Research Article

Morrey Meets Herz with Variable Exponent and Applications to Commutators of Homogeneous Fractional Integrals with Rough Kernels

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We define the new central Morrey space with variable exponent and investigate its relation to the Morrey-Herz spaces with variable exponent. As applications, we obtain the boundedness of the homogeneous fractional integral operator $T_{\Omega,\sigma}$ and its commutator $[b, T_{\Omega,\sigma}]$ on Morrey-Herz space with variable exponent, where $\Omega \in L^s(S^{n-1})$ for $s \geq 1$ is a homogeneous function of degree zero, $0 < \sigma < n$, and b is a BMO function.

1. Introduction

The Morrey spaces first appeared in 1938 in the work of Morrey [1] in relation to some problems in partial differential equations. In [2] the authors introduced central Morrey spaces. The Herz spaces are a class of function spaces introduced by Herz in the study of absolutely convergent Fourier transforms in 1968; see [3]. The complete theory of Herz spaces for the case of general indexes was established by Lu et al. in 2008; see [4]. Recently, Lu and Xu defined the homogeneous Morrey-Herz spaces in [5].

The theory of function spaces with variable exponent was extensively studied by researchers since the work of Kováčik and Rákosník [6] appeared in 1991; see [7, 8] and the references therein. Many applications of these spaces were given, for example, in the modeling of electrorheological fluids [9], in the study of image processing [10], and in differential equations with nonstandard growth [11]. In 2009, Izuki established the Herz spaces with variable exponent and Morrey-Herz spaces with variable exponent; see [12, 13]. In [14], the authors introduced the nonhomogeneous central Morrey spaces of variable exponent.

Suppose that S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ ($n \geq 2$) equipped with normalized Lebesgue measure. Let $\Omega \in L^s(S^{n-1})$ for $s \geq 1$ be a homogeneous function of degree zero. For $0 < \sigma < n$, the homogeneous fractional integral operator $T_{\Omega,\sigma}$ is defined by

$$T_{\Omega,\sigma} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\sigma}} f(y) dy. \quad (1)$$

Denote $T_\sigma = T_{\Omega,\sigma}$ when $\Omega \equiv 1$. In 1971, Muckenhoupt and Wheeden [15] gave the weighted (L^p, L^q) boundedness of $T_{\Omega,\sigma}$ with power weights. Recently, Tan and Liu [16] gave some boundedness of $T_{\Omega,\sigma}$ on function spaces with variable exponent. For the applications of fractional integral operator in fractional order system, we choose to refer to [17].

Let b be a locally integrable function; the commutator of homogeneous fractional integral operator $[b, T_{\Omega,\sigma}]$ is defined by

$$\begin{aligned} [b, T_{\Omega,\sigma}] f(x) &= \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\sigma}} (b(x) - b(y)) f(y) dy. \end{aligned} \quad (2)$$

Motivated by the above references, in the present paper we will study the boundedness for the homogeneous fractional integral operator $T_{\Omega,\sigma}$ and its commutator $[b, T_{\Omega,\sigma}]$ on the Morrey-Herz space with variable exponent.

Let us explain the outline of this article. In Section 2, we first briefly recall some standard notations and lemmas in variable Lebesgue spaces. In Section 3, we will give the relation between central Morrey spaces, Herz spaces, and Morrey-Herz spaces with variable exponent. We will establish the boundedness for the homogeneous fractional integral operator $T_{\Omega,\sigma}$ and its commutator $[b, T_{\Omega,\sigma}]$ on the Morrey-Herz space with variable exponent in Sections 4 and 5.

2. Variable Lebesgue Spaces

Given an open set $E \subset \mathbb{R}^n$ and a measurable function $p(\cdot) : E \rightarrow [1, \infty)$, $L^{p(\cdot)}(E)$ denotes the set of measurable functions f on E such that for some $\lambda > 0$

$$\int_E \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty. \quad (3)$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(E)} = \inf \left\{ \lambda > 0 : \int_E \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}. \quad (4)$$

These spaces are referred to as variable L^p spaces, since they generalized the standard L^p spaces: if $p(x) = p$ is constant, then $L^{p(\cdot)}(E)$ is isometrically isomorphic to $L^p(E)$.

The space $L_{\text{loc}}^{p(\cdot)}(E)$ is defined by

$$L_{\text{loc}}^{p(\cdot)}(E) := \{f : f \in L^{p(\cdot)}(F) \ \forall \text{ compact subsets } F \subset E\}. \quad (5)$$

Define $\mathcal{P}^0(E)$ to be set of $p(\cdot) : E \rightarrow (0, \infty)$ such that

$$\begin{aligned} p^- &= \text{ess inf } \{p(x) : x \in E\} > 0, \\ p^+ &= \text{ess sup } \{p(x) : x \in E\} < \infty. \end{aligned} \quad (6)$$

Define $\mathcal{P}(E)$ to be set of $p(\cdot) : E \rightarrow [1, \infty)$ such that

$$\begin{aligned} p^- &= \text{ess inf } \{p(x) : x \in E\} > 1, \\ p^+ &= \text{ess sup } \{p(x) : x \in E\} < \infty. \end{aligned} \quad (7)$$

Denote $p'(x) = p(x)/(p(x) - 1)$. In addition, we denote the Lebesgue measure and the characteristic function of a measurable set $A \subset \mathbb{R}^n$ by $|A|$ and χ_A , respectively. The notation $f \approx g$ means that there exist constants $C_1, C_2 > 0$ such that $C_1 g \leq f \leq C_2 g$.

In variable L^p spaces there are some important lemmas as follows.

Lemma 1 (see [18]). *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and satisfies*

$$\begin{aligned} |p(x) - p(y)| &\leq \frac{C}{-\log(|x - y|)}, \quad |x - y| \leq \frac{1}{2}, \\ |p(x) - p(y)| &\leq \frac{C}{\log(|x| + e)}, \quad |y| \geq |x|, \end{aligned} \quad (8)$$

then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$; that is, the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Lemma 2 (see [6]). *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}, \quad (9)$$

where

$$r_p = 1 + \frac{1}{p^-} - \frac{1}{p^+}. \quad (10)$$

This inequality is named the generalized Hölder inequality with respect to the variable L^p spaces.

Lemma 3 (see [19]). *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$*

$$\begin{aligned} \frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} &\leq C \frac{|B|}{|S|}, \\ \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} &\leq C \left(\frac{|S|}{|B|} \right)^{\delta_1}, \\ \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} &\leq C \left(\frac{|S|}{|B|} \right)^{\delta_2}, \end{aligned} \quad (11)$$

where $0 < \delta_1, \delta_2 < 1$ depend on $p(\cdot)$.

Throughout this paper δ_2 will change with $p(\cdot)$.

Lemma 4 (see [19]). *Suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$*

$$\frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C. \quad (12)$$

3. The Relation between Central Morrey Spaces, Herz Spaces, and Morrey-Herz Spaces with Variable Exponent

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $A_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Denote $\mathbb{Z}_+$ and $\mathbb{N}$ as the sets of all positive and nonnegative integers, $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$ and $\tilde{\chi}_0 = \chi_{B_0}$.

We can give the following definitions of function spaces with variable exponent.

Definition 5. Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. The homogeneous central Morrey space with variable exponent $\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)$ is defined by

$$\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n) = \{f \in L_{\text{loc}}^{q(\cdot)}(\mathbb{R}^n) : \|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)} < \infty\}, \quad (13)$$

where

$$\|f\|_{\dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \sum_{k=-\infty}^L \|f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \quad (14)$$

The nonhomogeneous central Morrey space with variable exponent $\mathcal{B}^{q(\cdot),\lambda}(\mathbb{R}^n)$ is defined by

$$\mathcal{B}^{q(\cdot),\lambda}(\mathbb{R}^n) = \left\{ f \in L_{\text{loc}}^{q(\cdot)}(\mathbb{R}^n) : \|f\|_{\mathcal{B}^{q(\cdot),\lambda}(\mathbb{R}^n)} < \infty \right\}, \quad (15)$$

where

$$\|f\|_{\mathcal{B}^{q(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \sum_{k=0}^L \|f\tilde{\chi}_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \quad (16)$$

Definition 6 (see [19]). Let $\alpha \in \mathbb{R}$, $0 < p \leq \infty$, and $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\begin{aligned} \dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) \\ = \left\{ f \in L_{\text{loc}}^{q(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} < \infty \right\}, \end{aligned} \quad (17)$$

where

$$\|f\|_{\dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p}. \quad (18)$$

The nonhomogeneous Herz space with variable exponent $K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L_{\text{loc}}^{q(\cdot)}(\mathbb{R}^n) : \|f\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} < \infty \right\}, \quad (19)$$

where

$$\|f\|_{K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)} = \left\{ \sum_{k=0}^{\infty} 2^{k\alpha p} \|f\tilde{\chi}_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p}. \quad (20)$$

Definition 7. Let $\alpha \in \mathbb{R}$, $0 < p \leq \infty$, $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $0 \leq \lambda < \infty$. The homogeneous Morrey-Herz space with variable exponent $\dot{M}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$ (see [13]) is defined by

$$\begin{aligned} \dot{M}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n) \\ = \left\{ f \in L_{\text{loc}}^{q(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{M}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)} < \infty \right\}, \end{aligned} \quad (21)$$

where

$$\begin{aligned} \|f\|_{\dot{M}_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)} \\ = \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=-\infty}^L 2^{k\alpha p} \|f\chi_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p}. \end{aligned} \quad (22)$$

The nonhomogeneous Morrey-Herz space with variable exponent $M_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)$ is defined by

$$\begin{aligned} M_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n) \\ = \left\{ f \in L_{\text{loc}}^{q(\cdot)}(\mathbb{R}^n) : \|f\|_{M_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)} < \infty \right\}, \end{aligned} \quad (23)$$

where

$$\|f\|_{M_{p,q(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)} = \sup_{L \in \mathbb{Z}} 2^{-L\lambda} \left\{ \sum_{k=0}^L 2^{k\alpha p} \|f\tilde{\chi}_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^p \right\}^{1/p}. \quad (24)$$

Remark 8. If $\alpha = 0$, $p = 1$, then $\dot{M}_{1,q(\cdot)}^{0,\lambda}(\mathbb{R}^n) = \dot{\mathcal{B}}^{q(\cdot),\lambda}(\mathbb{R}^n)$ and $M_{1,q(\cdot)}^{0,\lambda}(\mathbb{R}^n) = \mathcal{B}^{q(\cdot),\lambda}(\mathbb{R}^n)$.

Remark 9. If $\lambda = 0$, then $\dot{M}_{p,q(\cdot)}^{\alpha,0}(\mathbb{R}^n) = \dot{K}_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$ and $M_{p,q(\cdot)}^{\alpha,0}(\mathbb{R}^n) = K_{q(\cdot)}^{\alpha,p}(\mathbb{R}^n)$.

Remark 10. If $q(\cdot) = q$ is a constant, then $\dot{M}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n) = \dot{M}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$ and $M_{p,q}^{\alpha,\lambda}(\mathbb{R}^n) = M_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$.

4. Estimate for the Homogeneous Fractional Integral Operator

In this section we will prove the boundedness of the homogeneous fractional integral operators $T_{\Omega,\sigma}$ on Morrey-Herz spaces with variable exponent.

Theorem 11. Suppose that $0 < \nu \leq 1$, $0 < \sigma < n - \nu$, $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies conditions (8) in Lemma 1 with $q_1^+ < n/\sigma$, $1/q_1(x) - 1/q_2(x) = \sigma/n$, $\Omega \in L^s(S^{n-1})$ ($s > q_1^+$). Let $0 < p_1 \leq p_2 < \infty$ and $0 < \lambda < \alpha < n\delta_2 - \nu - n/s$ (or $0 < \lambda < \alpha_2 \leq \alpha_1 < n\delta_2 - \nu - n/s$). Then $T_{\Omega,\sigma}$ is bounded from $\dot{M}_{q_1(\cdot)}^{\alpha,p_1}(\mathbb{R}^n)$ (or $\dot{M}_{q_1(\cdot)}^{\alpha_1,p_1}(\mathbb{R}^n)$) to $\dot{M}_{q_2(\cdot)}^{\alpha,p_2}(\mathbb{R}^n)$ (or $\dot{M}_{q_2(\cdot)}^{\alpha_2,p_2}(\mathbb{R}^n)$).

To prove Theorem 11, we need the following lemmas.

Lemma 12 (see [15]). If $a > 0$, $1 \leq s \leq \infty$, $0 < d \leq s$, and $-n + (n-1)d/s < \nu < \infty$, then

$$\begin{aligned} \left(\int_{|y| \leq a|x|} |y|^\nu |\Omega(x-y)|^d dy \right)^{1/d} \\ \leq C |x|^{(\nu+n)/d} \|\Omega\|_{L^s(S^{n-1})}. \end{aligned} \quad (25)$$

Lemma 13 (see [20]). Define a variable exponent $\tilde{q}(\cdot)$ by $1/p(x) = 1/\tilde{q}(x) + 1/q$ ($x \in \mathbb{R}^n$). Then we have

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{L^{\tilde{q}(\cdot)}(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}, \quad (26)$$

for all measurable functions f and g .

Lemma 14 (see [8]). Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy conditions (8) in Lemma 1. Then

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{1/p(x)} & \text{if } |Q| \leq 2^n, \ x \in Q, \\ |Q|^{1/p(\infty)} & \text{if } |Q| \geq 1, \end{cases} \quad (27)$$

for every cube (or ball) $Q \subset \mathbb{R}^n$, where $p(\infty) = \lim_{x \rightarrow \infty} p(x)$.

Proof of Theorem 11. We only prove the homogeneous case. Similar to the method of [21], it is easy to prove that

$M\dot{K}_{q_2(\cdot)}^{\alpha_1, p_2}(\mathbb{R}^n) \subset M\dot{K}_{q_2(\cdot)}^{\alpha_2, p_2}(\mathbb{R}^n)$ for $0 < \alpha_2 \leq \alpha_1$. So the nonhomogeneous case can be proved in the same way. Let $f \in M\dot{K}_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)$. Denote $f_j = \chi_{B_j} f$ for each $j \in \mathbb{Z}$; then we have $f(x) = \sum_{j=-\infty}^{\infty} f_j(x)$. Note that $p_1 \leq p_2$; we have

$$\begin{aligned} \|T_{\Omega, \sigma}(f)\|_{M\dot{K}_{q_2(\cdot)}^{\alpha, p_2}(\mathbb{R}^n)}^{p_1} &= \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\ &\cdot \left\{ \sum_{k=-\infty}^L 2^{k\alpha p_2} \|T_{\Omega, \sigma}(f) \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_2} \right\}^{p_1/p_2} \\ &\leq \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L 2^{k\alpha p_1} \|T_{\Omega, \sigma}(f) \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_1} \leq C \\ &\cdot \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\ &\cdot \sum_{k=-\infty}^L 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} \|T_{\Omega, \sigma}(f_j) \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} + C \\ &\cdot \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\ &\cdot \sum_{k=-\infty}^L 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} \|T_{\Omega, \sigma}(f_j) \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} =: I_1 \\ &+ I_2. \end{aligned} \quad (28)$$

We first estimate I_1 . For each $k \in \mathbb{Z}$, $j \leq k-2$, and *a.e.* $x \in A_k$, using the generalized Hölder inequality, we have

$$\begin{aligned} |T_{\Omega, \sigma}(f_j)(x)| &\leq C \int_{B_j} \frac{|\Omega(x-y)|}{|x-y|^{n-\sigma}} |f_j(y)| dy \\ &\leq C 2^{k(\sigma-n)} \int_{B_j} |\Omega(x-y)| |f_j(y)| dy \\ &\leq C 2^{k(\sigma-n)} \|\Omega(x-\cdot) \chi_j(\cdot)\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad (29)$$

Noting $s > q_1'^-$, we denote $\widetilde{q_1'}(\cdot) > 1$ and $1/q_1'(x) = 1/\widetilde{q_1'}(x) + 1/s$. By Lemmas 12 and 13 we have

$$\begin{aligned} \|\Omega(x-\cdot) \chi_j(\cdot)\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} &\leq \|\Omega(x-\cdot) \chi_j(\cdot)\|_{L^s(\mathbb{R}^n)} \|\chi_j(\cdot)\|_{L^{\widetilde{q_1'}(\cdot)}(\mathbb{R}^n)} \\ &\leq \|\Omega(x-\cdot) \chi_j(\cdot)\|_{L^s(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{\widetilde{q_1'}(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{-j\nu} \left(\int_{A_j} |\Omega(x-y)|^s |y|^{s\nu} dy \right)^{1/s} \|\chi_{B_j}\|_{L^{\widetilde{q_1'}(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{-j\nu} 2^{k(\nu+n/s)} \|\Omega\|_{L^s(S^{n-1})} \|\chi_{B_j}\|_{L^{\widetilde{q_1'}(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad (30)$$

When $|B_j| \leq 2^n$ and $x_j \in B_j$, by Lemma 14, we have

$$\|\chi_{B_j}\|_{L^{\widetilde{q_1'}(\cdot)}(\mathbb{R}^n)} \approx |B_j|^{1/\widetilde{q_1'}(x_j)} \approx \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} |B_j|^{-1/s}. \quad (31)$$

When $|B_j| \geq 1$ we have

$$\|\chi_{B_j}\|_{L^{\widetilde{q_1'}(\cdot)}(\mathbb{R}^n)} \approx |B_j|^{1/\widetilde{q_1'}(\infty)} \approx \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} |B_j|^{-1/s}. \quad (32)$$

Since

$$T_{\sigma}(\chi_{B_k})(x) \geq \int_{B_k} \frac{dy}{|x-y|^{n-\sigma}} \chi_{B_k}(x) \geq C 2^{k\sigma} \chi_{B_k}(x), \quad (33)$$

by Lemmas 3 and 4, we have

$$\begin{aligned} \|T_{\Omega, \sigma}(f_j) \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} &\leq C 2^{k(\sigma-n)} 2^{-j\nu} 2^{k(\nu+n/s)} \|\Omega\|_{L^s(S^{n-1})} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\ &\cdot \|\chi_{B_j}\|_{L^{\widetilde{q_1'}(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{k(\sigma-n)} 2^{-j\nu} 2^{k(\nu+n/s)} \|\Omega\|_{L^s(S^{n-1})} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\ &\cdot \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} |B_j|^{-1/s} \|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{-kn} 2^{-j\nu} 2^{k(\nu+n/s)} \|\Omega\|_{L^s(S^{n-1})} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\ &\cdot \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} |B_j|^{-1/s} \|T_{\sigma}(\chi_{B_k})\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{-kn+(k-j)(\nu+n/s)} \|\Omega\|_{L^s(S^{n-1})} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\ &\cdot \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{(k-j)(\nu+n/s)} \|\Omega\|_{L^s(S^{n-1})} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\ &\cdot \frac{\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}} \leq C 2^{(j-k)(n\delta_2-\nu-n/s)} \|\Omega\|_{L^s(S^{n-1})} \\ &\cdot \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad (34)$$

So we have

$$\begin{aligned} I_1 &= C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\ &\cdot \sum_{k=-\infty}^L 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} \|T_{\Omega, \sigma}(f_j) \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\ &\leq C \|\Omega\|_{L^s(S^{n-1})}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\ &\cdot \sum_{k=-\infty}^L 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_2-\nu-n/s)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1}. \end{aligned} \quad (35)$$

When $1 < p_1 < \infty$, take $1/p_1 + 1/p_1' = 1$. Since $n\delta_2 - \nu - n/s - \alpha > 0$, by the Hölder inequality, we have

$$\begin{aligned}
I_1 &\leq C \|\Omega\|_{L^s(S^{n-1})}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
&\cdot \sum_{k=-\infty}^L \left(\sum_{j=-\infty}^{k-2} 2^{j\alpha p_1} 2^{(j-k)(n\delta_2 - \nu - n/s - \alpha)p_1/2} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \right) \\
&\times \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_2 - \nu - n/s - \alpha)p_1'/2} \right)^{p_1/p_1'} \leq C \|\Omega\|_{L^s(S^{n-1})}^{p_1} \\
&\cdot \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
&\cdot \sum_{k=-\infty}^L \sum_{j=-\infty}^{k-2} 2^{j\alpha p_1} 2^{(j-k)(n\delta_2 - \nu - n/s - \alpha)p_1/2} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \quad (36) \\
&\leq C \|\Omega\|_{L^s(S^{n-1})}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-2} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
&\cdot \sum_{k=j+2}^L 2^{(j-k)(n\delta_2 - \nu - n/s - \alpha)p_1/2} \leq C \|\Omega\|_{L^s(S^{n-1})}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
&\cdot \sum_{j=-\infty}^{L-2} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \leq C \|\Omega\|_{L^s(S^{n-1})}^{p_1} \|f\|_{MK_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1}.
\end{aligned}$$

When $0 < p_1 \leq 1$, we have

$$\begin{aligned}
I_1 &\leq C \|\Omega\|_{L^s(S^{n-1})}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L 2^{k\alpha p_1} \\
&\cdot \sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_2 - \nu - n/s)p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} = C \|\Omega\|_{L^s(S^{n-1})}^{p_1} \\
&\cdot \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-2} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \quad (37) \\
&\cdot \sum_{k=j+2}^L 2^{(j-k)(n\delta_2 - \nu - n/s - \alpha)p_1} \leq C \|\Omega\|_{L^s(S^{n-1})}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
&\cdot \sum_{j=-\infty}^{L-2} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \leq C \|\Omega\|_{L^s(S^{n-1})}^{p_1} \\
&\cdot \|f\|_{MK_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1}.
\end{aligned}$$

Next we estimate I_2 ; by the $(L^{q_1(\cdot)}(\mathbb{R}^n), L^{q_2(\cdot)}(\mathbb{R}^n))$ -boundedness of the commutator $T_{\Omega, \sigma}$ we have

$$\begin{aligned}
I_2 &= C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
&\cdot \sum_{k=-\infty}^L 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} \|T_{\Omega, \sigma}(f_j) \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \leq C
\end{aligned}$$

$$\begin{aligned}
&\cdot \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1} = C \\
&\cdot \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha} 2^{j\alpha} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1}. \quad (38)
\end{aligned}$$

If $0 < p_1 \leq 1$, then we have

$$\begin{aligned}
I_2 &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
&\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=k-1}^{L-1} 2^{(k-j)\alpha p_1} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \quad (39) \\
&+ C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
&=: I_{21} + I_{22}.
\end{aligned}$$

For I_{21} , we have

$$\begin{aligned}
I_{21} &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-1} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p_1} \\
&\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-1} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \quad (40) \\
&\leq C \|f\|_{MK_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1}.
\end{aligned}$$

For I_{22} , by $0 < \lambda < \alpha$ we have

$$\begin{aligned}
I_{22} &= C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \leq C \\
&\cdot \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
&\cdot \sum_{k=-\infty}^L \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\lambda p_1} 2^{-j\lambda p_1} \left(\sum_{m=-\infty}^j 2^{m\alpha p_1} \|f_m\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \right) \\
&\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\lambda p_1} \|f\|_{MK_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1} \quad (41) \\
&= C \|f\|_{MK_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L 2^{k\alpha p_1} \sum_{j=L}^{\infty} 2^{j(\lambda - \alpha)p_1} \\
&\leq C \|f\|_{MK_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} 2^{L\alpha p_1} 2^{L(\lambda - \alpha)p_1} \\
&= C \|f\|_{MK_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1}.
\end{aligned}$$

If $1 < p_1 < \infty$, noting $\lambda < \alpha$, we can take a constant $\eta > 1$ so that $\lambda - \alpha/\eta < 0$. By the Hölder inequality we have

$$\begin{aligned}
 I_2 &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
 &\cdot \sum_{k=-\infty}^L \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p_1/\eta} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \right) \\
 &\cdot \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p'_1(\eta-1)/\eta} \right)^{p_1/p'_1} \leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
 &\cdot \sum_{k=-\infty}^L \sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p_1/\eta} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \leq C \\
 &\cdot \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=L}^{L-1} 2^{(k-j)\alpha p_1/\eta} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
 &+ C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1/\eta} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
 &=: I_{23} + I_{24}.
 \end{aligned} \tag{42}$$

For I_{23} , we have

$$\begin{aligned}
 I_{23} &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-1} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p_1/\eta} \\
 &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-1} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
 &\leq C \|f\|_{MK_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1}.
 \end{aligned} \tag{43}$$

For I_{24} , by $0 < \lambda < \alpha/\eta$ we have

$$\begin{aligned}
 I_{24} &= C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1/\eta} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
 &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1/\eta} 2^{j\lambda p_1} \|f\|_{MK_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1} \\
 &= C \|f\|_{MK_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L 2^{k\alpha p_1/\eta} \sum_{j=L}^{\infty} 2^{j(\lambda - \alpha/\eta)p_1} \\
 &\leq C \|f\|_{MK_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} 2^{L\alpha p_1/\eta} 2^{L(\lambda - \alpha/\eta)p_1} \\
 &= C \|f\|_{MK_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1}.
 \end{aligned} \tag{44}$$

Thus, by (28), (37), and (39)–(44) we complete the proof of Theorem 11. $\square$

Remark 15. In [22], Abdalmonem et al. considered the boundedness of fractional integral with variable kernel on

variable exponent Herz-Morrey spaces (see Theorem A, pp. 790). It is different from Theorem 11. Theorem 11 is under the weaker conditions than Theorem A. For example, the conditions of Theorem A are $\Omega \in L^r(S^{n-1})$ ($r > p_2^+$) and satisfy $\int_0^1 \omega_r(\delta)/\delta^{1+\beta} d\delta < \infty$. But we do not need the condition $\int_0^1 \omega_r(\delta)/\delta^{1+\beta} d\delta < \infty$ in Theorem 11.

5. BMO Estimate for the Commutator of Homogeneous Fractional Integral Operator

Let us first recall that the space $BMO(\mathbb{R}^n)$ consists of all locally integrable functions f such that

$$\|f\|_* = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx < \infty, \tag{45}$$

where $f_Q = |Q|^{-1} \int_Q f(y) dy$, the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes, and $|Q|$ denotes the Lebesgue measure of Q .

A nonnegative locally integrable function $\omega(x)$ on $\mathbb{R}^n$ is said to belong to $A(p, q)$ ($1 < p, q < \infty$), if there is a constant $C > 0$ such that

$$\begin{aligned}
 &\sup_Q \left(\frac{1}{|Q|} \int_Q \omega(x)^q dx \right)^{1/q} \left(\frac{1}{|Q|} \int_Q \omega(x)^{-p'} dx \right)^{1/p'} \\
 &\leq C < \infty,
 \end{aligned} \tag{46}$$

where $p' = p/(p-1)$.

Let $b \in BMO(\mathbb{R}^n)$. The weighted (L^p, L^q) boundedness of $[b, T_{\Omega, \sigma}]$ has been proved by Segovia and Torrea [23], Ding [24], and Ding and Lu [25], respectively.

Lemma 16 (see [24]). *Suppose that $0 < \sigma < n$, $s' < p < n/\sigma$, $1/q = 1/p - \sigma/n$, $\Omega \in L^s(S^{n-1})$, and $\omega^{s'} \in A(p/s', q/s')$, then for $b \in BMO(\mathbb{R}^n)$, $m \in \mathbb{Z}$, there is a constant C , independent of f , such that*

$$\begin{aligned}
 &\left(\int_{\mathbb{R}^n} |[b, T_{\Omega, \sigma}]^m f(x) \omega(x)|^q dx \right)^{1/q} \\
 &\leq C \left(\int_{\mathbb{R}^n} |f(x) \omega(x)|^p dx \right)^{1/p}.
 \end{aligned} \tag{47}$$

We say that $\omega \in A_1$ if $M\omega(x) \leq C\omega(x)$ for a.e. x .

Lemma 17 (see [26]). *Given a family $\mathcal{F}$ and an open set $E \subset \mathbb{R}^n$, assume that, for some p_0 and q_0 , $0 < p_0 \leq q_0 < \infty$, and every weight $\omega \in A_1$*

$$\begin{aligned}
 &\left(\int_E f(x)^{q_0} \omega(x) dx \right)^{1/q_0} \\
 &\leq C_0 \left(\int_E g(x)^{p_0} \omega(x)^{p_0/q_0} dx \right)^{1/p_0}, \quad (f, g) \in \mathcal{F}.
 \end{aligned} \tag{48}$$

Given $p(\cdot) \in \mathcal{P}^0(E)$ such that $p_0 < p_- \leq p_+ < p_0 q_0/(q_0 - p_0)$, define the function $q(\cdot)$ by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}, \quad x \in E. \tag{49}$$

If $p(\cdot)$ satisfies (8) in Lemma 1, then for all $(f, g) \in \mathcal{F}$ such that $f \in L^{q(\cdot)}(E)$

$$\|f\|_{L^{q(\cdot)}(E)} \leq C \|g\|_{L^{p(\cdot)}(E)}. \quad (50)$$

Similar to the method of [16, Theorem 1.9], by Lemmas 16 and 17, it is easy to get the $(L^{p(\cdot)}(\mathbb{R}^n), L^{q(\cdot)}(\mathbb{R}^n))$ -boundedness of the commutator $[b, T_{\Omega, \sigma}]$.

Next, we will give the corresponding result about the commutator $[b, T_{\Omega, \sigma}]$ on Morrey-Herz spaces with variable exponent.

Theorem 18. Suppose that $b \in BMO(\mathbb{R}^n)$, $0 < \nu \leq 1$, $0 < \sigma < n - \nu$, $q_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfy conditions (8) in Lemma 1 with $q_1^+ < n/\sigma$, $1/q_1(x) - 1/q_2(x) = \sigma/n$, and $\Omega \in L^s(S^{n-1})$ ($s > q_1^+$). Let $0 < p_1 \leq p_2 < \infty$ and $0 < \lambda < \alpha < n\delta_2 - \nu - n/s$ (or $0 < \lambda < \alpha_2 \leq \alpha_1 < n\delta_2 - \nu - n/s$). Then $[b, T_{\Omega, \sigma}]$ is bounded from $M\dot{K}_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)$ (or $M\dot{K}_{q_1(\cdot)}^{\alpha_1, p_1}(\mathbb{R}^n)$) to $M\dot{K}_{q_2(\cdot)}^{\alpha, p_2}(\mathbb{R}^n)$ (or $M\dot{K}_{q_2(\cdot)}^{\alpha_2, p_2}(\mathbb{R}^n)$).

In the proof of Theorem 18, we also need the following lemmas.

Lemma 19 (see [27]). Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, k be a positive integer, and B be a ball in $\mathbb{R}^n$. Then we have that for all $b \in BMO(\mathbb{R}^n)$ and all $j, i \in \mathbb{Z}$ with $j > i$

$$\begin{aligned} \frac{1}{C} \|b\|_*^k &\leq \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b - b_B)^k \chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leq C \|b\|_*^k, \end{aligned} \quad (51)$$

$$\|(b - b_{B_i})^k \chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C (j - i)^k \|b\|_*^k \|\chi_{B_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $B_i = \{x \in \mathbb{R}^n : |x| \leq 2^i\}$ and $B_j = \{x \in \mathbb{R}^n : |x| \leq 2^j\}$.

Proof of Theorem 18. We only prove the homogeneous case. The nonhomogeneous case can be proved in the same way. Let $f \in M\dot{K}_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)$ and $b \in BMO(\mathbb{R}^n)$. Denote $f_j = f \chi_j$ for each $j \in \mathbb{Z}$; then we have $f(x) = \sum_{j=-\infty}^{\infty} f_j(x)$. Note that $p_1 \leq p_2$; we have

$$\begin{aligned} &\|[b, T_{\Omega, \sigma}](f)\|_{M\dot{K}_{q_2(\cdot)}^{\alpha, p_2}(\mathbb{R}^n)}^{p_1} \\ &= \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \left\{ \sum_{k=-\infty}^L 2^{k\alpha p_2} \|[b, T_{\Omega, \sigma}](f) \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_2} \right\}^{p_1/p_2} \\ &\leq \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L 2^{k\alpha p_1} \|[b, T_{\Omega, \sigma}](f) \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}^{p_1} \\ &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} \|[b, T_{\Omega, \sigma}](f_j) \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\ &\quad + C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} \|[b, T_{\Omega, \sigma}](f_j) \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\ &=: J_1 + J_2. \end{aligned} \quad (52)$$

We first estimate J_1 . For each $k \in \mathbb{Z}$, $j \leq k - 2$, and a.e. $x \in A_k$, using the generalized Hölder inequality we have

$$\begin{aligned} &|[b, T_{\Omega, \sigma}](f_j)(x)| \leq C \int_{B_j} \frac{|\Omega(x - y)|}{|x - y|^{n-\sigma}} |b(x) - b(y)| \\ &\quad \cdot |f_j(y)| dy \leq C 2^{k(\sigma-n)} \int_{B_j} |\Omega(x - y)| \\ &\quad \cdot |b(x) - b(y)| |f_j(y)| dy \\ &\leq C 2^{k(\sigma-n)} \left(|b(x) - b_{B_j}| \int_{B_j} |\Omega(x - y)| |f_j(y)| dy \right. \\ &\quad \left. + \int_{B_j} |\Omega(x - y)| |b_{B_j} - b(y)| |f_j(y)| dy \right) \\ &\leq C 2^{k(\sigma-n)} \left(|b(x) - b_{B_j}| \|\Omega(x - \cdot) \chi_j(\cdot)\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \right. \\ &\quad \cdot \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\ &\quad \left. + \|\Omega(x - \cdot) (b_{B_j} - b(\cdot)) \chi_j(\cdot)\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \right. \\ &\quad \left. \cdot \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right). \end{aligned} \quad (53)$$

Noting $s > q_1'^-$, we denote $\widetilde{q}_1'(\cdot) > 1$ and $1/\widetilde{q}_1'(x) = 1/q_1'(x) + 1/s$. By Lemmas 12 and 13 we have

$$\begin{aligned} &\|\Omega(x - \cdot) \chi_j(\cdot)\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \\ &\leq \|\Omega(x - \cdot) \chi_j(\cdot)\|_{L^s(\mathbb{R}^n)} \|\chi_j(\cdot)\|_{L^{\widetilde{q}_1'(\cdot)}(\mathbb{R}^n)} \\ &\leq \|\Omega(x - \cdot) \chi_j(\cdot)\|_{L^s(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{\widetilde{q}_1'(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{-j\nu} \left(\int_{A_j} |\Omega(x - y)|^s |y|^{s\nu} dy \right)^{1/s} \|\chi_{B_j}\|_{L^{\widetilde{q}_1'(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{-j\nu} 2^{k(\nu+n/s)} \|\Omega\|_{L^s(S^{n-1})} \|\chi_{B_j}\|_{L^{\widetilde{q}_1'(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad (54)$$

When $|B_j| \leq 2^n$ and $x_j \in B_j$, by Lemma 14 we have

$$\|\chi_{B_j}\|_{L^{\widetilde{q}_1'(\cdot)}(\mathbb{R}^n)} \approx |B_j|^{1/\widetilde{q}_1'(x_j)} \approx \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} |B_j|^{-1/s}. \quad (55)$$

When $|B_j| \geq 1$ we have

$$\|\chi_{B_j}\|_{L^{\widetilde{q}_1'(\cdot)}(\mathbb{R}^n)} \approx |B_j|^{1/\widetilde{q}_1'(\infty)} \approx \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} |B_j|^{-1/s}. \quad (56)$$

So we obtain $\|\chi_{B_j}\|_{L^{\widetilde{q}_1'(\cdot)}(\mathbb{R}^n)} \approx \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} |B_j|^{-1/s}$.

So we have

$$\begin{aligned} &\|\Omega(x - \cdot) \chi_j(\cdot)\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \\ &\leq C 2^{(k-j)(\nu+n/s)} \|\Omega\|_{L^s(S^{n-1})} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}. \end{aligned} \quad (57)$$

Similarly, by Lemma 19 we have

$$\begin{aligned}
& \left\| \Omega(x - \cdot) (b_{B_j} - b(\cdot)) \chi_j(\cdot) \right\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \\
& \leq \left\| \Omega(x - \cdot) \chi_j(\cdot) \right\|_{L^s(\mathbb{R}^n)} \left\| (b_{B_j} - b(\cdot)) \chi_j(\cdot) \right\|_{L^{\tilde{q}'_1(\cdot)}(\mathbb{R}^n)} \\
& \leq C \|b\|_* \left\| \chi_{B_j} \right\|_{L^{\tilde{q}'_1(\cdot)}(\mathbb{R}^n)} \left\| \Omega(x - \cdot) \chi_j(\cdot) \right\|_{L^s(\mathbb{R}^n)} \\
& \leq C \|b\|_* 2^{(k-j)(\nu+n/s)} \left\| \Omega \right\|_{L^s(S^{n-1})} \left\| \chi_{B_j} \right\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{58}$$

By (57), (58), and Lemmas 3, 4, and 19 we have

$$\begin{aligned}
& \left\| [b, T_{\Omega, \sigma}] (f_j) \chi_k \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
& \leq C 2^{k(\sigma-n)} \left(2^{(k-j)(\nu+n/s)} \left\| \Omega \right\|_{L^s(S^{n-1})} \left\| \chi_{B_j} \right\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \right. \\
& \quad \cdot \left\| (b(\cdot) - b_{B_j}) \chi_k(\cdot) \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \left\| f_j \right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} + \|b\|_* \\
& \quad \cdot 2^{(k-j)(\nu+n/s)} \left\| \Omega \right\|_{L^s(S^{n-1})} \left\| \chi_{B_j} \right\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \left\| \chi_k \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
& \quad \cdot \left\| f_j \right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \Big) \leq C 2^{k(\sigma-n)} \left((k-j) \|b\|_* \right. \\
& \quad \cdot 2^{(k-j)(\nu+n/s)} \left\| \Omega \right\|_{L^s(S^{n-1})} \left\| \chi_{B_j} \right\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \left\| \chi_{B_k} \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \\
& \quad \cdot \left\| f_j \right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} + \|b\|_* 2^{(k-j)(\nu+n/s)} \left\| \Omega \right\|_{L^s(S^{n-1})} \\
& \quad \cdot \left\| \chi_{B_j} \right\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \left\| \chi_{B_k} \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \left\| f_j \right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \Big) \leq C (k \\
& \quad - j) \|b\|_* 2^{k(\sigma-n)} 2^{(k-j)(\nu+n/s)} \left\| \Omega \right\|_{L^s(S^{n-1})} \left\| \chi_{B_j} \right\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \\
& \quad \cdot \left\| \chi_{B_k} \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \left\| f_j \right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \leq C (k-j) \|b\|_* \\
& \quad \cdot 2^{-kn} 2^{(k-j)(\nu+n/s)} \left\| \Omega \right\|_{L^s(S^{n-1})} \left\| \chi_{B_j} \right\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \\
& \quad \cdot \left\| T_\sigma (\chi_{B_k}) \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \left\| f_j \right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \leq C (k-j) \|b\|_* \\
& \quad \cdot 2^{-kn} 2^{(k-j)(\nu+n/s)} \left\| \Omega \right\|_{L^s(S^{n-1})} \left\| f_j \right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
& \quad \cdot \left\| \chi_{B_j} \right\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)} \left\| \chi_{B_k} \right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \leq C (k-j) \|b\|_* \\
& \quad \cdot 2^{(k-j)(\nu+n/s)} \left\| \Omega \right\|_{L^s(S^{n-1})} \left\| f_j \right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \frac{\left\| \chi_{B_j} \right\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}}{\left\| \chi_{B_k} \right\|_{L^{q'_1(\cdot)}(\mathbb{R}^n)}} \\
& \leq C \|b\|_* (k-j) 2^{(j-k)(n\delta_2 - \nu - n/s)} \left\| \Omega \right\|_{L^s(S^{n-1})} \\
& \quad \cdot \left\| f_j \right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{59}$$

So we have

$$\begin{aligned}
J_1 &= C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
& \cdot \sum_{k=-\infty}^L 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} \left\| [b, T_{\Omega, \sigma}] (f_j) \chi_k \right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\
& \leq C \|b\|_*^{p_1} \left\| \Omega \right\|_{L^s(S^{n-1})}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
& \cdot \sum_{k=-\infty}^L 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} (k-j) \right. \\
& \quad \cdot 2^{(j-k)(n\delta_2 - \nu - n/s)} \left\| f_j \right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \Big)^{p_1}.
\end{aligned} \tag{60}$$

When $1 < p_1 < \infty$, take $1/p_1 + 1/p'_1 = 1$. Since $n\delta_2 - \nu - n/s - \alpha > 0$, by the Hölder inequality, we have

$$\begin{aligned}
J_1 &\leq C \|b\|_*^{p_1} \left\| \Omega \right\|_{L^s(S^{n-1})}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
& \cdot \sum_{k=-\infty}^L \left(\sum_{j=-\infty}^{k-2} 2^{j\alpha p_1} 2^{(j-k)(n\delta_2 - \nu - n/s - \alpha)p_1/2} \left\| f_j \right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \right) \\
& \times \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_2 - \nu - n/s - \alpha)p'_1/2} (k-j)^{p'_1} \right)^{p_1/p'_1} \\
& \leq C \|b\|_*^{p_1} \left\| \Omega \right\|_{L^s(S^{n-1})}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
& \cdot \sum_{k=-\infty}^L \sum_{j=-\infty}^{k-2} 2^{j\alpha p_1} 2^{(j-k)(n\delta_2 - \nu - n/s - \alpha)p_1/2} \left\| f_j \right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
& \leq C \|b\|_*^{p_1} \left\| \Omega \right\|_{L^s(S^{n-1})}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
& \cdot \sum_{j=-\infty}^{L-2} 2^{j\alpha p_1} \left\| f_j \right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \sum_{k=j+2}^L 2^{(j-k)(n\delta_2 - \nu - n/s - \alpha)p_1/2} \\
& \leq C \|b\|_*^{p_1} \left\| \Omega \right\|_{L^s(S^{n-1})}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-2} 2^{j\alpha p_1} \left\| f_j \right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
& \leq C \|b\|_*^{p_1} \left\| \Omega \right\|_{L^s(S^{n-1})}^{p_1} \left\| f \right\|_{MK_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1}.
\end{aligned} \tag{61}$$

When $0 < p_1 \leq 1$, we have

$$\begin{aligned}
J_1 &\leq C \|b\|_*^{p_1} \left\| \Omega \right\|_{L^s(S^{n-1})}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
& \cdot \sum_{k=-\infty}^L 2^{k\alpha p_1} \sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_2 - \nu - n/s)p_1} (k-j)^{p_1}
\end{aligned}$$

$$\begin{aligned}
& \cdot \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} = C \|b\|_*^{p_1} \|\Omega\|_{L^s(S^{n-1})}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
& \cdot \sum_{j=-\infty}^{L-2} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
& \cdot \sum_{k=j+2}^L 2^{(j-k)(n\delta_2 - \nu - n/s - \alpha)p_1} (k-j)^{p_1} \leq C \|b\|_*^{p_1} \\
& \cdot \|\Omega\|_{L^s(S^{n-1})}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-2} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
& \leq C \|b\|_*^{p_1} \|\Omega\|_{L^s(S^{n-1})}^{p_1} \|f\|_{M\dot{K}_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1}.
\end{aligned} \tag{62}$$

Next we estimate J_2 ; by the $(L^{q_1(\cdot)}(\mathbb{R}^n), L^{q_2(\cdot)}(\mathbb{R}^n))$ -boundedness of the commutator $[b, T_{\Omega, \sigma}]$ we have

$$\begin{aligned}
J_2 &= C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
& \cdot \sum_{k=-\infty}^L 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} \|[b, T_{\Omega, \sigma}](f_j) \chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\
& \leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\
& = C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
& \cdot \sum_{k=-\infty}^L \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha} 2^{j\alpha} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1}.
\end{aligned} \tag{63}$$

If $0 < p_1 \leq 1$, then we have

$$\begin{aligned}
J_2 &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
&\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=k-1}^{L-1} 2^{(k-j)\alpha p_1} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
&\quad + C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1}
\end{aligned} \tag{64}$$

$$=: J_{21} + J_{22}.$$

For J_{21} , we have

$$\begin{aligned}
J_{21} &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-1} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p_1} \\
&\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-1} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
&\leq C \|f\|_{M\dot{K}_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1}.
\end{aligned} \tag{65}$$

For J_{22} , by $0 < \lambda < \alpha$ we have

$$\begin{aligned}
J_{22} &= C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \leq C \\
& \cdot \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
& \cdot \sum_{k=-\infty}^L \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\lambda p_1} 2^{-j\lambda p_1} \left(\sum_{m=-\infty}^j 2^{m\alpha p_1} \|f_m\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \right) \\
& \leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1} 2^{j\lambda p_1} \|f\|_{M\dot{K}_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1} \\
& = C \|f\|_{M\dot{K}_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L 2^{k\alpha p_1} \sum_{j=L}^{\infty} 2^{j(\lambda-\alpha)p_1} \\
& \leq C \|f\|_{M\dot{K}_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} 2^{L\alpha p_1} 2^{L(\lambda-\alpha)p_1} \\
& = C \|f\|_{M\dot{K}_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1}.
\end{aligned} \tag{66}$$

If $1 < p_1 < \infty$, noting $\lambda < \alpha$, we can take a constant $\eta > 1$ so that $\lambda - \alpha/\eta < 0$. By the Hölder inequality we have

$$\begin{aligned}
J_2 &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
& \cdot \sum_{k=-\infty}^L \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p_1/\eta} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \right) \\
& \cdot \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p_1'(\eta-1)/\eta} \right)^{p_1/p_1'} \leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
& \cdot \sum_{k=-\infty}^L \sum_{j=k-1}^{\infty} 2^{(k-j)\alpha p_1/\eta} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
& \leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \\
& \cdot \sum_{k=-\infty}^L \sum_{j=k-1}^{L-1} 2^{(k-j)\alpha p_1/\eta} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
& \quad + C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1/\eta} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
& =: J_{23} + J_{24}.
\end{aligned} \tag{67}$$

For J_{23} , we have

$$\begin{aligned}
J_{23} &\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-1} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p_1/\eta}
\end{aligned}$$

$$\begin{aligned}
&\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{j=-\infty}^{L-1} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
&\leq C \|f\|_{\dot{MK}_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1}.
\end{aligned} \tag{68}$$

For J_{24} , by $0 < \lambda < \alpha/\eta$ we have

$$\begin{aligned}
J_{24} &= C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1/\eta} 2^{j\alpha p_1} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}^{p_1} \\
&\leq C \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L \sum_{j=L}^{\infty} 2^{(k-j)\alpha p_1/\eta} 2^{j\lambda p_1} \|f\|_{\dot{MK}_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1} \\
&= C \|f\|_{\dot{MK}_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} \sum_{k=-\infty}^L 2^{k\alpha p_1/\eta} \sum_{j=L}^{\infty} 2^{j(\lambda-\alpha/\eta)p_1} \\
&\leq C \|f\|_{\dot{MK}_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1} \sup_{L \in \mathbb{Z}} 2^{-L\lambda p_1} 2^{L\alpha p_1/\eta} 2^{L(\lambda-\alpha/\eta)p_1} \\
&= C \|f\|_{\dot{MK}_{q_1(\cdot)}^{\alpha, p_1}(\mathbb{R}^n)}^{p_1}.
\end{aligned} \tag{69}$$

Thus, by (52), (61), (62), and (64)–(69) we complete the proof of Theorem 18. $\square$

Remark 20. In [28], the authors pointed out $\text{BMO} \subset \bigcap_{q>1} \text{CMO}^q$, so we will consider the corresponding results on CMO space in the following paper.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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