |\varphi(z)|} \right) (1 - |z|^2) \beta \left| (\varphi'(z))^2 \right|
\cdot g'(\varphi(z)) \leq \sup_{|\varphi(z)| > 1/2} \log \left( \frac{1}{1 - |\varphi(z)|} \right) (1 - |z|^2) \beta \left| (\varphi'(z))^2 \right|
\cdot g'(\varphi(z)) \leq C + \log \left( \frac{4}{3} \right) \left| (g' \circ \varphi)(\varphi'(z))^2 \right|_{\mathcal{L}^\beta} < \infty.
$$

On the other hand, from (15) and Lemma 4, we have

$$
\sup_{n \geq 0} \left( \log n \right) \left| (g' \circ \varphi)(\varphi'(z))^2 \right|_{\mathcal{L}^\beta} = \sup_{z \in D} (1 - |z|^2) \beta
\cdot \log \left( \frac{2}{1 - |\varphi(z)|} \right) \left| (g' \circ \varphi)(\varphi'(z))^2 \right|_{\mathcal{L}^\beta}.
$$

Hence (39) holds.

The converse implication can be shown as in the proof of (i).

(iii) Case $1 < \alpha < 2$. $(g' \circ \varphi)(\varphi'') + (g'' \circ \varphi)(\varphi')^2 \in H^\infty_\varphi$ has been proved above. We take the test function $f_{\varphi(z)}$ in (23) for
every \( z \in D \) with \(|\varphi(z)| > 1/2 \); by the same way as (ii), we can obtain that (40) holds.

The converse implication can be shown as in the proof of (i).

(iv) Case \( \alpha = 2 \). We have proved that (41) holds above. To prove (42), we consider another test function \( t_n(z) = \log(2/(1 - n z)) \). Clearly \( t_n \in \mathcal{Z}^2 \) and \( \sup_{|z| < 1} \| t_n \|_{\mathcal{X}^2} < \infty \). For every \( z \in D \) with \(|\varphi(z)| > 1/2 \), it follows that

\[
\sup_{z \in D} \left( 1 - |z|^2 \right)^\beta \left| \left( C_{\varphi} T g \right)' (z) \right| \\
\geq \sup_{|\varphi(z)| > 1/2} \log \left( \frac{1}{1 - |\varphi(z)|} \right) \left( 1 - |z|^2 \right)^\beta \\
\cdot \left| \left( g'(z) \right)^2 g''(\varphi(z)) + g'(\varphi(z)) \varphi''(z) \right| \\
+ \sup_{|\varphi(z)| > 1/2} \left( 1 - |z|^2 \right)^\beta \left| g'(\varphi(z)) \varphi'(z) \right|^2 .
\]

Applying (41) we get

\[
\sup_{|\varphi(z)| > 1/2} \log \left( \frac{1}{1 - |\varphi(z)|} \right) \left( 1 - |z|^2 \right)^\beta \\
\cdot \left| \left( g'(z) \right)^2 g''(\varphi(z)) + g'(\varphi(z)) \varphi''(z) \right| \\
\leq \sup_{|\varphi(z)| > 1/2} \left( 1 - |z|^2 \right)^\beta \left| g'(\varphi(z)) \varphi'(z) \right|^2 \\
+ \left\| C_{\varphi} T g \right\|_{\mathcal{X}^2} < \infty .
\]

Noting \( (g' \circ \varphi)(\varphi'(z)) + (g'' \circ \varphi)(\varphi'(z)^2) \in H^\infty_{\varphi(y)} \) and using Lemma 4 and (15), we conclude that (42) holds.

(v) Case \( \alpha > 2 \). We have proved that (40) holds above. Applying test function \( f_{\varphi(z)} \) in (23) for every \( z \in D \) with \(|\varphi(z)| > 1/2 \), we have

\[
S_1 = \sup_{|\varphi(z)| > 1/2} \left( 1 - |z|^2 \right)^\beta \left| \left( g'(z) \right)^2 g''(\varphi(z)) \right| \\
\cdot \left| \varphi'(z) \right|^2 \left| \varphi'(z) \right|^2 \\
\leq \sup_{|\varphi(z)| > 1/2} \left( 1 - |z|^2 \right)^\beta \left| g'(\varphi(z)) \right|^2 \left| \varphi'(z) \right|^2 \\
\cdot \left( 1 - |z|^2 \right)^\beta \left| \varphi'(z) \right|^2 \left( 1 - |\varphi(z)| \right)^{\alpha-1} \\
+ \left\| C_{\varphi} T g \right\|_{\mathcal{X}^2} < \infty .
\]

With the same calculation for test function \( t_{\varphi(z)}(\varphi(z)) = (1 - |\varphi(z)|^2)^2/(1 - |\varphi(z)|^2) \) with \(|\varphi(z)| > 1/2 \), then \( \sup_{|\varphi(z)| > 1/2} \| t_{\varphi(z)} \|_{\mathcal{X}^2} \leq C \), and we have that

\[
S_2 = \sup_{|\varphi(z)| > 1/2} \left( 1 - |z|^2 \right)^\beta \\
\cdot \left| \left( g'(z) \right)^2 g''(\varphi(z)) + g'(\varphi(z)) \varphi''(z) \right| \\
\cdot \left( 1 - |\varphi(z)| \right)^{\alpha-2} \\
+ \alpha \sup_{|\varphi(z)| > 1/2} \left( 1 - |\varphi(z)| \right)^{\alpha-1} \left| g'(\varphi(z)) \right|^2 \\
\cdot \left( 1 - |\varphi(z)| \right)^{\alpha-1} \leq \left\| C_{\varphi} T g \right\|_{\mathcal{X}^2} < \infty .
\]

Therefore,

\[
\sup_{|\varphi(z)| > 1/2} \left( 1 - |z|^2 \right)^\beta \\
\cdot \left| \left( g'(z) \right)^2 g''(\varphi(z)) + g'(\varphi(z)) \varphi''(z) \right| \leq S_2 \\
\cdot \left( 1 - |\varphi(z)| \right)^{\alpha-2} \\
+ \alpha \sup_{|\varphi(z)| > 1/2} \left( 1 - |\varphi(z)| \right)^{\alpha-1} \left| g'(\varphi(z)) \right|^2 \leq S_2 + \alpha S_1 \\
\leq \infty .
\]

Since \((g' \circ \varphi)(\varphi'(z)) + (g'' \circ \varphi)(\varphi'(z)^2) \in H^\infty_{\varphi(y)} \), we conclude that (43) holds.

\[\square\]

**Theorem 9.** Let \( \varphi \) be an analytic self-map of \( D \) and \( g \in H(D) \).

(i) If \( 0 < \alpha < 1 \), then \( T_C g : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta \) is a bounded operator if and only if \( g \varphi \in H^\infty_{\varphi(y)} \) and \( g'' \in H^\infty_{\varphi(y)} \).

(ii) If \( \alpha = 1 \), then \( T_C g : \mathcal{Z}^\alpha \to \mathcal{Z}^\beta \) is a bounded operator if and only if \( g \varphi \in H^\infty_{\varphi(y)} \) and

\[
\sup_{n \geq 0} \left( |g' \varphi'| \right)^n \left( \frac{1 - |\varphi(z)|^2}{1 - |\varphi(z)|^2} \right) \leq \infty .
\]
(iii) If $1 < \alpha < 2$, then $T g C_q : \mathcal{L}^\alpha \to \mathcal{L}^\beta$ is a bounded operator if and only if $g'' \in H_{v\beta}$ and
\[
\sup_{n \geq 0} (n + 1)^{\alpha - 1} \| (g'') \psi'' \|_{\mathcal{L}^\beta} \\
\quad \times \sup_{z \in D} \frac{(1 - |z|^2)\beta}{(1 - |\varphi (z)|^2)^{\alpha - 1}} |g'' (z)\psi'' (z)| < \infty.
\] (56)

(iv) If $\alpha = 2$, then $T g C_q : \mathcal{L}^\alpha \to \mathcal{L}^\beta$ is a bounded operator if and only if
\[
\sup_{n \geq 0} (n + 1)^{\alpha - 2} \| (g'') \psi'' \|_{\mathcal{L}^\beta} \\
\quad \times \sup_{z \in D} \frac{(1 - |z|^2)\beta}{(1 - |\varphi (z)|^2)^{\alpha - 2}} |g'' (z)\psi'' (z)| < \infty.
\] (57)

(v) If $\alpha > 2$, then $T g C_q : \mathcal{L}^\alpha \to \mathcal{L}^\beta$ is a bounded operator if and only if (56) holds and
\[
\sup_{n \geq 0} (n + 1)^{\alpha - 2} \| (g'') \psi'' \|_{\mathcal{L}^\beta} \\
\quad \times \sup_{z \in D} \frac{(1 - |z|^2)\beta}{(1 - |\varphi (z)|^2)^{\alpha - 2}} |g'' (z)\psi'' (z)| < \infty.
\] (58)

The proof is similar to that of Theorem 8, and the details are omitted.

3. Essential Norms

In this section we estimate the essential norms of these integral type operators on Zygmund type spaces in terms of $g$, $\varphi$, their derivatives, and the $n$th power $\varphi^n$ of $\varphi$.

Let $\mathcal{L}^\alpha$ = \{ $f \in \mathcal{L}^\alpha : f(0) = f'(0) = 0$ \} and $\mathcal{L}^\beta$ = \{ $f \in \mathcal{L}^\beta : f(0) = 0$ \}. We note that every compact operator $T \in \mathcal{K}(\mathcal{L}^\alpha, \mathcal{L}^\beta)$ can be extended to a compact operator $K \in \mathcal{K}(\mathcal{L}^\alpha, \mathcal{L}^\beta)$. In fact, for every $f \in \mathcal{L}^\alpha$, $f - f(0) - f'(0)z \in \mathcal{L}^\alpha$, and we can define $K(f) = T(f - f(0) - f'(0)z) + f(0) + f'(0)z$.

For $r \in (0, 1)$, we consider the compact operator $K_r : \mathcal{L}^\alpha \to \mathcal{L}^\beta$ defined by $K_rf(z) = (rz)$.

Lemma 10. If $X(g) C_q g C_q T g C_q$ is a bounded operator from $\mathcal{L}^\alpha$ to $\mathcal{L}^\beta$ space, then
\[
\|X\|_{\mathcal{L}^\alpha \to \mathcal{L}^\beta} = \|X\|_{\mathcal{L}^\alpha \to \mathcal{L}^\beta}.
\] (59)

Let $X(g) C_q g C_q T g C_q$ be given. Let $\{r \}$ be an increasing sequence in $(0, 1)$ converging to 1 and $\mathcal{G} = \{h : h = a + bz\}$, the closed subspace of $\mathcal{L}^\alpha$. Then
\[
\|X - T\|_{\mathcal{L}^\alpha \to \mathcal{L}^\beta} = \sup_{f \in \mathcal{L}^\alpha, \|f\|_1 \leq 1} \|X(f) - T(f)\|_{\mathcal{L}^\beta} \\
\quad \leq \sup_{f \in \mathcal{L}^\alpha, \|f\|_1 \leq 1} \|X(f - f(0) - f'(0)z) - T(f - f(0) - f'(0)z)\|_{\mathcal{L}^\beta} \\
\quad + \sup_{f \in \mathcal{L}^\alpha, \|f\|_1 \leq 1} \|X(f(0) + f'(0)z) - T(f(0) + f'(0)z)\|_{\mathcal{L}^\beta} \\
\quad + \sup_{g \in \mathcal{G}, \|g\|_1 \leq 1} \|X(g) - T(g)\|_{\mathcal{L}^\beta} \\
\quad + \sup_{h \in \mathcal{G}, \|h\|_1 \leq 1} \|X(h) - T\alpha(h)\|_{\mathcal{L}^\beta}.
\] (60)

Hence
\[
\inf_{T \in \mathcal{K}(\mathcal{L}^\alpha, \mathcal{L}^\beta)} \|X - T\|_{\mathcal{L}^\alpha \to \mathcal{L}^\beta} \\
\quad \leq \inf_{T \in \mathcal{K}(\mathcal{L}^\alpha, \mathcal{L}^\beta)} \|X - T\|_{\mathcal{L}^\alpha \to \mathcal{L}^\beta} \\
\quad + \inf_{T \in \mathcal{K}(\mathcal{L}^\alpha, \mathcal{L}^\beta)} \|X - T\|_{\mathcal{L}^\alpha \to \mathcal{L}^\beta} \\
\quad \leq \|X\|_{\mathcal{L}^\alpha \to \mathcal{L}^\beta} + \lim_{n \to \infty} \|X(I - K_\alpha)\|_{\mathcal{L}^\alpha \to \mathcal{L}^\beta}.
\] (61)

Since $\lim_{n \to \infty} \|X(I - K_\alpha)\|_{\mathcal{L}^\alpha \to \mathcal{L}^\beta} = 0$, we have $\|X\|_{\mathcal{L}^\alpha \to \mathcal{L}^\beta} \leq \|X\|_{\mathcal{L}^\alpha \to \mathcal{L}^\beta}$, and the proof is finished. \hfill $\Box$

Let $D_x : \mathcal{L}^\alpha \to \mathcal{B}_x$ and $S_x : \mathcal{B}^\alpha \to H_{v\beta}$ be the derivative operators. Then clearly $D_x$ and $S_x$ are linear isometries on $\mathcal{L}^\alpha$ and $\mathcal{B}^\alpha$, respectively, and
\[
S_x D_x g C_q g D_x^{-1} S_x^{-1} = g(\varphi') C_q + g' C_q S_x^{-1}.
\] (62)

Therefore
\[
\|I g C_q g C_q T g C_q\|_{\mathcal{L}^\alpha \to \mathcal{L}^\beta} \leq \|g' C_q g C_q T g C_q\|_{\mathcal{L}^\alpha \to \mathcal{L}^\beta} \\
\quad + \|g(\varphi') C_q g C_q T g C_q\|_{\mathcal{L}^\alpha \to \mathcal{L}^\beta}.
\] (63)

Similarly,
\[
S_x D_x g C_q g D_x^{-1} S_x^{-1} = (g \circ \varphi(\varphi')) C_q + (g' \circ \varphi(\varphi')) C_q S_x^{-1}.
\] (64)
Let \( I_\gamma C_{\varphi} \) be a bounded operator from \( \mathcal{X}^\alpha \) to \( \mathcal{X}^\beta \) space.

(i) If \( 0 < \alpha < 1 \), then

\[
\| I_\gamma C_{\varphi} \|_{\mathcal{X}^\alpha \to \mathcal{X}^\beta} \leq \limsup_{n \to \infty} \left( \frac{n + 1}{n} \right)^\alpha \left\| g \varphi' \right\|_{\mathcal{X}^\beta}.
\]

(ii) If \( \alpha = 1 \), then

\[
\| I_\gamma C_{\varphi} \|_{\mathcal{X}^\alpha \to \mathcal{X}^\beta} = \max \left\{ \limsup_{n \to \infty} \left( n + 1 \right)^\alpha \left\| \varphi' \right\|_{\mathcal{X}^\beta}, \limsup_{n \to \infty} \left( \log n \right) \left\| \varphi' \right\|_{\mathcal{X}^\beta} \right\}.
\]

(iii) If \( \alpha > 1 \), then

\[
\| I_\gamma C_{\varphi} \|_{\mathcal{X}^\alpha \to \mathcal{X}^\beta} = \max \left\{ \limsup_{n \to \infty} \left( n + 1 \right)^\alpha \left\| \varphi' \right\|_{\mathcal{X}^\beta} \right\},\]

\[
\| g \varphi' \|_{\mathcal{X}^\beta}, \limsup_{n \to \infty} \left( (n + 1)^\alpha - 1 \right) \left\| g \varphi' \right\|_{\mathcal{X}^\beta},\]

Proof. (i) We start with the upper bound. First we show that \( g \varphi' \) is a compact weighted composition operator for \( \mathcal{X}^\beta \) into \( \mathcal{X}^\alpha \). Suppose that \( \{ f_n \} \) is bounded sequence in \( \mathcal{X}^\alpha \). From Lemma 3.6 in [27], \( \{ f_n \} \) has a subsequence \( \{ f_{n_k} \} \) which converges uniformly on \( D \) to a function, which we can assume to be identically zero. Then it follows from Theorem 6 and Lemma 1 that

\[
\limsup_{k \to \infty} \frac{\| g \varphi' \|_{\mathcal{X}^\beta} + C \| (g \varphi')' \|_{\mathcal{X}^\beta}}{\| \varphi' \|_{\mathcal{X}^\beta}} = 0,
\]

which shows that \( g \varphi' : \mathcal{X}^\alpha \to \mathcal{X}^\beta \) is a compact operator and \( \| g \varphi' \|_{\mathcal{X}^\alpha \to \mathcal{X}^\beta} = 0 \). Applying (63), Lemmas 4, 5, and 10, we get that

\[
\| I_\gamma C_{\varphi} \|_{\mathcal{X}^\alpha \to \mathcal{X}^\beta} \leq \| g \varphi' \|_{\mathcal{X}^\beta} \leq C \| \varphi' \|_{\mathcal{X}^\beta}.
\]

For the lower bound, let \( \{ z_n \} \subseteq D \) with \( |\varphi(z_n)| > 1/2 \) and \( |\varphi(z_n)| \to 1 \) as \( n \to \infty \). Taking \( g_n = g_{\varphi(z_n)} \) defined in (25), we obtain that \( \{ g_n \} \) is bounded sequence in \( \mathcal{X}^\alpha \) converging to 0 uniformly on compact subset of \( D \) and \( \sup_{n \in \mathbb{N}} \| g_n \|_{\mathcal{X}^\alpha} \leq C \). For every compact operator \( T : \mathcal{X}^\alpha \to \mathcal{X}^\beta \),

\[
C \| I_\gamma C_{\varphi} - T \|_{\mathcal{X}^\alpha \to \mathcal{X}^\beta} \geq \limsup_{n \to \infty} \| I_\gamma C_{\varphi} g_n \|_{\mathcal{X}^\beta} - \limsup_{n \to \infty} \| T(g_n) \|_{\mathcal{X}^\beta},
\]

Now we use (14) and Lemma 4 to obtain that

\[
\| I_\gamma C_{\varphi} - T \|_{\mathcal{X}^\alpha \to \mathcal{X}^\beta} \geq \frac{\alpha}{C} \limsup_{n \to \infty} \| g \varphi' \|_{\mathcal{X}^\beta} \left( |\varphi(z_n)| - 1 \right),
\]

\[
\left( 1 - |\varphi(z_n)|^2 \right)^{\beta - 1} \left( 1 - |\varphi(z_n)|^2 \right)^{\alpha \beta},
\]

\[
\frac{\alpha}{C} \limsup_{n \to \infty} \| g \varphi' \|_{\mathcal{X}^\beta} \left( |\varphi(z_n)| - 1 \right),
\]

Hence (70) holds.
(ii) The boundedness of $I_\varphi C_\varphi$ guarantees that $(g\varphi')C_\varphi : H^1_v \to H^1_v$ and $(g')C_\varphi : B \to H^\infty_v$ are bounded weighted composition operators. Theorem 3.4 in [28] ensures that
\[
\|g' C_\varphi\|_{L^\infty \rightarrow H^\infty_v} = \lim_{\varphi(z) \to 1} \left( 1 - |z|^2 \right) \left| g'(z) \right| \log \frac{2}{1 - |\varphi(z)|^2}.
\]
Now we use Lemmas 4, 5 and (63) to conclude that
\[
\|I_\varphi C_\varphi\|_{L^1 \rightarrow L^1_v} \leq \|g' C_\varphi\|_{L^\infty \rightarrow H^\infty_v} + \|g(\varphi')\|
\leq C \limsup_{n \to \infty} \|g(\varphi')\varphi''\|_{L^1_v} + C \limsup_{n \to \infty} \|g'\varphi''\|_{L^1_v} = C \limsup_{n \to \infty} (\log n) \|g'\varphi''\|_{L^1_v}
+ \frac{Ce}{2} \limsup_{n \to \infty} (n + 1) \|g(\varphi')\varphi''\|_{L^1_v} \leq C \max \left\{ \limsup_{n \to \infty} (n + 1), \|g(\varphi')\varphi''\|_{L^1_v} \right\}.
\]

On the other hand, let $\{z_n\}$ be a sequence in $D$ such that $|\varphi(z_n)| > 1/2$ and $|\varphi(z_n)| \to 1$ as $n \to \infty$. Given
\[
h_n(z) = \frac{h(\varphi(z_n))}{\varphi(z_n)} \left( \log \frac{2}{1 - |\varphi(z_n)|^2} \right)^{1-2}
- \left( \log \frac{2}{1 - |\varphi(z_n)|^2} \right)^{-2},
\]
where $h(z) = (z - 1)(1 + \log(2/(1 - z)))^2 + 1$, from [11] we know that $\{h_n\}$ is a bounded sequence in $L^\infty_1$ which converges to zero uniformly on compact subsets of $D$, and
\[
\|h_n\|_{L^\infty_1} < \infty.
\]
For every compact operator $T : X \to X_\beta$, we have $\|T(h_n)\|_{X_\beta} \to 0$ as $n \to \infty$. By Lemmas 4 and 5, we obtain that
\[
C \|I_\varphi C_\varphi - T\|_{X \rightarrow X_\beta} \geq \|I_\varphi C_\varphi h_n\|_{X \rightarrow X_\beta}
\geq \limsup_{n \to \infty} \left( 1 - |z_n|^2 \right) \left| g'(z_n) \varphi'(z_n) \right| \left( \frac{|\varphi(z_n)|}{1 - |\varphi(z_n)|^2} \right)^{\beta}
\geq \limsup_{n \to \infty} \|g\varphi'\varphi''\|_{L^1_v}
= \frac{\alpha}{2} \limsup_{n \to \infty} (n + 1) \|g(\varphi')\varphi''\|_{L^1_v}.
\]

Now we take another function
\[
f_n = \frac{h(\varphi(z_n))}{\varphi(z_n)} \left( \log \frac{2}{1 - |\varphi(z_n)|^2} \right)^{-1}.
\]
From [11] we know that $\{f_n\}$ is a bounded sequence in $L^\infty_1$ which converges to zero uniformly on compact subsets of $D$, and $\sup_{n \in \mathbb{N}} \|f_n\|_{X_\beta} < \infty$. It follows from Lemmas 4 and 5 that
\[
C \|I_\varphi C_\varphi - I_\varphi f\|_{L^1 \rightarrow L^1_v} \geq \limsup_{n \to \infty} \|I_\varphi C_\varphi f_n\|_{L^1_v}
\geq \limsup_{n \to \infty} \left( 1 - |z_n|^2 \right) \left| g'(z_n) \varphi'(z_n) \right| \left( \frac{|\varphi(z_n)|}{1 - |\varphi(z_n)|^2} \right)^{\beta}
\geq \limsup_{n \to \infty} \|g(\varphi')\varphi''\|_{L^1_v}.
\]
Noting that $\limsup_{n \to \infty} (n + 1) \|g(\varphi')\varphi''\|_{L^1_v} \leq (2C/e)\|I_\varphi C_\varphi\|_{L^1 \rightarrow L^1_v}$, we obtain
\[
\left( C + \frac{2C}{e} \right) \|I_\varphi C_\varphi\|_{L^1 \rightarrow L^1_v} \geq \limsup_{n \to \infty} \left( 1 - |z_n|^2 \right) \left| g'(z_n) \varphi'(z_n) \right| \left( \frac{|\varphi(z_n)|}{1 - |\varphi(z_n)|^2} \right)^{\beta}
= \limsup_{n \to \infty} (n + 1) \|g(\varphi')\varphi''\|_{L^1_v}.
\]
Hence we have $\|I_\varphi C_\varphi\|_{L^1 \rightarrow L^1_v} \geq C \max \{\limsup_{n \to \infty} (n + 1) \|g(\varphi')\varphi''\|_{L^1_v}, \limsup_{n \to \infty} (\log n) \|g(\varphi')\varphi''\|_{L^1_v}\}$.

(iii) Let $\alpha > 1$. The proof of the upper bound is similar to that of (ii). From the proof of (i), we get that, for some constant $C$
\[
C \|I_\varphi C_\varphi\|_{L^1 \rightarrow L^1_v} \geq \limsup_{n \to \infty} \left( 1 - |z_n|^2 \right) \left| g'(z_n) \varphi'(z_n) \right| \left( \frac{|\varphi(z_n)|}{1 - |\varphi(z_n)|^2} \right)^{\beta}
\geq \limsup_{n \to \infty} \|g(\varphi')\varphi''\|_{L^1_v}.
\]
Now, let $\{z_n\}$ be as before and note that the function $f_n = f_{\varphi(z_n)}$ given in (23). Then $\{f_n\}$ is bounded sequence in

\[\text{Journal of Function Spaces}\]
Let $\varphi(z)$ be an analytic self-map of $D$ and $g \in H(D)$, and $C_{\varphi} T_g : \mathcal{X}^\alpha \to \mathcal{X}^\beta$ be a bounded operator.

(i) If $0 < \alpha < 1$, then
$$\| C_{\varphi} T_g \|_{\mathcal{X}^\alpha \to \mathcal{X}^\beta} = 0.$$  

(ii) If $\alpha = 1$, then
$$\| C_{\varphi} T_g \|_{\mathcal{X}^\alpha \to \mathcal{X}^\beta} = \limsup_{n \to \infty} (\log n) \left\| \left( (g' \circ \varphi) \left( \varphi' \right)^2 \right) \varphi^n \right\|_{\| \cdot \|_{\mathcal{Y}_g}}.$$  

(iii) If $1 < \alpha < 2$, then
$$\| C_{\varphi} T_g \|_{\mathcal{X}^\alpha \to \mathcal{X}^\beta} = \limsup_{n \to \infty} (n+1)^{\alpha-1} \left\| \left( (g' \circ \varphi) \left( \varphi' \right)^2 \right) \varphi^n \right\|_{\| \cdot \|_{\mathcal{Y}_g}}.$$  

(iv) If $\alpha = 2$, then
$$\| C_{\varphi} T_g \|_{\mathcal{X}^\alpha \to \mathcal{X}^\beta} = \max \left\{ \limsup_{n \to \infty} (n+1) \cdot \left\| \left( (g' \circ \varphi) \left( \varphi' \right)^2 \right) \varphi^n \right\|_{\| \cdot \|_{\mathcal{Y}_g}}, \limsup_{n \to \infty} (\log n) \left\| \left( (g' \circ \varphi) \left( \varphi' \right)^2 \right) \varphi^n \right\|_{\| \cdot \|_{\mathcal{Y}_g}} \right\}.$$  

(v) If $\alpha > 2$, then
$$\| C_{\varphi} T_g \|_{\mathcal{X}^\alpha \to \mathcal{X}^\beta} = \max \left\{ \limsup_{n \to \infty} (n+1)^{\alpha-1} \cdot \left\| \left( (g' \circ \varphi) \left( \varphi' \right)^2 \right) \varphi^n \right\|_{\| \cdot \|_{\mathcal{Y}_g}}, \limsup_{n \to \infty} (n+1)^{\alpha-2} \cdot \left\| \left( (g'' \circ \varphi) \left( \varphi' \right)^2 \right) \varphi^n \right\|_{\| \cdot \|_{\mathcal{Y}_g}} \right\}.$$
For every compact operator $T : \mathcal{X} \to \mathcal{X}^\beta$, we have $\|T(B_n)\|_{\mathcal{X}^\beta} \to 0$ as $n \to \infty$. Let $M = \sup_{n \in \mathbb{N}} \|g_n\|_{\mathcal{X}^\beta}$. It follows from Lemma 5 that

$$M \|C_{\varphi}T_g - T\|_{e,\mathcal{X} \to \mathcal{X}^\beta} \geq \limsup_{n \to \infty} \left(1 - |z_n|^2\right)^\beta \left|\varphi'(z_n)\right| \left|\varphi'(z_n)\right|^2 \left(\log \frac{1}{1 - |\varphi(z_n)|^2}\right) \|\varphi\|^n_{B_g},$$

$$\geq \limsup_{n \to \infty} \left(\|\varphi\|^n\right) \left(\left(\varphi'(z_n)\right)^2 \left|\varphi'(z_n)\right|^2 \right) \|\varphi\|^n_{B_g}.$$  \hspace{1cm} (98)

This completes the proof.

The proof of (iii) is the same as that of Theorem 11 (ii); we do not prove it again.

(iv) Let $\alpha = 2$. Applying Lemma 3 (ii) and Theorem 3.2 in [29], we get that

$$\left\|\left(\varphi' \circ \varphi\right)^2 C_{\varphi} \right\|_{e,\mathcal{X} \to \mathcal{X}^\beta} = \limsup_{n \to \infty} (n + 1) \left(\left(\varphi'(z_n)\right)^2 \left|\varphi'(z_n)\right|^2 \right) \|\varphi\|^n_{B_g},$$

$$\left\|\left(\varphi'' \circ \varphi\right) \left(\varphi'(z_n)\right)^2 + \left(\varphi' \circ \varphi\right)^2 \varphi'' \left(\varphi'(z_n)\right)^2 \right\|_{B_g} \geq \limsup_{n \to \infty} (n + 1) \left(\left(\varphi'(z_n)\right)^2 \left|\varphi'(z_n)\right|^2 \right) \|\varphi\|^n_{B_g}.$$  \hspace{1cm} (99)

which yields the upper bound by (67).

With the same arguments as in the proof of Theorems 8 and 11, for some constant $C$, we have

$$C \left\|C_{\varphi}T_g \right\|_{e,\mathcal{X} \to \mathcal{X}^\beta} \geq \limsup_{n \to \infty} (n + 1) \left(\left(\varphi'(z_n)\right)^2 \left|\varphi'(z_n)\right|^2 \right) \|\varphi\|^n_{B_g}.$$  \hspace{1cm} (100)

On the other hand, let $\{z_n\} \subseteq D$ with $|\varphi(z_n)| > 1/2$ and $|\varphi(z_n)| \to 1$ as $n \to \infty$. Let the test function

$$O_n(z) = \left(1 + \left(\log \frac{2}{1 - |\varphi(z_n)|^2}\right)^2\right)^{\alpha - 2} \left(\log \frac{2}{1 - |\varphi(z_n)|^2}\right)^{-1},$$

From [8] we obtain that $\{O_n\}$ is a bounded sequence in $\mathcal{X}^2_0$ which converges to zero uniformly on compact subsets of $D$, and

$$\lim_{n \to \infty} \left(1 - |z_n|^2\right)^\beta \left|\varphi'(z_n)\right| \left(\varphi'(z_n)\right)^2 \left|O'\right| \left(\varphi'(z_n)\right)^2 = 2 \lim_{n \to \infty} \left(1 - |z_n|^2\right)^\beta \left|\varphi'(z_n)\right| \left(\varphi'(z_n)\right)^2 \left|O'\right| \left(\varphi'(z_n)\right)^2$$

$$\cdot \left|\varphi(z_n)\right| \leq C \left\|C_{\varphi}T_g \right\|_{e,\mathcal{X} \to \mathcal{X}^\beta}.$$  \hspace{1cm} (102)

Applying Theorem 8 we get

$$C \left\|C_{\varphi}T_g \right\|_{e,\mathcal{X} \to \mathcal{X}^\beta} \geq \limsup_{n \to \infty} \left(1 - |z_n|^2\right)^\beta \left|\varphi'(z_n)\right| \left(\varphi'(z_n)\right)^2 \left|O'\right| \left(\varphi'(z_n)\right)^2 \left|O\right| \left(\varphi'(z_n)\right)$$

$$\cdot \left|\varphi(z_n)\right| \leq C \left\|C_{\varphi}T_g \right\|_{e,\mathcal{X} \to \mathcal{X}^\beta}.$$  \hspace{1cm} (103)

Hence

$$\lim_{n \to \infty} \left(1 - |z_n|^2\right)^\beta \left|\varphi'(z_n)\right| \left(\varphi'(z_n)\right)^2 \left|O'\right| \left(\varphi'(z_n)\right)^2 \left|O\right| \left(\varphi'(z_n)\right)$$

$$\cdot \left|\varphi(z_n)\right| \leq C \left\|C_{\varphi}T_g \right\|_{e,\mathcal{X} \to \mathcal{X}^\beta}.$$  \hspace{1cm} (104)

On the other hand, the lower bound can be easily proved by Lemmas 4 and 5.

If $\alpha > 2$, the proof is similar to that of (iv) except that we now choose the test function $t_a(z) = (1 - |\varphi(z_n)|^2)^\alpha/(1 - \varphi(z_n)^2)$ instead of $O_n(z)$. This completes the proof of Theorem 12.

Using the same methods of Theorems 11 and 12, we can have the following results.

Theorem 13. Let $C_{\varphi}T_g$ be a bounded operator from $\mathcal{X}^\alpha$ to $\mathcal{X}^\beta$ space.

(i) If $0 < \alpha < 1$, then

$$\left\|C_{\varphi}T_g \right\|_{e,\mathcal{X}^\alpha \to \mathcal{X}^\beta}$$

$$= \limsup_{n \to \infty} (n + 1)^\alpha \left(\left(\varphi'(z_n)\right)^2 \varphi'' \left(\varphi'(z_n)\right)^2 \right) \|\varphi\|^n_{B_g}.$$  \hspace{1cm} (105)
Let $\mathbf{H}(\mathbb{D})$ regarding the publication of this paper. The authors declare that there are no conflicts of interest.

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