Research Article

Hyperstability of Some Functional Equations on Restricted Domain

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The paper concerns functions which approximately satisfy, not necessarily on the whole linear space, a generalization of linear functional equation. A Hyers-Ulam stability result is proved and next applied to give conditions implying the hyperstability of the equation. The results may be used as tools in stability studies on restricted domains for various functional equations. We use the main theorem to obtain a few hyperstability results of Fréchet equation on restricted domain for different control functions.

1. Introduction

The famous Cauchy equation

$$f(x + y) = f(x) + f(y) \quad (1)$$

has a very long history, which started with A. M. Legendre in 1791 and C. F. Gauss in 1809, but the first important result about the solutions of (1) was proved by Cauchy (see [1]) and due to this outcome (1) is today named after him, as is mentioned in [2]. In the meantime, many authors have dealt with this equation, and next Fréchet in [3] studied an important generalization of Cauchy equation, characterizing the polynomials among the continuous mappings, namely, the equation

$$\Delta_{h_1 \dots h_n}^n f(x) = 0, \quad (2)$$

where Δ denotes the difference operator defined as usual by

$$\begin{aligned} \Delta_h f(x) &= \Delta_h^1 f(x) := f(x + h) - f(x), \\ \Delta_{h_1 \dots h_{n+1}}^{n+1} f(x) &= \Delta_{h_{n+1}} (\Delta_{h_1 \dots h_n}^n f(x)), \end{aligned} \quad (3)$$

for $x, h, h_1, \dots, h_{n+1} \in \mathbb{R}$, $n \in \mathbb{N}$. From the result of Fréchet it follows that a continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies (2) for all x and h_j if and only if f is a polynomial of degree smaller

than or equal to $n - 1$ (cf. [2, 4]). As a generalization we can consider the functional equation

$$\sum_{i=1}^m A_i g \left(\sum_{j=1}^n a_{ij} x_j \right) + A = 0, \quad (4)$$

where $g: X \rightarrow Y$ and X, Y are the linear spaces over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $A, a_{ij} \in \mathbb{F}$, and $A_i \in \mathbb{F} \setminus \{0\}$, $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$.

The stability and hyperstability of the above functional equation and its particular cases were studied by many mathematicians (cf., e.g., [5–16]). Let us recall (see [9]) that the theory of Hyers-Ulam stability, in general, was motivated by a problem raised in 1940 by Ulam [17] and a partial answer to it was provided by Hyers [18]. The term *hyperstability*, in the meaning used here, was introduced presumably in [19] (cf. also [20]).

Our basic assumptions are the following. X is a normed space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, E_1, E_2 are the nonempty subsets of X such that $E_2 \subseteq E_1 \subseteq X$, and D is a nonempty subset of $\mathbb{F}$ such that $1 \in D$, $\mathbb{R}_+ := [0, +\infty)$. Denote $\mathbb{N}_{\leq k} := \{1, \dots, k\}$ and $\mathbb{N}_k := \{l \in \mathbb{N} : l \geq k\}$, for $k \in \mathbb{N}$, and a sum of numbers over an empty set is defined to be zero. In the sequel, $\mathcal{P}(A) = \{B \subset A : A \neq \emptyset\}$ for nonempty set A .

We say that the equation

$$\sum_{i=1}^m A_i g \left(\sum_{j=1}^n a_{ij} x_j \right) + A = 0, \quad (5)$$

$$x_1, \dots, x_n \in E_2, \sum_{j=1}^n a_{1j} x_j, \dots, \sum_{j=1}^n a_{mj} x_j \in E_1$$

is θ -hyperstable in the class of functions $g : E_1 \rightarrow Y$ (with a control function $\theta : E_2^n \rightarrow \mathbb{R}_+$), if $g : E_1 \rightarrow Y$ satisfies the inequality

$$\left\| \sum_{i=1}^m A_i g \left(\sum_{j=1}^n a_{ij} x_j \right) + A \right\| \leq \theta(x_1, \dots, x_n), \quad (6)$$

for all $x_1, \dots, x_n \in E_2$, and $\sum_{j=1}^n a_{1j} x_j, \dots, \sum_{j=1}^n a_{mj} x_j \in E_1$ fulfills (5).

The above notion of hyperstability for functions defined on subsets of a linear space X (not necessarily on the whole space but on the restricted domain) is a counterpart of hyperstability considered in [8]. In this paper we obtain, applying the fixed point theorem, some results concerning stability and hyperstability for (4) on restricted domain. Our outcomes allow studying the behavior of functions approximately fulfilling functional equations of this linear type on different subsets of the linear space, and this also applies to domain restrictions for control functions. This way we give a tool for the further Hyers-Ulam stability research. Its application is presented in the last section where some new as well as generalizations of known results on hyperstability of Cauchy and Fréchet equation are obtained.

2. Auxiliary Result

First we establish a counterpart of stability result from [7] on restricted domain, and to do this we use the fixed point theorem. This result will be needed to show our main theorem which provides criteria for hyperstability of many functional equations of the linear type on a restricted domain.

Lemma 1. Assume that Y is a Banach space, $A = 0$ or " $A \neq 0$ and $\sum_{i=1}^m A_i \neq 0$," and $\omega : D \rightarrow \mathbb{R}_+$. Suppose that there exist $\emptyset \neq I \subset \mathbb{N}_{\leq m}$ and $c_1, \dots, c_n \in D$ such that

$$\begin{aligned} \beta_i &:= \sum_{j=1}^n a_{ij} c_j \in D, \\ c_j x &\in E_2, \beta_i x \in E_2, \quad i \in \mathbb{N}_{\leq m}, \quad j \in \mathbb{N}_{\leq n}, \quad x \in E_2, \\ \beta_l &= 1, \quad l \in I, \end{aligned} \quad (7)$$

$$A_I := \sum_{l \in I} A_l \neq 0,$$

$$\gamma := \sum_{i \notin I} \left| \frac{A_i}{A_I} \right| \omega(\beta_i) < 1.$$

Let $\theta : E_2^n \rightarrow \mathbb{R}_+$ satisfy the inequality

$$\theta(\beta_i x_1, \dots, \beta_i x_n) \leq \omega(\beta_i) \theta(x_1, \dots, x_n), \quad (8)$$

$$i \in \mathbb{N}_{\leq m}, \quad x_1, \dots, x_n \in E_2.$$

If $g : E_1 \rightarrow Y$ fulfills the estimation

$$\left\| \sum_{i=1}^m A_i g \left(\sum_{j=1}^n a_{ij} x_j \right) + A \right\| \leq \theta(x_1, \dots, x_n), \quad (9)$$

$$x_1, \dots, x_n \in E_2, \sum_{j=1}^n a_{1j} x_j, \dots, \sum_{j=1}^n a_{mj} x_j \in E_1,$$

then there exists a unique function $G : E_2 \rightarrow Y$ satisfying (5) such that

$$\|g(x) - G(x)\| \leq \frac{\theta(c_1 x, \dots, c_n x)}{|A_I|(1-\gamma)}, \quad x \in E_2. \quad (10)$$

Proof. Take $\emptyset \neq I \subset \mathbb{N}_{\leq m}$ and $c_1, \dots, c_n \in D$ such that assumptions (7) and (8) hold.

First consider the case $A = 0$. Taking $x \in E_2$ and substituting $x_j = c_j x$, $j \in \mathbb{N}_{\leq n}$, in (9) we have

$$\left\| g(x) - \sum_{i \notin I} \frac{-A_i}{A_I} g(\beta_i x) \right\| \leq \frac{\theta(c_1 x, \dots, c_n x)}{|A_I|}, \quad x \in E_2. \quad (11)$$

Define

$$(\mathcal{T}\xi)(x) := \sum_{i \notin I} \frac{-A_i}{A_I} \xi(\beta_i x), \quad \xi \in Y^{E_2}, \quad x \in E_2, \quad (12)$$

$$(\Lambda\delta)(x) := \sum_{i \notin I} \left| \frac{-A_i}{A_I} \right| \delta(\beta_i x), \quad \delta \in \mathbb{R}_+^{E_2}, \quad x \in E_2, \quad (13)$$

$$\varepsilon(x) := \frac{\theta(c_1 x, \dots, c_n x)}{|A_I|}, \quad x \in E_2. \quad (14)$$

Then (11) may be rewritten as follows:

$$\|g(x) - \mathcal{T}g(x)\| \leq \varepsilon(x), \quad x \in E_2. \quad (15)$$

By induction, for every $x \in E_2$ and $k \in \mathbb{N}_0$

$$\Lambda^k \varepsilon(x) \leq \varepsilon(x) \gamma^k. \quad (16)$$

Since $\gamma < 1$, we see that

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(x) \leq \varepsilon(x) \sum_{n=0}^{\infty} \gamma^n = \frac{\varepsilon(x)}{1-\gamma}, \quad (17)$$

$$x \in E_2.$$

Notice that all assumptions of Theorem 1 in [21] are satisfied for operators (12) and (13) and $X = E_2$. According to this fixed point theorem, the function $G : E_2 \rightarrow Y$ given by $G(x) = \lim_{n \rightarrow \infty} (\mathcal{T}^n g)(x)$ for $x \in E_2$ is a fixed point of $\mathcal{T}$ fulfilling (10).

It is sufficient to show that G satisfies (5) (with $A = 0$). First we prove that for every $k \in \mathbb{N} \cup \{0\}$ and every $x_1, \dots, x_n \in E_2$ such that $\sum_{j=1}^n a_{1j}x_j, \dots, \sum_{j=1}^n a_{mj}x_j \in E_1$

$$\left\| \sum_{i=1}^m A_i (\mathcal{T}^k g) \left(\sum_{j=1}^n a_{ij}x_j \right) \right\| \leq \gamma^k \theta(x_1, \dots, x_n). \quad (18)$$

Obviously, the case $k = 0$ is just (9). Next, fix $k \in \mathbb{N} \cup \{0\}$, $x_1, \dots, x_n \in E_2$, and $\sum_{j=1}^n a_{1j}x_j, \dots, \sum_{j=1}^n a_{mj}x_j \in E_1$ and assume that (18) holds. Then

$$\begin{aligned} & \sum_{i=1}^m A_i (\mathcal{T}^{k+1} g) \left(\sum_{j=1}^n a_{ij}x_j \right) \\ &= \sum_{i=1}^m A_i \mathcal{T} (\mathcal{T}^k g) \left(\sum_{j=1}^n a_{ij}x_j \right) \\ &= \sum_{i=1}^m A_i \sum_{p \notin I} \frac{-A_p}{A_I} (\mathcal{T}^k g) \left(\beta_p \sum_{j=1}^n a_{ij}x_j \right) \\ &= \sum_{p \notin I} \frac{-A_p}{A_I} \left[\sum_{i=1}^m A_i (\mathcal{T}^k g) \left(\sum_{j=1}^n a_{ij}(\beta_p x_j) \right) \right]. \end{aligned} \quad (19)$$

Thus by induction hypothesis and (8) we have

$$\begin{aligned} & \left\| \sum_{i=1}^m A_i (\mathcal{T}^{k+1} g) \left(\sum_{j=1}^n a_{ij}x_j \right) \right\| \\ & \leq \sum_{p \notin I} \left| \frac{-A_p}{A_I} \right| \gamma^k \theta(\beta_p x_1, \dots, \beta_p x_n) \\ & \leq \sum_{p \notin I} \left| \frac{-A_p}{A_I} \right| \gamma^k \omega(\beta_p) \theta(x_1, \dots, x_n) \\ & = \gamma^{k+1} \theta(x_1, \dots, x_n), \end{aligned} \quad (20)$$

which finishes the proof of (18). Letting $k \rightarrow \infty$ in (18), we prove that G satisfies (5), which completes the proof in the first case.

In the case where $A \neq 0$ and $\sum_{i=1}^m A_i \neq 0$, the function $h(x) := g(x) + A / \sum_{i=1}^m A_i$, $x \in E_1$, satisfies

$$\left\| \sum_{i=1}^m A_i h \left(\sum_{j=1}^n a_{ij}x_j \right) \right\| \leq \theta(x_1, \dots, x_n), \quad (21)$$

for $x_1, \dots, x_n \in E_2$, and $\sum_{j=1}^n a_{1j}x_j, \dots, \sum_{j=1}^n a_{mj}x_j \in E_1$. Consequently, we conclude that there exists a unique function $H : E_2 \rightarrow Y$ such that

$$\|h(x) - H(x)\| \leq \frac{\theta(c_1 x, \dots, c_n x)}{|A_I|(1 - \gamma)}, \quad x \in E_2. \quad (22)$$

Obviously $G(x) := H(x) - A / \sum_{i=1}^m A_i$ and $x \in E_2$ is the desired function. $\square$

Denote

$$\begin{aligned} X_0 &:= X \setminus \{0\}, \\ \mathbb{F}_0 &:= \mathbb{F} \setminus \{0\}. \end{aligned} \quad (23)$$

Remark 2. It is easily seen that functions

- (i) $\theta_1(x_1, \dots, x_n) = C$;
- (ii) $\theta_2(x_1, \dots, x_n) = C \sum_{j=1}^n \|c_j x_j\|^{k_j}$;
- (iii) $\theta_3(x_1, \dots, x_n) = C \max \{\|c_j x_j\|^{k_j} : j \in \{1, \dots, n\}\}$;
- (iv) $\theta_4(x_1, \dots, x_n) = C \prod_{j=1}^n \|x_j\|^{k_j}$;
- (v) $\theta_5(x_1, \dots, x_n) = C \prod_{j=1}^n \|x_j\|^{t_j} + S \sum_{j=1}^n \|c_j x_j\|^{k_j}$;

with some $C, S \in [0, +\infty)$, $C + S > 0$, $c_j \in \mathbb{F}_0$, and $k_j, t_j \neq 0$ fulfill condition (8) for some function ω . For instance, in case (iv) we can use $\omega_4(\beta) = |\beta|^{\sum_{j=1}^n k_j}$ for $\beta \in D$, and in case (v) we may take

$$\omega_5(\beta) := \begin{cases} |\beta|^{\max\{k_1, \dots, k_n, t\}} & \text{for } |\beta| \geq 1 \\ |\beta|^{\min\{k_1, \dots, k_n, t\}} & \text{for } |\beta| < 1, \end{cases} \quad (24)$$

where $t = \sum_{j=1}^n t_j$. If there exists at least one $j_0 \in \mathbb{N}_{\leq n}$ such that $k_{j_0} < 0$ or $t_{j_0} < 0$ then $0 \notin D$.

3. The Main Result

We can now state a counterpart (on restricted domain) of Theorem 2.1 in [8], concerning θ -hyperstability of (5).

Theorem 3. Let Y be a normed space over $\mathbb{F}$, $A = 0$ or " $A \neq 0$ and $\sum_{i=1}^m A_i \neq 0$," $g : E_1 \rightarrow Y$, $\omega : D \rightarrow \mathbb{R}_+$, and $\theta : E_2^n \rightarrow \mathbb{R}_+$ satisfy (9). If there exist $\emptyset \neq I \subset \mathbb{N}_{\leq m}$ and a sequence $\{(c_1^k, \dots, c_n^k)\}_{k \in \mathbb{N}}$ of elements of D^n such that

$$\begin{aligned} \beta_i^k &:= \sum_{j=1}^n a_{ij} c_j^k \in D, \\ c_j^k x &\in E_2, \end{aligned} \quad (25)$$

$$\begin{aligned} \beta_i^k x &\in E_2, \\ i &\in \mathbb{N}_{\leq m}, \quad j \in \mathbb{N}_{\leq n}, \quad k \in \mathbb{N}, \quad x \in E_2, \\ \beta_l^k &= 1, \quad l \in I, \\ A_I &:= \sum_{l \in I} A_l \neq 0, \end{aligned} \quad (26)$$

$$\begin{aligned} \lim_{k \rightarrow \infty} \sum_{i \notin I} \left| \frac{A_i}{A_I} \right| \omega(\beta_i^k) &< 1, \\ \theta(\beta_i^k x_1, \dots, \beta_i^k x_n) &\leq \omega(\beta_i^k) \theta(x_1, \dots, x_n) \end{aligned} \quad (27)$$

$$i \in \mathbb{N}_{\leq m}, \quad k \in \mathbb{N}, \quad x_1, \dots, x_n \in E_2,$$

$$\lim_{k \rightarrow \infty} \theta(c_1^k x, \dots, c_n^k x) = 0, \quad (28)$$

then g satisfies (5).

Proof. We can certainly assume that Y is a Banach space, for if not we replace it by its completion.

Take $\emptyset \neq I \subset \mathbb{N}_{\leq m}$ and the sequence $\{(c_1^k, \dots, c_n^k)\}_{k \in \mathbb{N}}$ of the elements of D^n such that conditions (25), (26), and (27) hold. Hence we can find $\varepsilon \in (0, 1)$ and $k_0 \in \mathbb{N}$ such that

$$\gamma_k := \sum_{i \notin I} \left| \frac{A_i}{A_I} \right| \omega(\beta_i^k) < 1 - \varepsilon, \quad k \geq k_0. \quad (29)$$

Therefore, according to Lemma 1, for each $k \in \mathbb{N}_{k_0}$ there exists a unique function $G_k : E_2 \rightarrow Y$ satisfying equation (5) such that

$$\|g(x) - G_k(x)\| \leq \frac{\theta(c_1^k x, \dots, c_n^k x)}{|A_I|(1 - \gamma_k)}, \quad x \in E_2. \quad (30)$$

In this way we obtain the sequence $\{G_k\}_{k \in \mathbb{N}_{k_0}}$ of functions satisfying (5). Letting in (30) $k \rightarrow \infty$, we get, using (28), that g satisfies (5). $\square$

4. Applications

Using the above outcome we can get the results for the particular form of (5), as, for instance, for Fréchet functional equation, presented in this section.

Corollary 4. Assume that Y is a normed space over $\mathbb{F}$, $n \in \mathbb{N}_2$, E is a nonempty subset of X_0 such that there exists a positive integer l_0 with

$$-x, lx \in E, \quad x \in E, \quad l \in \mathbb{N}_{l_0}. \quad (31)$$

If $g : E \rightarrow Y$ fulfills the inequality

$$\begin{aligned} \|\Delta_{x_2 \dots x_n}^{n-1} g(x_1)\| &\leq C \prod_{j=1}^n \|x_j\|^{k_j}, \\ \sum_{j \in T} x_j &\in E, \quad T \in \mathcal{P}(\mathbb{N}_{\leq n}), \end{aligned} \quad (32)$$

with $C > 0$ and $\sum_{j=1}^n k_j < 0$, then g is a solution of the equation

$$\begin{aligned} \Delta_{x_2 \dots x_n}^{n-1} g(x_1) &= 0, \\ \sum_{j \in T} x_j &\in E, \quad T \in \mathcal{P}(\mathbb{N}_{\leq n}). \end{aligned} \quad (33)$$

Proof. Put $E := E_2 = E_1$, $D := \mathbb{Z}_0$, and $\omega(\beta) := |\beta|^{\sum_{j=1}^n k_j}$ for $\beta \in \mathbb{Z}_0$ and consider two cases:

- (a) $\sum_{j=2}^n k_j < 0$,
- (b) $\sum_{j=2}^n k_j \geq 0$.

In the case (a) we define the sequence $\{(c_1^k, \dots, c_n^k)\}_{k \in \mathbb{N}}$ as follows:

$$c_j^k := \begin{cases} k + l_0 & \text{for } j \neq 1 \\ 1 & \text{for } j = 1. \end{cases} \quad (34)$$

It follows easily that the assumptions of Theorem 3 are satisfied with $m = 2^n$ and $I = \{2^n\}$.

In case (b) we can find $j_0 \in \{2, \dots, n\}$ such that $k_{j_0} \geq 0$. Define

$$c_j^k := \begin{cases} k + l_0 & \text{for } j \neq j_0 \\ 1 - k - l_0 & \text{for } j = j_0. \end{cases} \quad (35)$$

There exists $i_0 \in \{1, \dots, 2^n\}$ such that $a_{i_0 1} = a_{i_0 j_0} = 1$ and $a_{i_0 j} = 0$ for $j \in \{2, \dots, n\} \setminus \{1, j_0\}$. We will verify that we can apply Theorem 3 for $I = \{i_0\}$. Namely,

$$\sum_{j=1}^n a_{i_0 j} c_j^k = k + l_0 + 1 - k - l_0 = 1, \quad (36)$$

$$k + l_0 \leq \sum_{j=1}^n a_{ij} c_j^k \leq (k + l_0)(n - 1) \quad \text{for } i \neq i_0.$$

Therefore

$$\begin{aligned} \lim_{k \rightarrow \infty} \sum_{i \neq i_0} \left| \sum_{j=1}^n a_{ij} c_j^k \right|^{\sum_{j=1}^n k_j} &= 0, \\ \lim_{k \rightarrow \infty} C \prod_{j=1}^n \|c_j^k x\|^{k_j} &= \lim_{k \rightarrow \infty} |k + l_0|^{\sum_{j \neq j_0} k_j} |k + l_0 - 1|^{k_{j_0}} \|x\|^{\sum_{j=1}^n k_j} \\ &\leq \lim_{k \rightarrow \infty} |k + l_0|^{\sum_{j=1}^n k_j} \|x\|^{\sum_{j=1}^n k_j} = 0. \end{aligned} \quad (37)$$

Thus conditions (26) and (28) hold, which finishes the proof. $\square$

Corollary 5. Assume that Y is a normed space over $\mathbb{F}$, $n \in \mathbb{N}_2$, and E is a nonempty subset of X_0 such that there exists a positive integer l_0 with (31).

If $g : E \rightarrow Y$ fulfills the inequality

$$\begin{aligned} \|\Delta_{x_2 \dots x_n}^{n-1} g(x_1)\| &\leq \left(\sum_{j=1}^n C_j \|x_j\|^{k_j} \right)^w, \\ \sum_{j \in T} x_j &\in E, \quad T \in \mathcal{P}(\mathbb{N}_{\leq n}), \end{aligned} \quad (38)$$

with $C_j > 0$ and $wk_j < 0$, for $j \in \{1, \dots, n\}$, then g is a solution of (33).

Proof. Putting $E := E_2 = E_1$, $D := \mathbb{Z}_0$, and $\omega(\beta) := |\beta|^{-v|w|}$ for $\beta \in \mathbb{Z}_0$, where $v := \min\{|k_1|, \dots, |k_n|\}$, $I = \{1\}$, and

$$c_j^k := \begin{cases} (n - 1)(k + l_0) + 1 & \text{for } j = 1 \\ -k - l_0 & \text{for } j > 1, \end{cases} \quad (39)$$

we get that

$$\begin{aligned} \beta_1^k &= 1, \\ k + l_0 + 1 &\leq \beta_i^k \leq (n - 1)(k + l_0) + 1 \quad \text{for } i > 1, \end{aligned} \quad (40)$$

and therefore $\lim_{k \rightarrow \infty} |\beta_i^k| = +\infty$ for $i \neq 1$, and consequently condition (26) is satisfied. Also, condition (28) holds, because

$$\lim_{k \rightarrow \infty} \left(C_1 (|(n-1)(k+l_0) + 1| \|x\|)^{k_1} + \sum_{j=2}^n C_j (|k+l_0| \|x\|)^{k_j} \right)^w = 0, \quad (41)$$

which with Theorem 3 completes the proof. $\square$

Denote

$$F(f, x_1, \dots, x_n) = \Delta_{x_1 \dots x_n}^n f(0) + (-1)^{n+1} f(0), \quad (42)$$

$$f \in Y^X, \quad x_1, \dots, x_n \in X.$$

Corollary 6. Assume that Y is a normed space over $\mathbb{F}$, $n \in \mathbb{N}_2$, and E is a nonempty subset of X_0 such that there exists a positive integer l_0 with

$$lx \in E, \quad x \in E, \quad l \in \mathbb{N}_{l_0}. \quad (43)$$

If $g : E \rightarrow Y$ fulfills the inequality

$$\|F(g, x_1, \dots, x_n)\| \leq C \prod_{j=1}^n \|x_j\|^{k_j}, \quad (44)$$

$$\sum_{j \in T} x_j \in E, \quad T \in \mathcal{P}(\mathbb{N}_{\leq n}),$$

with $C > 0$ and $\sum_{j=1}^n k_j < 0$, then g is a solution of the equation

$$F(g, x_1, \dots, x_n) = 0, \quad \sum_{j \in T} x_j \in E, \quad T \in \mathcal{P}(\mathbb{N}_{\leq n}). \quad (45)$$

Proof. Put $E := E_2 = E_1$, $D := \mathbb{Z}_0$, and $\omega(\beta) := |\beta| \sum_{j=1}^n k_j$ for $\beta \in \mathbb{Z}_0$. There exist $j_0 \in \{1, \dots, n\}$ and $i_0 \in \{1, \dots, 2^n - 1\}$, such that $\sum_{j \neq j_0}^n k_j < 0$, $a_{i_0 j_0} = 1$, and $a_{i_0 j} = 0$ for $j \neq j_0$. Define the sequence $\{(c_1^k, \dots, c_n^k)\}_{k \in \mathbb{N}}$ as follows:

$$c_j^k := \begin{cases} k + l_0 & \text{for } j \neq j_0 \\ 1 & \text{for } j = j_0. \end{cases} \quad (46)$$

The assumptions of Theorem 3 are fulfilled with $m = 2^n - 1$ and $I = \{i_0\}$, as is easy to check, which finishes the proof. $\square$

In the same way as Corollary 5 we can prove the following corollary.

Corollary 7. Assume that Y is a normed space over $\mathbb{F}$, $n \in \mathbb{N}_2$, and E is a nonempty subset of X_0 such that there exists a positive integer l_0 with (31).

If $g : E \rightarrow Y$ fulfills the inequality

$$\|F(g, x_1, \dots, x_n)\| \leq \left(\sum_{j=1}^n C_j \|x_j\|^{k_j} \right)^w, \quad (47)$$

$$\sum_{j \in T} x_j \in E, \quad T \in \mathcal{P}(\mathbb{N}_{\leq n})$$

with $C_j > 0$ and $wk_j < 0$ for $j \in \{1, \dots, n\}$, then g is a solution of (45).

Remark 8. Observe that putting in the above two corollaries $n = 2$ we get hyperstability results for Cauchy equation. Corollary 7 generalizes the result from [9] (see Theorem 1.2), where $n = 2$ and $k_1 = k_2 < 0$ (in our considerations $k_1, k_2 < 0$ may be different). Corollaries 6 and 7 generalize the outcomes from [6].

We end the paper with an example, which shows that the additional assumption on E , appearing in the above corollaries, is essential.

Example 9. Let $X = Y = \mathbb{R}$, $E = [-1, 0) \cup (0, 1]$, and $f : E \rightarrow \mathbb{R}$ be defined by $f(x) = \sin x/2^n$, $x \in E$. Then f satisfies the estimations

$$\|\Delta_{x_2 \dots x_n}^{n-1} f(x_1)\| < 1 \leq \prod_{j=1}^n \|x_j\|^{-1},$$

$$\|F(f, x_1, \dots, x_n)\| < 1 \leq \prod_{j=1}^n \|x_j\|^{-1},$$

$$\|\Delta_{x_2 \dots x_n}^{n-1} f(x_1)\| < 1 \leq \sum_{j=1}^n \|x_j\|^{-1}, \quad (48)$$

$$\|F(f, x_1, \dots, x_n)\| < 1 \leq \sum_{j=1}^n \|x_j\|^{-1},$$

for $\sum_{j \in T} x_j \in E$, and $T \in \mathcal{P}(\mathbb{N}_{\leq n})$ but f satisfies neither (33) nor (45).

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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