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Research Article

Existence Results for Integral Equations and Boundary Value Problems via Fixed Point Theorems for Generalized F-Contractions in b-Metric-Like Spaces

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In this paper, we introduce some new classes of generalized F-contractions and we establish certain fixed point results for such mappings in the setting of b-metric-like spaces. Some examples will illustrate the results and the corresponding computer simulations are suggestive from the output point of view. A second purpose of the paper is to apply the abstract results in the study of the existence of a solution for an integral equation problem and for a boundary value problem related to a real life mathematical model, namely, the problem of conversion of solar energy to electrical energy. Our study is concluded with an open problem, related to an integrodifferential equation arising in the study of electrical and electronics circuit analysis.

1. Introduction and Preliminaries

There are many extensions and generalizations of the metric space concept. In 1989, Bakhtin [1] introduced the notion of b-metric space, while Czerwik ([2, 3]) extensively used the concept of b-metric space for proving fixed point theorems for single-valued and multivalued mappings. On the other hand, the concept of partial metric space was introduced by Matthews [4].

More recently, Amini-Harandi [5] generalized the concept of partial metric space by introducing the metric-like spaces. After that, in [6], Alghamdi et al. introduced b-metric-like spaces, which extends the notions of partial metric spaces, b-metric spaces, and metric-like spaces. There are many other types of generalized metric spaces (see [7, 8]), introduced by adapting and developing new metric axioms. These generalized metric spaces frequently appear to be metrizable and the contraction conditions may be conserved under various particular transforms. Hence, fixed point theory in such spaces may be an outcome of the fixed point theory in classical metric spaces. However, it is not true that all generalized fixed point results become obvious in this way. More specifically, these results are based on some contraction type conditions, and some of these conditions do not remain authentic when one considers the problem in the associated metric space; see, for example, the well-written papers [9, 10].

On the other hand, in 2012, Wardowski [11] introduced a new contraction mapping, called F-contraction, and proved a fixed point result as a generalization of the Banach contraction principle. After this, Abbas et al. [12] generalized the idea of F-contraction and proved certain fixed and common fixed point theorems. Recently, Secelean [13] described a large class of functions using the condition $(F2')$ instead of the condition $(F2)$ in the definition of F-contraction presented by Wardowski [11]. Very recently, Piri and Kumam [14] improved the result of Secelean [13], by using the condition $(F3')$ instead of the condition $(F3)$.

In this paper, we consider the notions of ϕ-F-contraction and Suzuki-Berinde type F-contraction in the context of b-metric-like spaces in order to prove certain fixed point results. Some illustrative examples are considered, which validate the hypothesis of proved results. Moreover, some applications to integral equations and a boundary value problem related to a mathematical model of conversion of solar energy to electrical energy are also given. Finally, an open problem is also suggested for the utilization of our results to some engineering problems.

In this paper, $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{R}^+ := [0, \infty)$ will denote the set of all real numbers, natural numbers, and the set of all real nonnegative numbers, respectively.

For the beginning, some necessary definitions and fundamental results, which will be used in the sequel, are presented here.

Definition 1 (see [2]). Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

(b_1) $d(x, y) = 0$ iff $x = y$;

(b_2) $d(x, y) = d(y, x)$;

(b_3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X, d).

Definition 2 (see [5]). A function $\sigma : X \times X \to [0, \infty)$ is called a metric-like if, for all $x, y, z \in X$, the following conditions are satisfied:

($\sigma 1$) $\sigma(x, y) = 0$ implies $x = y$;

($\sigma 2$) $\sigma(x, y) = \sigma(y, x)$;

($\sigma 3$) $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.

The pair (X, σ) is called a metric-like space.

In the following definition, Alghamdi et al. [6] extended Definition 2 in order to introduce the new notion of b-metric-like space.

Definition 3 (see [6]). Let X be a nonempty set and $s \geq 1$ be a given real number. A function $\sigma_b : X \times X \to [0, \infty)$ is called a b-metric-like if, for all $x, y, z \in X$, the following conditions are satisfied:

($\sigma_b 1$) $\sigma_b(x, y) = 0$ implies $x = y$;

($\sigma_b 2$) $\sigma_b(x, y) = \sigma_b(y, x)$;

($\sigma_b 3$) $\sigma_b(x, y) \leq s[\sigma_b(x, z) + \sigma_b(z, y)]$.

The pair (X, σ_b) is called a b-metric-like space. The number $s \geq 1$ is called the coefficient of (X, σ_b).

Example 4 (see [6]). Let $X = \mathbb{R}^+$ and the mapping $\sigma_b : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma_b(x, y) = [\max\{x, y\}]^2, \tag{1}$$

for all $x, y \in X$. Then (X, σ_b) is a b-metric-like space with the coefficient $s = 2 > 1$, but it is neither a b-metric nor a metric-like space.

Remark 5. The class of b-metric-like space (X, σ_b) is larger than the class of metric-like space, since a metric-like space is a special case of b-metric-like space (X, p_b) when $s = 1$. Also, the class of b-metric-like space (X, σ_b) is effectively larger than the class of b-metric space, since a b-metric space is a special case of a b-metric-like space (X, σ_b) when the self-distance $\sigma_b(x, x) = 0$.

Each b-metric-like σ_b on X generalizes a topology τ_{σ_b} on X whose base is the family of open σ_b-balls $B - \sigma_b(x, \epsilon) = \{y \in X : |\sigma_b(x, y) - \sigma_b(x, x)| < \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Definition 6 (see [6]). A sequence $\{x_n\}$ in a b-metric-like space (X, σ_b) is said to be

(1) σ_b convergent to a point $x \in X$ if $\sigma_b(x, x) = \lim_{n\to\infty} \sigma_b(x, x_n)$;

(2) a σ_b-Cauchy sequence if $\lim_{n,m\to\infty} \sigma_b(x_n, x_m)$ exists (and is finite).

Definition 7 (see [6]). A b-metric-like space (X, σ_b) is said to be σ_b-complete if every σ_b-Cauchy sequence $\{x_n\}$ in X, σ_b-converges to a point $x \in X$, such that

$$\sigma_b(x, x) = \lim_{n,m\to\infty} \sigma_b(x_n, x_m) = \lim_{n\to\infty} \sigma_b(x, x_n). \tag{2}$$

Definition 8 (see [6]). Suppose that (X, σ_b) is a b-metric-like space. A mapping $T : X \to X$ is said to be continuous at $x \in X$, if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $T(B_{\sigma_b}(x, \delta)) \subset B_{\sigma_b}(Tx, \epsilon)$. We say that T is continuous on X if T is continuous at all $x \in X$.

Lemma 9 (see [6]). *Let $\{y_n\}$ be a sequence in a b-metric-like space (X, σ_b) such that*

$$\sigma_b(y_n, y_{n+1}) \leq \lambda \sigma_b(y_{n-1}, y_n), \tag{3}$$

for some λ, $0 < \lambda < 1/s$, and each $n \in \mathbb{N}$. Then $\{y_n\}$ is a Cauchy sequence in X and $\lim_{n,m\to\infty} \sigma_b(y_n, y_m) = 0$.

Remark 10 (see [6]). Let (X, σ_b) be a b-metric-like space with constant $s \geq 1$. Then it is clear that

$$\sigma_b^s(x, y) = |2\sigma_b(x, y) - \sigma_b(x, x) - \sigma_b(y, y)| \tag{4}$$

satisfies $\sigma_b^s(x, x) = 0$, for all $x \in X$. So it is considered to be a b-metric induced by b-metric-like spaces.

Remark 11 (see [15]). Let (X, σ_b) be a b-metric-like space and let $f : X \to X$ be a continuous mapping. Then

$$\lim_{n\to\infty} \sigma_b(x_n, x) = \sigma_b(x, x) \Longrightarrow \lim_{n\to\infty} \sigma_b(fx, fx_n) = \sigma_b(x, x). \tag{5}$$

Wardowski [11] introduced the F-contraction as follows.

Definition 12. Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying

($F1$) F is strictly increasing, that is, for $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$ implies $F(\alpha) < F(\beta)$;

($F2$) for each sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

($F3$) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Denote the set of all functions satisfying $(F1)$–$(F3)$ by $\mathfrak{F}$. In [13], Secelean changed the condition $(F2)$ by an equivalent but a more simple condition $(F2')$.

$(F2')$ $\inf F = -\infty$,

or also by

$(F2'')$ there exists a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers with $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Recently, Piri and Kumam [14] used the following condition $(F3')$ instead of $(F3)$.

$(F3')$ F is continuous on $(0, \infty)$.

In our subsequent discussion, condition $(F2')$ is dropped out. Thus we utilize the functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfy $(F1)$ and $(F3')$. The class of all functions satisfying $(F1)$ and $(F3')$ is denoted by Δ_F.

Let Φ be the set of functions $\phi : [0, \infty) \to [0, \infty)$ such that

(1) ϕ is monotonic increasing, that is, $t_1 \le t_2 \Rightarrow \phi(t_1) \le \phi(t_2)$;

(2) ϕ is continuous and $\phi(t) < t$ for each $t > 0$.

Let Ψ denote the set of all continuous functions $\psi : (0, \infty) \to (0, \infty)$.

Remark 13. For recent interesting fixed point results for α-F-contractions, see [16–19].

2. Fixed Point Results for ϕ-F Contractive Mappings

We introduce the following concept.

Definition 14. Let (X, σ_b) be a b-metric-like space. A self-mapping $T : X \to X$ is said to be generalized ϕ-F contraction if $F \in \Delta_F$ and

$$\sigma_b(Tx, Ty) > 0 \Longrightarrow$$
$$\psi(\sigma_b(x, y)) + F(\sigma_b(Tx, Ty)) \le F\left(\phi\left(\alpha\sigma_b(x, y) + \beta\frac{\sigma_b(x, Ty)}{2s} + \gamma\frac{\sigma_b(y, Tx)}{2s}\right)\right), \tag{6}$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \in [0, 1]$ (not all zero simultaneously), such that $\alpha + \beta + \gamma \le 1$, $\phi \in \Phi$ and $\psi \in \Psi$.

For illustrating the above definition, the following example is presented.

Example 15. Let $X = [0, 25]$ and let the function $\sigma_b : X \times X \to [0, \infty)$ be defined by

$$\sigma_b(x, y) = \left(\max\{x, y\}^{3/2}\right), \tag{7}$$

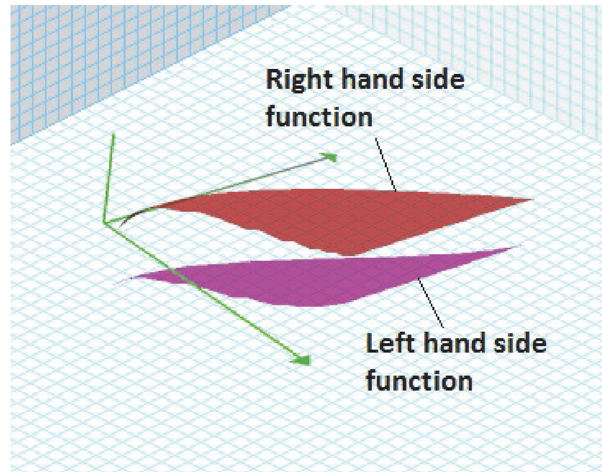

Figure 1: Right hand side superimposes left hand side in Case 1.

for all $x, y \in X$. It is obvious that (X, σ_b) is a complete b-metric-like space with $s = \sqrt{2}$. Define the mapping $T : X \to X$ by

$$Tx = \tan\left(\frac{1}{20}(\log(10 + x) + x)\right). \tag{8}$$

In order to verify the Condition (6) with $\alpha = 0.6$, $\beta = 0.2$, and $\gamma = 0.2$ such that $\alpha + \beta + \gamma \le 1$ and $F(k) = \log k$, for all $k \in \mathbb{R}^+$, we see that $F \in \Delta_F$. Let $\phi : [0, \infty) \to [0, \infty)$ be given by $\phi(t) = 5t/6$ and functions $\psi : (0, \infty) \to (0, \infty)$ are defined by $\psi(t) = 1/(t + 10)$.

Here we note that

$$\sigma_b(Tx, Ty) > 0 \quad \forall x, y \in X. \tag{9}$$

Without loss of generality, assume that $x \ge y$. Then, the following cases arise.

Case 1. When $x \ge y > Tx$, calculating various terms appearing in the inequality (6), we conclude that left hand side of (6) comes out

$$\psi(\sigma_b(x, y)) + F(\sigma_b(Tx, Ty)) = \frac{1}{x^{3/2} + 10} + \log\left[\tan\left(\frac{1}{20}(\log(10 + x) + x)\right)\right]^{3/2}, \tag{10}$$

and right hand side of (6) becomes

$$\log\left[\frac{5}{6}\left(\alpha x^{3/2} + \frac{\beta}{2\sqrt{2}}x^{3/2} + \frac{\gamma}{2\sqrt{2}}y^{3/2}\right)\right]. \tag{11}$$

It is evident from Figure 1 that the surface representing right hand side is dominating the surface representing left hand side. This concludes that, in this case, the condition (6) is verified.

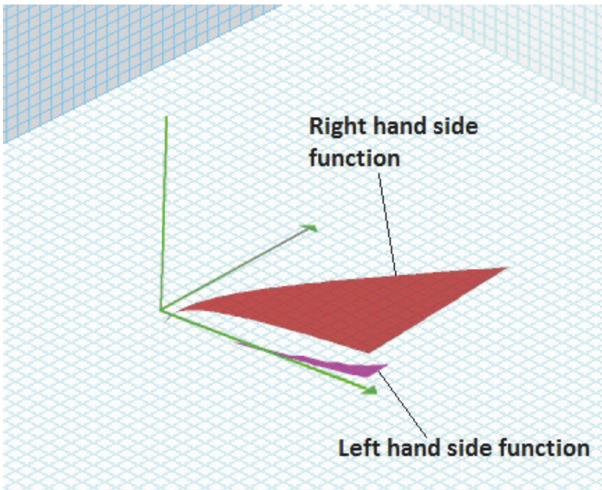

Figure 2: Right hand side is dominating left hand side in Case 2.

Case 2. When $x > Tx \geq y$, with this assumption, evaluating the terms involved in Condition (6), we obtain the left hand side as

$$\psi(\sigma_b(x,y)) + F(\sigma_b(Tx,Ty)) = \frac{1}{x^{3/2}+10} + \log\left[\tan\left(\frac{1}{20}(\log(10+x)+x)\right)\right]^{3/2}, \tag{12}$$

and right hand side of (6) becomes

$$\log\left[\frac{5}{6}\left(\alpha x^{3/2} + \frac{\beta}{2\sqrt{2}}x^{3/2} + \frac{\gamma}{2\sqrt{2}}\left\{\tan\left(\frac{1}{20}(\log(10+x)+x)\right)\right\}^{3/2}\right)\right]. \tag{13}$$

Figure 2 shows that right hand side expression is superimposing the left hand side expression, which validates our condition in this case.

Thus all the hypothesis of Definition 14 are fulfilled and therefore T is a ϕ-F contraction mapping.

Our main result runs as follows.

Theorem 16. *Let (X, σ_b) be a complete b-metric-like space and T be a continuous generalized ϕ-F contraction. If $\sigma_b(Tx,Tx) \leq \sigma_b(x,x)$, for all $x \in X$, then T has a unique fixed point in X.*

Proof. Let x_0 be an arbitrary point in X. Set $Tx_0 = x_1$ and define a sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}. \tag{14}$$

If there exists $n_0 \in \mathbb{N}$ such that

$$\sigma_b(x_{n_0}, x_{n_0+1}) = 0, \tag{15}$$

then x_{n_0} is the fixed point of T, which complete the proof.

Consequently, we suppose $\sigma_b(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}$. Then we have

$$0 < \sigma_b(x_n, x_{n+1}) = \sigma_b(Tx_{n-1}, Tx_n), \quad \forall n \in \mathbb{N}, \tag{16}$$

and by (6) we obtain

$$\begin{aligned}&\psi(\sigma_b(x_{n-1},x_n)) + F(\sigma_b(Tx_{n-1},Tx_n)) \\ &\quad \leq F\left(\phi\left(\alpha\sigma_b(x_{n-1},x_n) + \frac{\beta}{2s}\sigma_b(x_{n-1},Tx_n) + \frac{\gamma}{2s}\sigma_b(x_n,Tx_{n-1})\right)\right).\end{aligned} \tag{17}$$

Now, we claim that

$$\sigma_b(x_n, Tx_n) < \sigma_b(x_{n-1}, Tx_{n-1}), \quad \forall n \in \mathbb{N}. \tag{18}$$

Suppose, on the contrary, that there exists $n_0 \in \mathbb{N}$, such that

$$\sigma_b(x_{n_0}, Tx_{n_0}) \geq \sigma_b(x_{n_0-1}, Tx_{n_0-1}). \tag{19}$$

Then, by (6), one gets

$$\begin{aligned}&\psi(\sigma_b(x_{n_0-1},x_{n_0})) + F(\sigma_b(x_{n_0},Tx_{n_0})) \\ &\quad = \psi(\sigma_b(x_{n_0-1},x_{n_0})) + F(\sigma_b(Tx_{n_0-1},Tx_{n_0})) \\ &\quad \leq F\left(\phi\left(\alpha\sigma_b(x_{n_0-1},x_{n_0}) + \frac{\beta}{2s}\sigma_b(x_{n_0-1},Tx_{n_0}) + \frac{\gamma}{2s}\sigma_b(x_{n_0},Tx_{n_0-1})\right)\right) \leq F\left(\phi\left(\alpha\sigma_b(x_{n_0-1},x_{n_0})\right.\right. \\ &\quad + \frac{\beta}{2s}\left(s\sigma_b(x_{n_0-1},x_{n_0}) + s\sigma_b(x_{n_0},Tx_{n_0})\right) \\ &\quad \left.\left.+ \frac{\gamma}{2s}\left(s\sigma_b(x_{n_0},x_{n_0-1}) + s\sigma_b(x_{n_0-1},Tx_{n_0-1})\right)\right)\right) \\ &\quad = F\left(\phi\left(\alpha\sigma_b(x_{n_0-1},Tx_{n_0-1})\right.\right. \\ &\quad + \frac{\beta}{2}\left(\sigma_b(x_{n_0-1},Tx_{n_0-1}) + \sigma_b(x_{n_0},Tx_{n_0})\right) \\ &\quad \left.\left.+ \frac{\gamma}{2}\left(\sigma_b(Tx_{n_0-1},x_{n_0-1}) + \sigma_b(x_{n_0-1},Tx_{n_0-1})\right)\right)\right) \\ &\quad \leq F\left(\phi\left(\alpha\sigma_b(x_{n_0-1},Tx_{n_0-1}) + \beta\sigma_b(x_{n_0},Tx_{n_0})\right.\right. \\ &\quad \left.\left.+ \gamma\sigma_b(x_{n_0-1},Tx_{n_0-1})\right)\right).\end{aligned} \tag{20}$$

In view of the properties of Φ, Ψ, and $(F1)$, we obtain that

$$\begin{aligned}\sigma(x_{n_0},Tx_{n_0}) &< \alpha\sigma_b(x_{n_0-1},Tx_{n_0-1}) + \beta\sigma_b(x_{n_0},Tx_{n_0}) \\ &\quad + \gamma\sigma_b(x_{n_0-1},Tx_{n_0-1}),\end{aligned} \tag{21}$$

which shows

$$\sigma_b(x_{n_0},Tx_{n_0}) < \left(\frac{\alpha+\gamma}{1-\beta}\right)\sigma_b(x_{n_0-1},Tx_{n_0-1}); \tag{22}$$

this implies

$$\sigma_b\left(x_{n_0}, Tx_{n_0}\right) < \sigma_b\left(x_{n_0-1}, Tx_{n_0-1}\right), \quad \text{since } \left(\frac{\alpha+\gamma}{1-\beta}\right) \leq 1. \tag{23}$$

This is a contradiction and hence (18) holds; that is, $\sigma_b(x_n, Tx_n) < \sigma_b(x_{n-1}, Tx_{n-1})$. So $\{\sigma_b(x_n, Tx_n)\}$ is a decreasing sequence in $\mathbb{R}^+$ and is bounded below at 0; consequently it is convergent to some point, say $p \in \mathbb{R}^+$. Now we assert that $p = 0$. On the contrary suppose $p > 0$.

On the similar approach as discussed earlier, we conclude that

$$\begin{aligned}\psi\left(\left(x_{n-1}, x_n\right)\right) &+ F\left(\sigma_b\left(Tx_{n-1}, Tx_n\right)\right) \\ &\leq F\left(\left(\alpha\sigma_b\left(x_{n-1}, Tx_{n-1}\right) + \beta\sigma_b\left(x_{n-1}, Tx_{n-1}\right)\right.\right. \\ &\left.\left.+ \gamma\sigma_b\left(x_{n-1}, Tx_{n-1}\right)\right)\right).\end{aligned} \tag{24}$$

Letting $n \to \infty$ and utilizing $(F3')$, we have

$$\psi(p) + F(p) \leq F((\alpha+\beta+\gamma)p). \tag{25}$$

This is a contradiction, in view of the properties of Ψ and $(F1)$ and the fact that $\alpha+\beta+\gamma \leq 1$.

So, we must have $p = 0$; that is,

$$\lim_{n\to\infty}\sigma_b\left(x_n, Tx_n\right) = 0. \tag{26}$$

Now we shall prove that $\{x_n\}$ is a Cauchy sequence. In fact, we will establish that

$$\lim_{n,m\to\infty}\sigma_b\left(x_n, x_m\right) = 0. \tag{27}$$

On the contrary, suppose that there exists $\epsilon > 0$, and two sequences $\{p_n\}$ and $\{q_n\}$ on natural number such that $p(n) > q(n) > n$,

$$\begin{aligned}\sigma_b\left(x_{p(n)}, x_{q(n)}\right) &\geq \epsilon, \\ \sigma_b\left(x_{p(n)-1}, x_{q(n)}\right) &< \epsilon, \\ &\forall n \in \mathbb{N}.\end{aligned} \tag{28}$$

From the triangle inequality, one can obtain

$$\begin{aligned}\sigma_b\left(x_{p(n)-1}, x_{q(n)-1}\right) &< s\sigma_b\left(x_{p(n)-1}, x_{q(n)}\right) \\ &\quad + \sigma_b\left(x_{q(n)}, x_{q(n)-1}\right) \\ &< s\sigma_b\left(x_{q(n)}, x_{q(n)-1}\right) + s\epsilon \\ &= s\sigma_b\left(Tx_{q(n)-1}, x_{q(n)-1}\right) + s\epsilon, \\ &\qquad \forall n \in \mathbb{N}.\end{aligned} \tag{29}$$

From (26), there exists $N_1 \in \mathbb{N}$, such that

$$\begin{aligned}\sigma_b\left(x_{p(n)-1}, Tx_{p(n)-1}\right) &< \epsilon, \\ \sigma_b\left(x_{q(n)-1}, Tx_{q(n)-1}\right) &< \epsilon, \\ &\forall n \geq N_1,\end{aligned} \tag{30}$$

which, together with (29), shows

$$\sigma_b\left(x_{p(n)-1}, x_{q(n)-1}\right) < 2\epsilon s, \quad \forall n \geq N_1; \tag{31}$$

therefore

$$F\left(\sigma_b\left(x_{p(n)-1}, x_{q(n)-1}\right)\right) < F(2s\epsilon), \quad \forall n \geq N_1. \tag{32}$$

From (28), we get

$$\epsilon \leq \sigma_b\left(x_{p(n)}, x_{q(n)}\right) = \sigma_b\left(Tx_{p(n)-1}, Tx_{q(n)-1}\right). \tag{33}$$

Employing (6), one acquires

$$\begin{aligned}\psi\left(\sigma_b\left(x_{p(n)-1}, x_{q(n)-1}\right)\right) &+ F\left(\sigma_b\left(Tx_{p(n)-1}, Tx_{q(n)-1}\right)\right) \\ &\leq F\left(\phi\left(\alpha\sigma_b\left(x_{p(n)-1}, x_{q(n)-1}\right) + \frac{\beta}{2s}\right.\right. \\ &\cdot \sigma_b\left(x_{p(n)-1}, Tx_{q(n)-1}\right) + \frac{\gamma}{2s} \\ &\left.\left.\cdot \sigma_b\left(x_{q(n)-1}, Tx_{p(n)-1}\right)\right)\right) \\ &\leq F\left(\phi\left(\alpha\sigma_b\left(x_{p(n)-1}, x_{q(n)-1}\right)\right.\right. \\ &+ \frac{\beta}{2}\left[\sigma_b\left(x_{p(n)-1}, x_{q(n)-1}\right) + \sigma_b\left(x_{q(n)-1}, Tx_{q(n)-1}\right)\right] \\ &+ \frac{\gamma}{2}\left[\sigma\left(x_{q(n)-1}, x_{p(n)-1}\right)\right. \\ &\left.\left.\left.+ \sigma_b\left(x_{p(n)-1}, Tx_{p(n)-1}\right)\right]\right)\right), \quad \forall n \geq N_1.\end{aligned} \tag{34}$$

In view of (30), (31), and (32), we get

$$\begin{aligned}&\psi(2s\epsilon) + F\left(\sigma_b\left(Tx_{p(n-1)}, Tx_{q(n-1)}\right)\right) \\ &\quad < F\left[\alpha(2s\epsilon) + \frac{\beta}{2}(2s\epsilon+\epsilon) + \frac{\gamma}{2}(2s\epsilon+\epsilon)\right] \Longrightarrow \\ &\sigma_b\left(Tx_{p(n-1)}, Tx_{q(n-1)}\right) \\ &\quad < \alpha(2s\epsilon) + \frac{\beta}{2}(2s\epsilon+\epsilon) + \frac{\gamma}{2}(2s\epsilon+\epsilon).\end{aligned} \tag{35}$$

This amounts to say that $\lim_{n\to\infty}\sigma_b(Tx_{p(n)-1}, Tx_{q(n)-1}) = 0$. Hence $\lim_{n\to\infty}\sigma_b(x_{p(n)}, x_{q(n)}) = 0$, which is a contradiction with (28). This validates (27). Therefore $\{x_n\}$ is a Cauchy sequence in X. Since (X, σ_b) is complete, there exists $w \in X$, such that

$$\sigma_b(w, w) = \lim_{n\to\infty}\sigma_b\left(x_n, w\right) = \lim_{n,m\to\infty}\sigma_b\left(x_n, x_m\right) = 0. \tag{36}$$

Since T is continuous, we get

$$\begin{aligned}\sigma_b(Tw, Tw) &= \lim_{n\to\infty}\sigma_b\left(Tx_n, Tw\right) \\ &= \lim_{n\to\infty}\sigma_b\left(x_{n+1}, Tw\right) = 0.\end{aligned} \tag{37}$$

Due to the fact that $\sigma(Tw, Tw) \le \sigma(w, w)$, from above, we have $\lim_{n\to\infty}\sigma_b(x_n, Tw) = 0$. Since $\sigma_b(w, Tw) \le s[\sigma_b(w, x_n) + \sigma_b(x_n, Tw)]$, letting $n \to \infty$, we obtain that $\sigma_b(w, Tw) = 0$. Thus $w = Tw$ and so T has a fixed point.

In order to show the uniqueness of fixed point, suppose v is another fixed point such that $w \ne v$. Then we have

$$\begin{aligned}
&\sigma_b(Tw, Tu) > 0 \Longrightarrow \\
&F(\sigma_b(w, v)) = F(\sigma_b(Tw, Tv)) \le F\Bigg(\phi\Bigg(\alpha\sigma_b(w, v) \\
&\quad + \beta\frac{\sigma_b(w, Tv)}{2s} + \gamma\frac{\sigma_b(v, Tw)}{2s}\Bigg)\Bigg) - \psi(\sigma_b(w, v)) \\
&\quad \le F\left(\left(\alpha + \frac{\beta}{2s} + \frac{\gamma}{2s}\right)\sigma_b(w, v)\right) - \psi(\sigma_b(w, v)).
\end{aligned} \tag{38}$$

This is a contradiction, in view of $(F1)$ and Ψ. Thus we have $w = v$. Hence T has a unique fixed point. This completes the proof. □

In order to illustrate our result, we present the following example.

Example 17. Let $X = [0.0477177, \infty)$ and let the function $\sigma_b : X\times X \to [0, \infty)$ be defined by $\sigma_b(x, y) = x^2 + y^2 + |x - y|^2$, for all $x, y \in X$. It is obvious that (X, σ_b) is a complete b-metric-like space with $s = 2$. Let the mapping $T : X \to X$ be defined by

$$Tx = \frac{e^{x^2/(1+x)}}{20 + e^{x^2/(1+x)}}. \tag{39}$$

We verify the condition (6) with $\alpha = 0.8$, $\beta = 0.1$, and $\gamma = 0.1$ (clearly $\alpha + \beta + \gamma \le 1$) and $F(p) = \log p + p$, for all $p \in \mathbb{R}^+$. Notice that $F \in \Delta_F$ and $\sigma_b(Tx, Ty) > 0$, for all $x, y \in X$. Consider $\phi, \psi : [0, \infty) \to [0, \infty)$ given by $\phi(t) = t/2$ and $\psi(t) = 1/(t^2 + 10)$.

Various terms involved in the inequality (6) are calculated as follows:

$$\begin{aligned}
\sigma_b(Tx, Ty) &= \left(\frac{e^{x^2/(1+x)}}{20 + e^{x^2/(1+x)}}\right)^2 + \left(\frac{e^{y^2/(1+y)}}{20 + e^{y^2/(1+y)}}\right)^2 \\
&\quad + \left|\frac{e^{x^2/(1+x)}}{20 + e^{x^2/(1+x)}} - \frac{e^{y^2/(1+y)}}{20 + e^{y^2/(1+y)}}\right|^2; \\
\sigma_b(x, y) &= x^2 + y^2 + |x - y|^2; \\
\sigma_b(x, Ty) &= x^2 + \left(\frac{e^{y^2/(1+y)}}{20 + e^{y^2/(1+y)}}\right)^2 + \left|x - \frac{e^{y^2/(1+y)}}{20 + e^{y^2/(1+y)}}\right|^2; \\
\sigma_b(y, Tx) &= y^2 + \left(\frac{e^{x^2/(1+x)}}{20 + e^{x^2/(1+x)}}\right)^2 + \left|y - \frac{e^{x^2/(1+x)}}{20 + e^{x^2/(1+x)}}\right|^2.
\end{aligned} \tag{40}$$

Utilizing aforementioned values, the left hand side of (6) becomes

$$\begin{aligned}
&\frac{1}{\left(x^2 + y^2 + |x - y|^2\right)^2 + 10} + \log\Bigg[\left(\frac{e^{x^2/(1+x)}}{20 + e^{x^2/(1+x)}}\right)^2 \\
&\quad + \left(\frac{e^{y^2/(1+y)}}{20 + e^{y^2/(1+y)}}\right)^2 \\
&\quad + \left|\frac{e^{x^2/(1+x)}}{20 + e^{x^2/(1+x)}} - \frac{e^{y^2/(1+y)}}{20 + e^{y^2/(1+y)}}\right|^2\Bigg] \\
&\quad + \left(\frac{e^{x^2/(1+x)}}{20 + e^{x^2/(1+x)}}\right)^2 + \left(\frac{e^{y^2/(1+y)}}{20 + e^{y^2/(1+y)}}\right)^2 \\
&\quad + \left|\frac{e^{x^2/(1+x)}}{20 + e^{x^2/(1+x)}} - \frac{e^{y^2/(1+y)}}{20 + e^{y^2/(1+y)}}\right|^2,
\end{aligned} \tag{41}$$

and the right hand side is obtained as

$$\begin{aligned}
&\log\Bigg[\frac{1}{2}\Bigg\{\alpha\left(x^2 + y^2 + |x - y|^2\right) \\
&\quad + \frac{\beta}{4}\left(x^2 + \left(\frac{e^{y^2/(1+y)}}{20 + e^{y^2/(1+y)}}\right)^2 + \left|x - \frac{e^{y^2/(1+y)}}{20 + e^{y^2/(1+y)}}\right|^2\right) \\
&\quad + \frac{\gamma}{4}\left(y^2 + \left(\frac{e^{x^2/(1+x)}}{20 + e^{x^2/(1+x)}}\right)^2 + \left|y - \frac{e^{x^2/(1+x)}}{20 + e^{x^2/(1+x)}}\right|^2\right)\Bigg\}\Bigg] \\
&\quad + \frac{1}{2}\Bigg\{\alpha\left(x^2 + y^2 + |x - y|^2\right) + \frac{\beta}{4}\Bigg(x^2 \\
&\quad + \left(\frac{e^{y^2/(1+y)}}{20 + e^{y^2/(1+y)}}\right)^2 + \left|x - \frac{e^{y^2/(1+y)}}{20 + e^{y^2/(1+y)}}\right|^2\Bigg) + \frac{\gamma}{4}\Bigg(y^2 \\
&\quad + \left(\frac{e^{x^2/(1+x)}}{20 + e^{x^2/(1+x)}}\right)^2 + \left|y - \frac{e^{x^2/(1+x)}}{20 + e^{x^2/(1+x)}}\right|^2\Bigg)\Bigg\}.
\end{aligned} \tag{42}$$

By Figure 3 it is obvious that the surface representing right hand side function is dominating the surface representing left hand side function. So, the condition (6) is verified.

Furthermore, T is continuous and we also have that $\sigma_b(Tx, Tx) \le \sigma_b(x, x)$ for all $x \in X$.

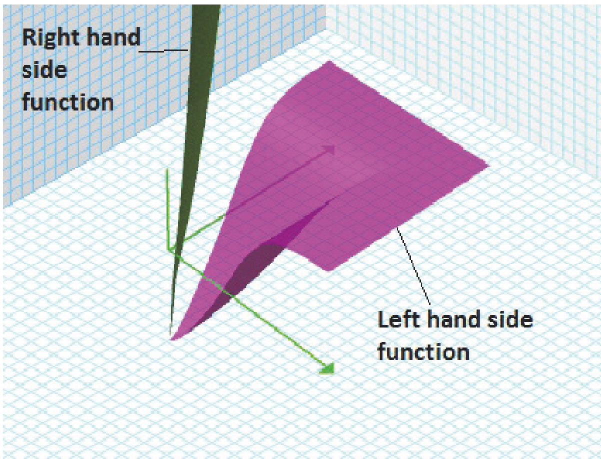

Figure 3: Right hand side function superimposes left hand side function.

Thus, all the conditions of Theorem 16 are satisfied and, consequently, the mapping T has a unique fixed point as $x = 0.0477177$. This is also demonstrated by Figure 4, where the mapping T and the first diagonal are represented.

If we choose $\psi(t) = \tau$ in Theorem 16, then the following corollary is obtained.

Corollary 18. *Let (X, σ_b) be a b-metric-like space and $T : X \to X$ be a continuous self-mapping. If there exist $\tau > 0$ and $F \in \Delta_F$ such that for all $x, y \in X$,*

$$\sigma_b(Tx, Ty) > 0 \Longrightarrow$$
$$\tau + F(\sigma_b(Tx, Ty)) \leq F\left(\phi\left(\alpha\sigma_b(x, y) + \beta\frac{\sigma_b(x, Ty)}{2s} + \gamma\frac{\sigma_b(y, Tx)}{2s}\right)\right), \tag{43}$$

where $\alpha, \beta, \gamma \in [0, 1]$ (not all zero simultaneously) such that $\alpha + \beta + \gamma \leq 1$ and $\phi \in \Phi$. Then T has a unique fixed point.

Replacing $\beta = \gamma = 0$ in Corollary 18, subsequent result is obtained.

Corollary 19. *Let (X, σ_b) be a b-metric-like space and $T : X \to X$ be a continuous self-mapping. If there exist $\tau > 0$ and $F \in \Delta_F$ such that for all $x, y \in X$,*

$$\sigma_b(Tx, Ty) > 0 \Longrightarrow$$
$$\tau + F(\sigma_b(Tx, Ty)) \leq F(\phi(\alpha\sigma_b(x, y))), \tag{44}$$

where $\alpha \in (0, 1]$ and $\phi \in \Phi$. Then T has a unique fixed point.

3. Results via Suzuki-Berinde Type F-Contractions

Berinde initiated some new mappings, called weak contraction mappings in a metric space [20–22]. He demonstrated that Banach's, Kannan's, and Chatterjea's mappings are weak contractions. Afterward, a lot of generalizations of these results in several spaces appeared in the literature. Berinde type weak contractions are usually called almost contractions. Clubbing the ideas of Berinde, Suzuki and the notion of F-contraction, Suzuki-Berinde type F-contractive mapping is defined in the framework of b-metric-like spaces.

Definition 20. Let (X, σ_b) be a b-metric-like space with $s \geq 1$. A self-mapping $T : X \to X$ is said to be Suzuki-Berinde type F-contraction, if there exists $F \in \Delta_F$ such that, for all $x, y \in X$ with $x \neq y$,

$$\frac{1}{2s}\sigma_b(x, Tx) < \sigma_b(x, y) \Longrightarrow$$
$$F\left(s^2\sigma_b(Tx, Ty)\right) \leq F\left(\frac{1}{s}M(x, y)\right) - \psi(M(x, y)) + L(N(x, y)), \tag{45}$$

where

$$M(x, y) = \max\left\{\sigma_b(x, y), \sigma_b(x, Tx), \sigma_b(y, Ty), \frac{\sigma_b(x, Ty) + \sigma_b(y, Tx)}{4s}\right\},$$
$$N(x, y) = \min\{\sigma_b^s(x, Ty), \sigma_b^s(y, Tx), \sigma_b(x, Tx), \sigma_b(y, Ty)\} \tag{46}$$

with $L \geq 0$ and $\psi \in \Psi$.

Theorem 21. *Let (X, σ_b) be a complete b-metric-like space and $T : X \to X$ be a continuous Suzuki-Berinde type F contraction. Then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$ be any arbitrary point. We construct a sequence $\{x_n\}$ in X in such a way that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.

Suppose that $\sigma_b(x_{n_0}, x_{n_0+1}) = \sigma(x_{n_0}, Tx_{n_0}) = 0$, for some $n_0 \geq 0$. Then one can get $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ and then x_{n_0} is a required fixed point. So, we are done in this case. Thus, from now on we assume that $\sigma(x_n, Tx_n) > 0$, for all $n \in \mathbb{N}$. Consequently, we have

$$\frac{1}{2s}\sigma_b(x_n, Tx_n) < \sigma_b(x_n, Tx_n) = \sigma_b(x_n, x_{n+1}), \quad \forall n \in \mathbb{N}. \tag{47}$$

Then by the Definition 20 with $x = x_n$ and $y = x_{n+1}$, we have

$$F\left(s^2\sigma_b(Tx_n, Tx_{n+1})\right) \leq F\left(\frac{1}{s}M(x_n, x_{n+1})\right) - \psi(M(x_n, x_{n+1})) + L(N(x_n, x_{n+1})). \tag{48}$$

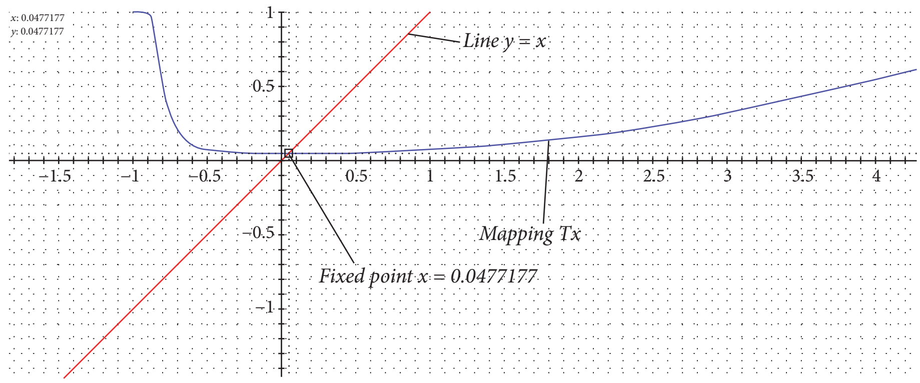

Figure 4: Fixed point of the mapping T.

Notice that

$$\begin{aligned} &\frac{\sigma_b(x_n, x_{n+2}) + \sigma_b(x_{n+1}, x_{n+1})}{4s} \\ &\quad \leq \frac{s[\sigma_b(x_n, x_{n+1}) + 3\sigma_b(x_{n+1}, x_{n+2})]}{4s} \\ &\quad \leq \max\{\sigma_b(x_n, x_{n+1}), \sigma_b(x_{n+1}, x_{n+2})\}. \end{aligned} \tag{49}$$

Thus, we have

$$M(x_n, x_{n+1}) = \max\{\sigma_b(x_n, x_{n+1}), \sigma_b(x_{n+1}, x_{n+2})\}. \tag{50}$$

If $M(x_n, x_{n+1}) = \sigma_b(x_{n+1}, x_{n+2})$ then, from (48), we have

$$\begin{aligned} F(s^2\sigma_b(x_{n+1}, x_{n+2})) \leq{}& F\left(\frac{1}{s}\sigma_b(x_{n+1}, x_{n+2})\right) \\ &- \psi(\sigma_b(x_{n+1}, x_{n+2})). \end{aligned} \tag{51}$$

This leads to a contradiction, in view of $(F1)$ and the hypothesis of ψ. Then we arrive at

$$\begin{aligned} F(s^2\sigma_b(x_{n+1}, x_{n+2})) \leq{}& F\left(\frac{1}{s}\sigma_b(x_n, x_{n+1})\right) \\ &- \psi(\sigma_b(x_n, x_{n+1})) \\ \leq{}& F\left(\frac{1}{s}\sigma_b(x_n, x_{n+1})\right). \end{aligned} \tag{52}$$

Thus, from (52) and $(F1)$, we get that

$$s^2\sigma_b(x_{n+1}, x_{n+2}) < \frac{1}{s}\sigma_b(x_n, x_{n+1}), \tag{53}$$

or equivalently

$$\sigma_b(x_n, x_{n+1}) < \sigma_b(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}, \tag{54}$$

as $s \geq 1$. Therefore $\{\sigma_b(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence of real numbers and is bounded below at 0, consequently convergent to some point $p \in \mathbb{R}^+$; now we claim that $p = 0$. Let us suppose that $p > 0$.

Letting $n \to \infty$ in (52), we have

$$F(s^2 p) \leq F\left(\frac{1}{s}p\right) - \psi(p), \tag{55}$$

which is a contradiction in view of $(F1)$ and the properties of ψ. Thus we have $p = 0$.

Consequently, we have

$$\lim_{n\to\infty} \sigma_b(x_n, Tx_n) = \lim_{n\to\infty} \sigma_b(x_n, x_{n+1}) = 0. \tag{56}$$

Now we will show that $\{x_n\}$ is a Cauchy sequence.

Case 1. When $s > 1$, rewrite (53) as

$$\sigma_b(x_n, x_{n+1}) < \frac{1}{s^3}\sigma_b(x_{n-1}, x_n). \tag{57}$$

Since $1/s^3 > 0$ and $1/s^3 < 1/s$, as $s > 1$, then, by Lemma 9, the sequence $\{x_n\}$ is a Cauchy sequence.

Case 2. When $s = 1$, following the same approach as in Theorem 16 and utilizing the condition (45), it is easy to show that $\{x_n\}$ is a Cauchy sequence in this case. Also the rest of the proof can be obtained with the similar approach as in Theorem 16. □

We now discuss the following consequences of Theorem 21.

If we set $L = 0$ in Theorem 21, then fixed point theorem for Suzuki-type generalized F-contraction in the setting of b-metric-like spaces is obtained.

Corollary 22. *Let (X, σ_b) be a b-metric-like space with $s \geq 1$; let $T : X \to X$ be a continuous mapping. If there exist $F \in \Delta_F$ and $\psi \in \Psi$ such that for all $x, y \in X$ with $x \neq y$,*

$$\begin{aligned} &\frac{1}{2s}\sigma_b(x, Tx) < \sigma_b(x, y) \Longrightarrow \\ &F(s^2\sigma_b(Tx, Ty)) \leq F\left(\frac{1}{s}M(x, y)\right) - \psi(M(x, y)), \end{aligned} \tag{58}$$

where

$$\begin{aligned} M(x, y) = \max\Bigg\{&\sigma_b(x, y), \sigma_b(x, Tx), \sigma_b(y, Ty), \\ &\frac{\sigma_b(x, Ty) + \sigma_b(y, Tx)}{4s}\Bigg\}. \end{aligned} \tag{59}$$

Then T has a unique fixed point.

If we choose $\psi(t) = \tau > 0$, then Berinde-Wardowski type fixed point result in the framework of b-metric-like spaces is acquired.

Corollary 23. *Let (X, σ_b) be a b-metric-like space with $s \geq 1$ and let $T : X \to X$ be a continuous mapping. If there exists $F \in \Delta_F$ such that for all $x, y \in X$ with $x \neq y$,*

$$\begin{aligned} &\sigma_b(Tx, Ty) > 0 \Longrightarrow \\ &\tau + F\left(s^2 \sigma_b(Tx, Ty)\right) \\ &\quad \leq F\left(\frac{1}{s} M(x, y)\right) + L(N(x, y)), \end{aligned} \tag{60}$$

where

$$\begin{aligned} M(x, y) = \max &\left\{\sigma_b(x, y), \sigma_b(x, Tx), \sigma_b(y, Ty),\right. \\ &\left.\frac{\sigma_b(x, Ty) + \sigma_b(y, Tx)}{4s}\right\}, \\ N(x, y) = \min &\{\sigma_b^s(x, Ty), \sigma_b^s(y, Tx), \sigma_b(x, Tx), \\ &\sigma_b(y, Ty)\} \end{aligned} \tag{61}$$

with $L \geq 0$. Then T has a unique fixed point in X.

Next, we present an example which substantiates the hypothesis of Theorem 21.

Example 24. Let $X = [0, 2]$ and let the function $\sigma_b : X \times X \to [0, \infty)$ be defined by $\sigma_b(x, y) = (\max\{x, y\}^2)$, for all $x, y \in X$. It is obvious that (X, σ_b) is a complete b-metric-like space with $s = 2$. Define the mapping $T : X \to X$ by $Tx = (1/20)(\log(1 + x^3) + x^3)$.

In order to show that T is a Suzuki-Berinde type F-contraction mapping, we first verify the condition (45) with $F(t) = \log t$, for all $t \in \mathbb{R}^+$. Clearly $F \in \Delta_F$. Let $\psi : (0, \infty) \to (0, \infty)$ be given by $\psi(t) = t/100$.

Consider

$$\frac{1}{2s}\sigma_b(x, Tx) = \frac{x^2}{4} \leq x^2 \leq \sigma_b(x, y); \quad \forall x, y \in X. \tag{62}$$

Next, we discuss subsequent possible cases for $x, y \in X$.

Case 1. If $x \geq y > Tx \geq Ty$, then values of terms appearing in (45) are evaluated as follows:

$$\begin{aligned} \sigma_b(Tx, Ty) &= \frac{2}{400}\left(\log\left(1 + x^3\right) + x^3\right)^2; \\ \sigma_b(x, y) &= x^2; \\ \sigma_b(x, Tx) &= x^2; \\ \sigma_b(y, Ty) &= y^2; \\ \sigma_b(x, Ty) &= x^2; \\ \sigma_b(y, Tx) &= y^2; \\ \sigma_b^s(x, Ty) &= \left|x^2 - (Ty)^2\right|; \\ \sigma_b^s(y, Tx) &= \left|y^2 - (Tx)^2\right|. \end{aligned} \tag{63}$$

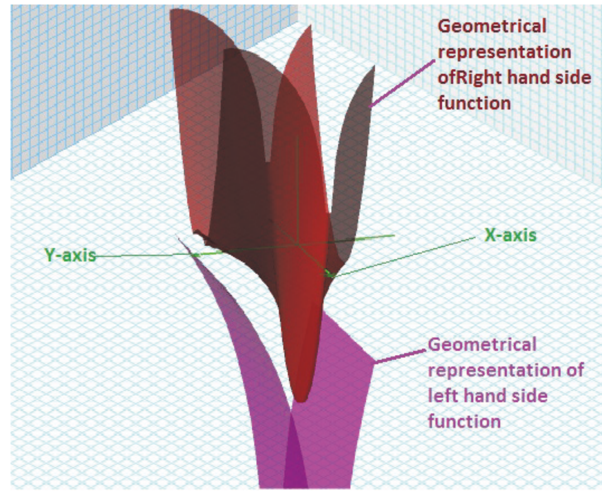

Figure 5: Plot of inequality for verification of Condition (45).

Employing aforementioned values to the left hand side of (45), we get

$$F\left(s^2 \sigma_b(Tx, Ty)\right) = \log\left(\frac{4}{400}\left(\log\left(1 + x^3\right) + x^3\right)^2\right), \tag{64}$$

and right hand side of (45) is obtained as

$$\begin{aligned} &F\left(\frac{1}{s} M(x, y)\right) - \psi(M(x, y)) + L(N(x, y)) \\ &\quad = \log\left(\frac{x^2}{2}\right) - \frac{x^2}{100} \\ &\qquad + L\left|y^2 - \frac{1}{400}\left(\log\left(1 + x^3\right) + x^3\right)^2\right|. \end{aligned} \tag{65}$$

It is very easy to verify that

$$\begin{aligned} F\left(s^2 \sigma_b(Tx, Ty)\right) \leq F\left(\frac{1}{s} M(x, y)\right) - \psi(M(x, y)) \\ + L(N(x, y)), \quad \text{for } L \geq 0, \end{aligned} \tag{66}$$

which is pictorially justified by Figure 5, in which we see that the surface showing right hand side expression is dominating the surface representing left hand side expression for $L = 1$, which validates condition (45).

Case 2. When $x > Tx > y > Ty$, then the same conclusion will be obtained as in Case 1.

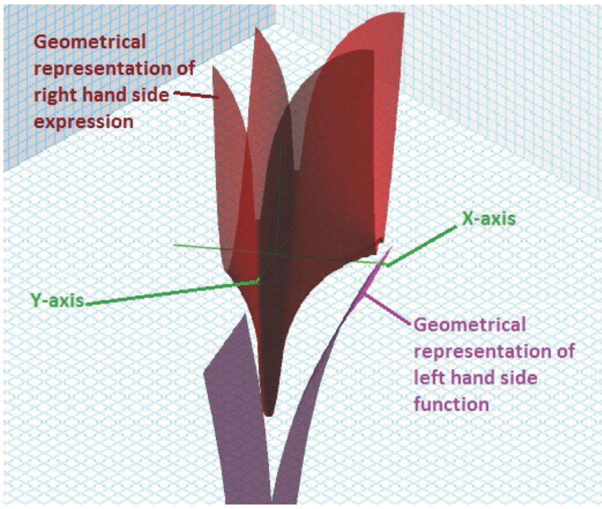

Figure 6: Plot showing domination of right hand side function over left hand side function.

Case 3. If $y > x > Ty > Tx$, then after calculating the terms involved in (45), the left hand side comes out

$$F\left(s^2\sigma_b(Tx,Ty)\right) = \log\left(\frac{4}{400}\left(\log\left(1+y^3\right)+y^3\right)^2\right) \quad (67)$$

and right hand side of (45) is obtained as

$$\begin{aligned} &F\left(\frac{1}{s}M(x,y)\right) - \psi(M(x,y)) + L(N(x,y)) \\ &= \log\left(\frac{y^2}{2}\right) - \frac{y^2}{100} \\ &+ L\left|x^2 - \frac{1}{400}\left(\log\left(1+y^3\right)+y^3\right)^2\right|. \end{aligned} \quad (68)$$

By Figure 6, it is clear that Condition (45) is satisfied for all $x, y \in X$ with $y > x > Ty > Tx$.

Same result will be obtained when $y > Ty > x > Tx$.

Moreover, the mapping T is continuous. Then all the conditions of Theorem 21 are satisfied and hence T has a fixed point $x = 0$, which is indeed unique, as demonstrated by Figure 7.

4. Applications

4.1. An Application to Integral Equations. In this section, we obtain the solution of the subsequent integral equation for an unknown function u:

$$u(t) = p(t) + \int_a^b G(t,z)\, f(z,u(z))\,dz, \quad t \in [a,b], \quad (69)$$

where $u : \mathbb{R}^+ \to \mathbb{R}$ is a nondecreasing function, $f : [a,b]\times\mathbb{R}^+ \to \mathbb{R}$ is a nonincreasing continuous function, and $G : [a,b]\times[a,b] \to [0,\infty)$ is an nondecreasing continuous function. Let $p : [a,b] \to \mathbb{R}$ be given continuous function.

Let X be the set $C[a,b]$ of real continuous functions on $[a,b]$ and let $\sigma_b : X\times X \to [0,\infty)$ be given by

$$\sigma_b(u,v) = \max_{a\le t\le b} |u(t) - v(t)|^2. \quad (70)$$

One can easily see that (X,σ_b) is a complete b-metric-like space (in view of Remark 5, since it a complete b-metric space). Let the mapping $T : X \to X$ be defined by

$$Tu(t) = p(t) + \int_a^b G(t,z)\, f(z,u(z))\,dz, \quad t \in [a,b]; \quad (71)$$

then $u(t)$ is a solution of (69) if and only if it is a fixed point of T. Now, we prove the following theorem to show the existence of solution of integral equation.

Theorem 25. *Assume that the following assumptions hold:*

(1)

$$\max_{a\le t\le b}\int_a^b |G(t,z)|^2\,dz \le \frac{1}{b-a}; \quad (72)$$

(2) *For all $x, y \in \mathbb{R}^+$, the following inequality holds:*

$$|f(z,x) - f(z,y)|^2 \le \frac{1}{2}|x-y|^2 e^{-\tau}. \quad (73)$$

Then the integral equation (69) has a solution.

Proof. By the conditions (1)-(2) and taking into account the integral equation (69), we have

$$\begin{aligned} \sigma_b(Tu_1,Tu_2) &= \max_{a\le t\le b}|Tu_1(t) - Tu_2(t)|^2 \\ &= \max_{a\le t\le b}\left|p(t) + \int_a^b G(t,z)\, f(z,u_1(z))\,dz - \left(p(t) + \int_a^b G(t,z)\, f(z,u_2(z))\,dz\right)\right|^2 \\ &= \max_{a\le t\le b}\left\{\left|\int_a^b (G(t,z)\, f(z,u_1(z)) - G(t,z)\, f(z,u_2(z)))\,dz\right|^2\right\} \\ &\le \max_{a\le t\le b}\left\{\int_a^b |G(t,z)|^2\,dz \cdot \int_a^b |f(z,u_1(z)) - f(z,u_2(z))|^2\,dz\right\} \end{aligned}$$

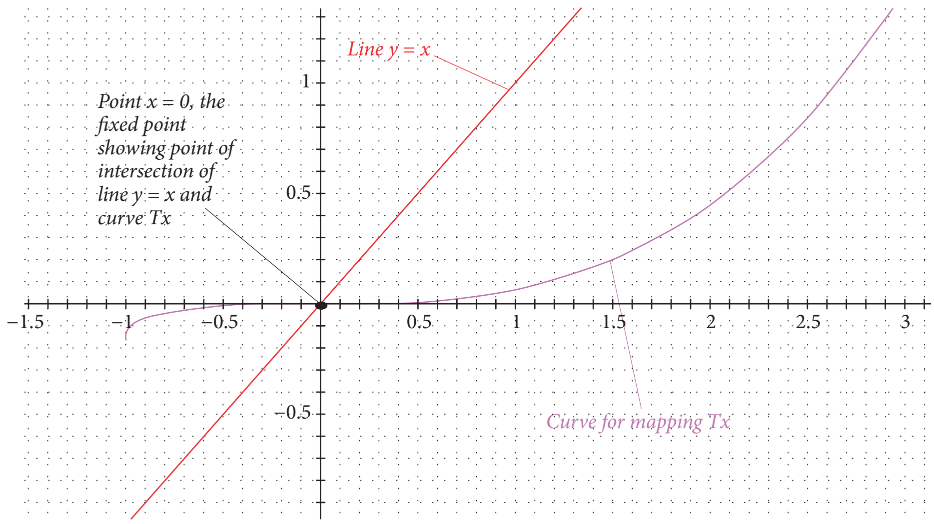

FIGURE 7: Plot showing Fixed point of Tx.

$$
\begin{aligned}
&= \left\{\max_{a\le t\le b}\int_a^b |G(t,z)|^2\,dz\right\}\cdot\left\{\int_a^b \left|f(z,u_1(z)) - f(z,u_2(z))\right|^2 dz\right\}\\
&\le \left\{\frac{1}{b-a}\right\}\cdot\left\{\int_a^b \frac{1}{2}\left[\left(\left|u_1(z)-u_2(z)\right|^2\right)e^{-\tau}\right]dz\right\} \le \frac{1}{2(b-a)}\int_a^b \max_{a\le t\le b}\left[\left|u_1(t)-u_2(t)\right|^2 e^{-\tau}\right]dz\\
&= \max_{a\le t\le b}\frac{1}{2}\left|u_1(t)-u_2(t)\right|^2 e^{-\tau} = \frac{1}{2}\sigma_b(u_1,u_2)\,e^{-\tau};
\end{aligned}
\tag{74}
$$

that is,

$$\sigma_b(Tu_1,Tu_2) \le \frac{\sigma_b(u_1,u_2)\,e^{-\tau}}{2}. \tag{75}$$

Consequently, by passing to logarithms, we get $\tau + \log(\sigma_b(Tu_1,Tu_2)) \le \log(\sigma_b(u_1,u_2)/2)$, and this turns into $\tau + F(\sigma_b(Tu_1,Tu_2)) \le F(\phi(\sigma_b(u_1,u_2)))$, for $F(t) = \log t$, $t > 0$, $\alpha = 1$, and $\phi(t) = 1/2$. Thus, all the conditions of Corollary 19 are satisfied. Hence, we conclude that T has a unique fixed point u^* in X which yields the fact that integral equation (69) has a unique solution which belongs to $X = C[a,b]$. □

The following example demonstrates the validity of hypothesis of Theorem 25.

Example 26. Consider the subsequent integral equation in $X = C([0,1],\mathbb{R})$.

$$u(t) = \frac{t}{t+1} + \frac{1}{2}\int_0^1 \frac{s}{(t+1)}\frac{1}{(1+u(s))}ds; \quad t\in[0,1]. \tag{76}$$

For obtaining the existence of solution of integral equation (76), we will show that u is a fixed point of T, that is, $Tu = u$, where

$$Tu(t) = \frac{t}{t+1} + \frac{1}{2}\int_0^1 \frac{s}{(t+1)}\frac{1}{(1+u(s))}ds; \quad t\in[0,1]. \tag{77}$$

We notice that the integral equation (76) is a particular case of (69), in which $p(t) = t/(t+1)$, $G(t,s) = s/(t+1)$, and $f(s,u(s)) = 1/2(1+u(s))$.

Indeed, the functions p, G, and f are continuous. Moreover, the function G is nondecreasing with respect to s; the active variable under integral and function f is nonincreasing for $u(s)$ which is considered to be a nondecreasing function. Thus the assumptions with respect to functions are satisfied. Further, for all $u, v \in \mathbb{R}$, we get

$$
\begin{aligned}
0 &\le \left|f(s,u) - f(s,v)\right|^2 \le \left|\frac{1}{2(1+u)} - \frac{1}{2(1+v)}\right|^2\\
&\le \frac{1}{4}|v-u|^2 \le \frac{1}{2}\left(|v-u|^2\right)e^{-0.2} \le \frac{1}{2}|v-u|^2 e^{-\tau},
\end{aligned}
\tag{78}
$$

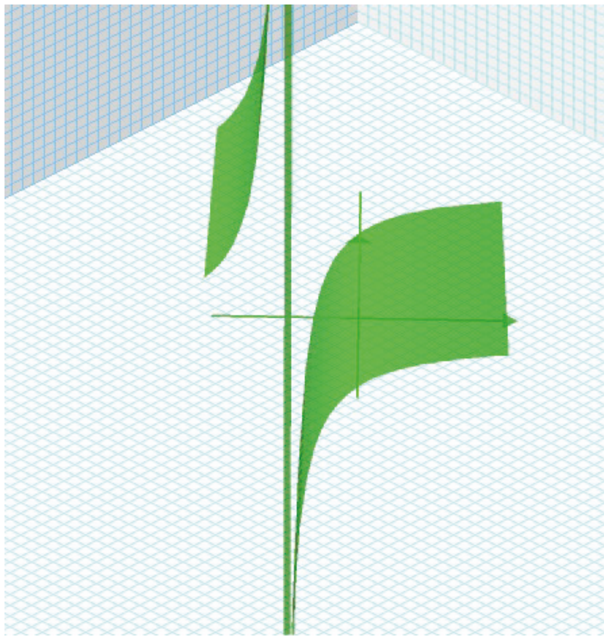

Figure 8: Geometrical representation of approximate solution of (76).

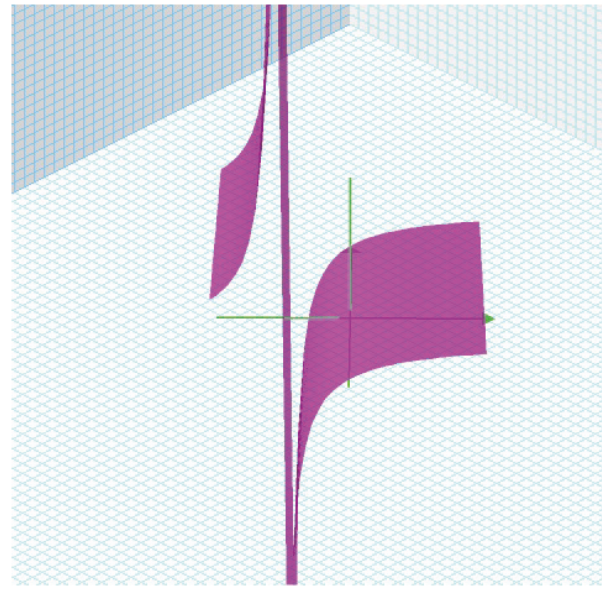

Figure 9: Plot of $Tu(t)$ represented by (81).

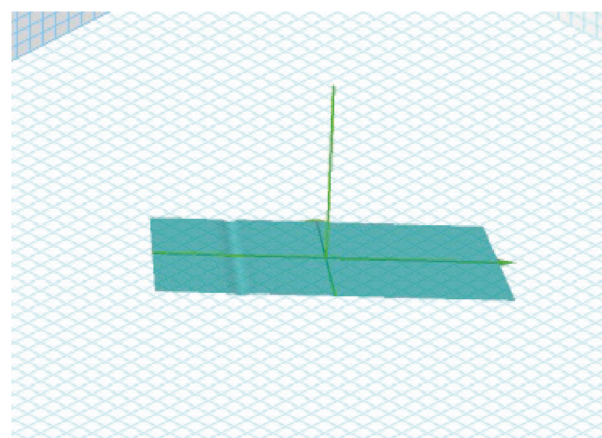

Figure 10: Error between solution and integral equation.

for $\tau = 0.2$. Therefore, condition (2) of Theorem 25 is fulfilled. For condition (1), we have

$$\begin{aligned} \max_{a\le t\le b} \int_0^1 |G(t,s)|^2 \, ds &= \max_{0\le t\le 1} \int_0^1 \frac{s^2}{(t+1)^2} ds \\ &= \max_{0\le t\le 1} \frac{1}{3(t+1)^2} \le 1. \end{aligned} \tag{79}$$

Thus, condition (1) is fulfilled.

Subsequently, we conclude that all the conditions of Theorem 25 are satisfied. Hence, the integral equation (76) has a solution in $X = C([0,1],\mathbb{R})$. Moreover, the approximate solution of the integral equation (76) is

$$u(t) = \frac{2t + 0.3356}{2(t+1)}. \tag{80}$$

The approximate solution of the integral equation (76) is represented geometrically by Figure 8.

Utilizing the obtained approximate solution and (77), one can get

$$Tu(t) = \frac{t}{t+1} + \frac{1}{2(t+1)} \int_0^1 \frac{2s(s+1)}{4s + 2.3356} ds; \quad t \in [0,1]. \tag{81}$$

Subsequent is the plot of $Tu(t)$, mentioned in (81). By Figures 8 and 9 one can easily deduce that the plot of approximate solution with green surface almost coincides with the plot of $Tu(t)$ with purple surface. This shows that approximate solution mentioned in (80) is a fixed point of (76) and hence is a solution of the integral equation (76). Also the error between the approximate solution and the value of $u(t)$ is given by Figure 10.

4.2. Application to Conversion of Solar Energy to Electrical Energy. Solar panels currently are being produced and marketed in mass to counteract the dependency humans have on the less forgiving fossil fuels. In 2007, 18.8 trillion kilowatt hours of electricity were produced globally [23]. In comparison, the sunlight received on the Earth's surface in one hour is enough to power the entire world for a year [24]. The question is, how do those radiant warm rays of light become electricity? With a basic understanding of how light is transformed into electricity, a mathematical model can be presented of the electric current in an RLC parallel circuit [25], also known as a "tuning" circuit. Such problems mathematically modeled as a Cauchy problem attached to differential equation are represented by

$$\frac{d^2 I}{dt^2} + \frac{R}{L}\frac{dI}{dt} = K(t, I(t)); \quad I(0) = 0, \; I'(0) = 0, \tag{82}$$

where $K : [0,1] \times R^+ \to R$ is a continuous function.

The above problem is equivalent to the integral equation

$$u(t)=\int_0^t G(t,s)\,K(s,I(s))\,ds,\quad t\in[0,1],\tag{83}$$

where $G(t,s)$ is Green's function, given by

$$G(t,s)=\begin{cases}(t-s)\,e^{\tau(t-s)}, & 0\le s\le t\le 1;\\ 0, & 0\le t\le s\le 1.\end{cases}\tag{84}$$

Here $\tau>0$ is a constant obtained by the values of R and L, mentioned in (82).

Let $X=C([0,1],R^{+})$ be the set of all nonnegative continuous real functions defined on $[0,1]$. We endow X with the b-metric-like

$$\sigma_b(I_1,I_2)=\sup_{t\in[0,1]}\left(|I_1(t)|+|I_2(t)|\right)^2e^{-2\tau t},\quad \forall I_1,I_2\in X \text{ with } \tau>0.\tag{85}$$

Then clearly (X,σ_b) is a complete b-metric-like space with $s=2$.

Consider the self-map $A:X\times X$, defined by

$$AI(t)=\int_0^t G(t,s)\,K(s,I(s))\,ds,\quad t\in[0,1].\tag{86}$$

It is obvious that I^* is a solution of (83) if and only if I^* is a fixed point of A.

The existence of a fixed point of A is obtained in our next theorem.

Theorem 27. *Consider the Cauchy problem (82). Suppose there exists $\tau>0$ such that $|K(s,I_1)|+|K(s,I_2)|\le e^{-\tau/2}(\sqrt{\alpha}(|I_1|+|I_2|)/\sqrt{2})$, for all $s\in[0,1]$, $I_1,I_2\in R^{+}$, where $0<\alpha\le 1$.*

Then the integral equation (83) has a solution.

Proof. Already note that $(C([0,1],R^{+}),\sigma_b)$ is a complete b-metric-like space, where $\sigma_b(I_1,I_2)$ is given by (85).

Next, for all $I_1,I_2\in X$ such that $FI_1(t)\ne FI_2(t)$, we have

$$\begin{aligned}\left(|AI_1(t)|+|AI_2(t)|\right)^2&\le\left(\left|\int_0^t G(t,s)\,K(s,I_1(s))\,ds\right|+\left|\int_0^t G(t,s)\,K(s,I_2(s))\,ds\right|\right)^2\le\left(\int_0^t|G(t,s)|\cdot\left(|K(s,I_1(s))|+|K(s,I_2(s))|\right)ds\right)^2\\&\le\left(\int_0^t G(t,s)\,e^{-\tau/2}\frac{\sqrt{\alpha}\left(|I_1(s)|+|I_2(s)|\right)}{\sqrt{2}}ds\right)^2=\left(\int_0^t G(t,s)\,e^{-\tau/2}\frac{\sqrt{\alpha}\left(|I_1(s)|+|I_2(s)|\right)}{\sqrt{2}}e^{-\tau s}e^{\tau s}ds\right)^2\\&\le e^{-\tau}\frac{\alpha}{2}\sigma_b(I_1,I_2)\left(\int_0^t e^{\tau s}(t-s)\,e^{\tau(t-s)}ds\right)^2=e^{-\tau}\frac{\alpha}{2}\cdot\sigma_b(I_1,I_2)\,e^{2\tau t}\left(\int_0^t(t-s)\,ds\right)^2=e^{-\tau}\frac{\alpha}{2}\sigma_b(I_1,I_2)\cdot e^{2\tau t}\frac{t^4}{4}\le e^{-\tau}\frac{\alpha}{2}e^{2\tau t}\sigma_b(I_1,I_2).\end{aligned}\tag{87}$$

Since $t^4/4\le 1$ as $t\in[0,1]$, this implies that

$$\left(|AI_1(t)|+|AI_2(t)|\right)^2e^{-2\tau t}\le e^{-\tau}\frac{\alpha}{2}\sigma_b(I_1,I_2).\tag{88}$$

This amounts to say that

$$\sigma_b(AI_1,AI_2)\le e^{-\tau}\frac{\alpha}{2}\sigma_b(I_1,I_2).\tag{89}$$

Consequently, by passing to logarithms, one can obtain

$$\log\left(\sigma_b(AI_1,AI_2)\right)\le\log\left(e^{-\tau}\frac{\alpha}{2}\sigma_b(I_1,I_2)\right).\tag{90}$$

Or

$$\tau+\log\left(\sigma_b(AI_1,AI_2)\right)\le\log\left(\phi\left(\alpha\sigma_b(I_1,I_2)\right)\right).\tag{91}$$

Here, we notice that the function $F:R^{+}\to R$ defined by $F(w)=\log(w)$, for each $w\in C([0,1],R^{+})$ and for $\tau>0$, is in Δ_F. Consequently all the conditions of Corollary 19 are satisfied by operator A with $\phi=t/2$ with $0<\alpha\le 1$. Consequently mapping A has a fixed point which is the solution of integral equation (83) and hence the equation which represented the conversion of solar energy to electric energy has a solution. □

Remark 28. We also notice that Theorem 16 can be utilized to study the existence of the solution of following real time problems:

(i) Solution of equation generated by the motion of pendulum;
(ii) Problems related to simple harmonic motion;
(iii) Solution of equations of vibrations.

Open Problem. For further applications of the presented results, an open problem is suggested as follows.

In electrical and electronics circuits analysis, the following integrodifferential equation appears:

$$I(t)=p(t)+\int_a^t f\left(t,s,I(s),I'(s)\right)ds.\tag{92}$$

It is an open question, whether the existence of solution of aforementioned integrodifferential equation can be established from our results, proved in this article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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