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Research Article

On the Power of Simulation and Admissible Functions in Metric Fixed Point Theory

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We investigate the existence and uniqueness of certain operators which form a new contractive condition via the combining of the notions of admissible function and simulation function contained in the context of complete b-metric spaces. The given results not only unify but also generalize a number of existing results on the topic in the corresponding literature.

1. Introduction

The crucial notion of this research is the simulation function which is defined by Khojasteh et al. [1]. After that, Argoubi et al. [2] relaxed the conditions of the notion of simulation function a little bit to guarantee that the considered set is nonempty.

In this manuscript, we respond to the question, how do we guarantee the existence of fixed points of the new contraction defined by the help of the admissible function and the simulation function in the frame of complete b-metric spaces? The presented main theorem of the paper covers and unifies a huge number of published results on the topic in the related literature.

Definition 1 (see [2], cf. [1]). Let $\sigma : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ be a mapping that satisfies the following inequality and the condition below:

($\mathcal{S}_1$) $\sigma(r,s) < s - r$ for each $r, s > 0$.

($\mathcal{S}_2$) if $\{r_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n > 0$, then

$$\limsup_{n\to\infty} \sigma(r_n, s_n) < 0. \quad (1)$$

We shall use the letter $\mathcal{S}$ to indicate the class of all simulation functions $\sigma : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$. It is obvious from the axiom ($\mathcal{S}_2$) that

$$\sigma(r,r) < 0 \quad \text{for every } r > 0. \quad (2)$$

Note that the condition $\sigma(0,0) = 0$ in the original definition of the simulation function is removed in Definition 1. Indeed, this condition gives a contradiction when one takes $s = r$ in the first condition ($\mathcal{S}_1$). For further detail on the discussion, see, for example, [2].

Throughout the paper, we shall use $\mathbb{R}_0^+$ to represent nonnegative real numbers.

The following example [1, 3, 4] shall be helpful to illustrate the worth of the notion of simulation function.

Example 2. Suppose that Φ denotes the set of all continuous functions $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $\phi(r) = 0$ if, and only if, $r = 0$. The following functions $\sigma_1 - \sigma_6$ form a simulation function.

(i) $\sigma_1(t,s) = \phi_1(s) - \phi_2(t)$ for each $t, s \in \mathbb{R}_0^+$ and $\phi_1, \phi_2 \in \Phi$, where $\phi_1(t) < t \le \phi_2(t)$ for each $t > 0$.

(ii) Let $R, T : \mathbb{R}_0^+ \to (0, \infty)$ be two continuous functions with respect to each variable and the inequality $R(t,s) > T(t,s)$ holds for each $t, s > 0$. Then,

$$\sigma_2(t,s) = s - \frac{R(t,s)}{T(t,s)}t \quad \text{for each } t, s \in \mathbb{R}_0^+. \tag{3}$$

(iii) $\sigma_3(t,s) = s - \phi_3(s) - t$ for each $t, s \in [0, \infty)$.

(iv) For each $s, t \in \mathbb{R}_0^+$,

$$\sigma_4(t,s) = s\varphi(s) - t, \tag{4}$$

where $\varphi : \mathbb{R}_0^+ \to [0,1)$ with $\limsup_{t \to r^+} \varphi(t) < 1$ for each $r > 0$.

(v) For each $s, t \in \mathbb{R}_0^+$

$$\sigma_5(t,s) = \eta(s) - t, \tag{5}$$

where $\eta \in \Phi$ and it is upper semicontinuous.

(vi) For each $s, t \in [0, \infty)$,

$$\sigma_6(t,s) = s - \int_0^t \phi(u)\,du, \tag{6}$$

where ϕ is a self-mapping on $\mathbb{R}_0^+$ with the following properties:

(1) $\int_0^\varepsilon \phi(u)du$ exists,

(2) for every $\varepsilon > 0$, $\int_0^\varepsilon \phi(u)du > \varepsilon$.

For the further attracted simulation function examples see, for example, [1, 3, 4].

In 1993 Czerwik [5] proposed a more general frame for the notion of standard metric, so called a b-metric.

Definition 3. For $M \neq \emptyset$, let $d_b : M \times M \to \mathbb{R}_0^+$ be a function satisfying the following conditions:

(1) $d_b(p,q) = 0$ if and only if $p = q$.

(2) $d_b(p,q) = d_b(q,p)$ for each $p, q \in M$.

(3) $d_b(p,q) \le s[d_b(p,r) + d_b(r,q)]$ for each $p, q, r \in M$, where $s \ge 1$.

Here, d_b is called a b-metric. Further, the triple (M, d_b, s) is called a b-metric space.

For the special case of $s = 1$, the notion of b-metric turns into the standard metric. Consequently, the notion of b-metric is more general than the standard metric.

For the sake of completeness, we recollect standard but interesting three examples of b-metric spaces; see, for example, [6, 7] and the related references therein.

Example 4. Let $M = \mathbb{R}$. For all $p, q \in M$, we define a function d_b as

$$d_b(p,q) = |p - q|^2. \tag{7}$$

Then, d_b is a b-metric on real numbers. The first two axioms are fulfilled in a straightforward way. The last axiom is satisfied for $s = 2$:

$$|p - q|^2 \le 2\left[|p - r|^2 + |r - q|^2\right]. \tag{8}$$

Example 5. For a fixed $p \in (0,1)$, consider

$$M = l_p(\mathbb{R}) = \left\{ t = \{t_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |t_n|^p < \infty \right\}. \tag{9}$$

We introduce the corresponding distance functions as

$$d_b(t,s) = \left(\sum_{n=1}^{\infty} |t_n - s_n|^p \right)^{1/p} \quad \text{for each } t, s \in M. \tag{10}$$

Then, (M, d_b, s) forms a b-metric space with the constant $s = 2^{1/p}$.

Example 6. Suppose that E is a Banach space with the zero vector 0_E of E. Take P as a cone in E such that $\text{int}(P) \neq \emptyset$ and further, $\preceq$ is partial ordering with respect to P. For a nonempty set M, we define a mapping $c : M \times M \to E$ as follows:

$(M1)$ $0 \preceq c(p,q)$ for each $p, q \in M$.

$(M2)$ $c(p,q) = 0$ if and only if $p = q$.

$(M3)$ $c(p,q) \preceq c(x,z) + c(z,y)$, for each $p, q \in M$.

$(M4)$ $c(p,q) = c(q,p)$ for each $p, q \in M$.

Then, the mapping c is called cone metric on M. Moreover, the pair (M, c) is said to be a cone metric space.

If a normal cone P in E is normal with the normality constant K, then, the mapping $D : M \times M \to \mathbb{R}_0^+$, defined by $D_b(x,y) = \|c(x,y)\|$, forms a b-metric space where the function $c : M \times M \to E$ is a cone metric. Moreover, the triple (M, D_b, s) forms a b-metric space with the constant $s := K \ge 1$.

Suppose that (M, d_b, s) is a b-metric space. A self-mapping T on M is said to be a *$\mathcal{S}$-contraction* with respect to σ [1], if the following inequality is fulfilled:

$$\sigma\left(d_b(Tp, Tq),\ d_b(p,q)\right) \ge 0 \qquad \text{for each } p, q \in M,\ \sigma \in \mathcal{S}. \tag{11}$$

On account of (σ_2), we derive that

$$d_b(Tp, Tq) \neq d_b(p,q) \quad \text{for each distinct } p, q \in M. \tag{12}$$

Taking (12) into account, we find that T cannot be an isometry whenever T is a $\mathcal{S}$-contraction. Moreover, if T is a $\mathcal{S}$-contraction in the setting of b-metric space with a fixed point, then the desired fixed point is necessarily unique.

Theorem 7. *In a complete b-metric space, each $\mathcal{S}$-contraction has a unique fixed point.*

This theorem can be stated also as follows: each $\mathcal{S}$-contraction yields a Picard sequence that converges to a unique fixed point.

For a family $\Psi := \{\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+\}$, if the following two conditions are fulfilled,

(i) each function $\psi \in \Psi$ is nondecreasing;

(ii) there exist $a \in (0,1)$ and $k_0 \in \mathbb{N}$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that for any $t \in \mathbb{R}^+$ and for $k \geq k_0$ we have

$$\psi^{k+1}(t) \leq a\psi^k(t) + v_k. \quad (13)$$

Here, Ψ is called the class of (c)-comparison functions (see [8]). For a $\psi \in \Psi$ the notation ψ^n indicates the nth iteration of the function ψ. The following lemma is recollected from [8].

Lemma 8. *For a $\psi \in \Psi$, we have*

(i) *for each $t \in \mathbb{R}^+$, the sequence $(\psi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$;*

(ii) *for any $t \in \mathbb{R}^+$, the inequalities $\psi(t) < t$ are fulfilled;*

(iii) *each auxiliary function ψ is continuous at 0;*

(iv) *for any $t \in \mathbb{R}^+$, the series $\sum_{k=1}^{\infty} \psi^k(t)$ is convergent.*

Berinde [9] characterized (c)-comparison functions to use for the contraction mappings in the setting of b-metric spaces, as follows.

Definition 9. Fix a real number $s \geq 1$. An increasing function $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be (b)-comparison if there exist a convergent nonnegative series $\sum_{k=1}^{\infty} v_k$, $k_0 \in \mathbb{N}$, and $a \in (0,1)$ such that for any $t \geq 0$ and for $k \geq k_0$,

$$s^{k+1}\phi^{k+1}(t) \leq as^k\phi^k(t) + v_k. \quad (14)$$

Lemma 10 (see [10]). *Let $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a (b)-comparison function. Define a self-mapping b_s on $\mathbb{R}_0^+$ as $b_s = \sum_{k=0}^{\infty} s^k\phi^k(t)$:*

(1) *For any $t \in \mathbb{R}_0^+$, the series $\sum_{k=0}^{\infty} s^k\phi^k(t)$ is convergent.*

(2) *The function b_s is increasing and continuous at $t = 0$.*

Remark 11. It is obvious that each (b)-comparison function is a comparison function. Consequently, on account of Lemma 8, we deduce that any (b)-comparison function ψ satisfies the inequality $\psi(t) < t$.

Popescu [11] introduces the notion of the α-orbital admissible as follows.

Definition 12 (see [11]). Suppose that T is a self-mapping over a nonempty set M and $\alpha : M \times M \to \mathbb{R}_0^+$ is a function. The mapping T is called an α-orbital admissible if the following implication is provided:

$$\alpha(x, Tx) \geq 1 \Longrightarrow \alpha\left(Tx, T^2x\right) \geq 1. \quad (15)$$

We should mention the notion of the α-orbital admissible [11, 12] inspired from the notion of the α-orbital admissible notion defined in [13, 14].

In this paper, by combining the notion of the simulation function together with the admissible functions, we shall consider a new type contractive mapping in the frame of complete b-metric spaces. Accordingly, our results improve and extend the main results in [15] in twofold: first, we investigate the existence and uniqueness of a fixed point in b-metric spaces instead of standard metric space. Secondly, we extend the condition $\sigma(\alpha(p,q)d_b(Tp,Tq),(d_b(p,q)) \geq 0$ for each $p,q \in M$ by adding an auxiliary function ϕ into account. Consequently, we investigate the existence and uniqueness of a fixed point in the new extended condition $\sigma(\alpha(p,q)d_b(Tp,Tq),\phi(d_b(p,q))) \geq 0$ for each $p,q \in M$. We illustrate that the class of the new contractive mapping covers several well-known contractive mappings.

2. Main Results

We start this section by defining the $(\alpha - \phi)$-type $\mathcal{S}$-*contraction* which is a generalization of the notion of $\mathcal{S}$-*contraction*.

Definition 13. Let M be a nonempty set $s \geq 1$ and $\alpha : M \times M \to [0,\infty)$ be function. Suppose that T is a self-mapping defined over a b-metric space (M, d_b, s). The self-mapping T is called an $(\alpha - \phi)$-type $\mathcal{S}$-*contraction* with respect to σ if there are $\sigma \in \mathcal{S}$ and $\phi \in \Phi$ such that

$$\sigma\left(\alpha(p,q)\, d_b(Tp,Tq), \phi\left(d_b(p,q)\right)\right) \geq 0 \quad \text{for each } p,q \in M. \quad (16)$$

Before stating our main theorem, we shall give lemmas that have a crucial role in the proof of the main result.

Lemma 14. *Let M be a nonempty set. Suppose that $\alpha : M \times M \to \mathbb{R}_0^+$ is a function and $T : M \to M$ is an α-orbital admissible mapping. If there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geq 1$ and $p_n = Tp_{n-1}$ for $n = 0, 1, \ldots$, then, we have*

$$\alpha\left(p_n, p_{n+1}\right) \geq 1, \quad \text{for each } n = 0, 1, \ldots. \quad (17)$$

Proof. On account of the assumptions of the theorem, there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geq 1$. Owing to the fact that T is α-orbital admissible, we find

$$\alpha\left(p_0, p_1\right) = \alpha\left(p_0, Tp_0\right) \geq 1 \Longrightarrow \alpha\left(Tp_0, Tp_1\right) = \alpha\left(p_1, p_2\right) \geq 1. \quad (18)$$

By iterating the above inequality, we derive that

$$\alpha\left(p_n, p_{n+1}\right) = \alpha\left(Tp_{n-1}, Tp_n\right) \geq 1, \quad \text{for each } n = 0, 1, \ldots. \quad (19)$$

□

Theorem 15. *Let M be a nonempty set, $s \geq 1$, and $\alpha : M \times M \to \mathbb{R}_0^+$ be a function. Suppose that a continuous self-mapping T over a complete b-metric space (M, d_b, s) is α-orbital admissible. Suppose also the mapping T forms an $(\alpha - \phi)$-type $\mathcal{S}$-contraction with respect to σ. If there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geq 1$, then there exists $p \in M$ such that $Tp = p$.*

Proof. Based on the assumption, there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geq 1$. We shall construct an iterative sequence $\{p_n\}$ in M by setting $p_{n+1} = Tp_n$ for each $n \geq 0$. By Lemma 14, we have (17); that is,

$$\alpha(p_n, p_{n+1}) \geq 1, \quad \text{for each } n = 0, 1, \ldots. \tag{20}$$

Taking (16) and (17) into account, for each $n \geq 1$, we derive that

$$\begin{aligned} 0 &\leq \sigma(\alpha(p_n, p_{n-1}) d_b(Tp_n, Tp_{n-1}), \phi(d_b(p_n, p_{n-1}))) \\ &= \sigma(\alpha(p_n, p_{n-1}) d_b(p_{n+1}, p_n), \phi(d_b(p_n, p_{n-1}))) \\ &< \phi(d_b(p_n, p_{n-1})) - \alpha(p_n, p_{n-1}) d_b(p_{n+1}, p_n). \end{aligned} \tag{21}$$

Accordingly, from (16) and (21) we conclude that

$$\begin{aligned} d_b(p_n, p_{n+1}) &\leq \alpha(p_n, p_{n-1}) d_b(p_n, p_{n+1}) \\ &\leq \phi(d_b(p_n, p_{n-1})) < d_b(p_n, p_{n-1}) \\ &\qquad \text{for each } n = 1, 2, \ldots. \end{aligned} \tag{22}$$

Hence, we conclude that the constructive sequence $\{d_b(p_n, p_{n-1})\}$ is bounded from below by zero, and moreover, it is nondecreasing. Hereby, there exists $\theta \geq 0$ such that $\lim_{n\to\infty} d_b(p_n, p_{n-1}) = \theta \geq 0$. We shall indicate that

$$\lim_{n\to\infty} d_b(p_n, p_{n-1}) = 0. \tag{23}$$

Suppose, on the contrary, that $\theta > 0$. On account of inequality (22), we find that

$$\begin{aligned} \lim_{n\to\infty} \alpha(p_n, p_{n-1}) d_b(p_n, p_{n+1}) &= \theta, \\ \lim_{n\to\infty} \phi(d_b(p_n, p_{n-1})) &= \theta. \end{aligned} \tag{24}$$

Taking $s_n = \alpha(p_n, p_{n-1}) d_b(p_n, p_{n+1})$ and $t_n = d_b(p_n, p_{n-1})$ together with the condition (σ_3), we derive that

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \sigma(\alpha(p_n, \\ &\quad p_{n-1}) d_b(p_{n+1}, p_n), \phi(d_b(p_n, p_{n-1}))) \\ &< 0, \end{aligned} \tag{25}$$

a contradiction. Consequently, we have $\theta = 0$.

In the next step, we shall show that the constructive sequence $\{p_n\}$ is Cauchy. By iteration on the inequality (22), we derive that

$$d_b(p_{n+1}, p_n) \leq \psi^n(d_b(p_1, p_0)), \quad \text{for each } n \geq 1. \tag{26}$$

From (26) and using the triangular inequality, for each $p \geq 1$, we have

$$\begin{aligned} d_b(p_n, p_{n+p}) &\leq s \cdot d_b(p_n, p_{n+1}) + s^2 \cdot d_b(p_{n+1}, p_{n+2}) \\ &\quad + \cdots + s^{p-2} \cdot d_b(p_{n+p-3}, p_{n+p-2}) + s^{p-1} \\ &\quad \cdot d_b(p_{n+p-2}, p_{n+p-1}) + s^p \cdot d_b(p_{n+p-1}, p_{n+p}) \leq s \\ &\quad \cdot d_b(p_0, p_1) + s^2 \cdot d_b(p_0, p_1) + \cdots + s^{p-2} \\ &\quad \cdot d_b(p_0, p_1) + s^{p-1} \cdot d_b(p_0, p_1) + s^p \cdot d_b(p_0, p_1) \\ &= \frac{1}{s^n} \cdot \left[s^{n+1} \cdot d_b(p_0, p_1) + \cdots + s^{n+p-1}\right. \\ &\quad \left.\cdot d_b(p_0, p_1) + s^{n+p} \cdot d_b(p_0, p_1)\right] \leq \frac{1}{s^n} \cdot \left[s^{n+1}\right. \\ &\quad \left.\cdot d_b(p_0, p_1) + \cdots + s^{n+p} \cdot d_b(p_0, p_1)\right] = \frac{1}{s^n} \\ &\quad \cdot \sum_{i=n+1}^{n+p} s^i \cdot d_b(p_0, p_1) < \frac{1}{s^n} \cdot \sum_{i=n+1}^{\infty} s^i \cdot d_b(p_0, p_1). \end{aligned} \tag{27}$$

The precedent inequality is

$$\begin{aligned} d_b(p_n, p_{n+p}) &< \frac{1}{s^n} \cdot \sum_{i=n+1}^{\infty} s^i \cdot d_b(p_0, p_1) \longrightarrow 0 \\ &\qquad \text{as } n \longrightarrow \infty, \end{aligned} \tag{28}$$

which yields that $\{p_n\}$ is a Cauchy sequence in (M, d_b, s). Since (M, d_b, s) is complete, there exists $p \in M$ such that

$$\lim_{n\to\infty} d_b(p_n, p) = 0. \tag{29}$$

Since T is continuous, we obtain from (29) that

$$\lim_{n\to\infty} d_b(p_{n+1}, Tp) = \lim_{n\to\infty} d_b(Tp_n, Tp) = 0. \tag{30}$$

Combining the uniqueness of the limit together with (29) amd (30), we find that p forms a fixed point of T; that is, $Tp = p$. □

The continuity condition can be relaxed in Theorem 15 by replacing a suitable condition like the given below.

Definition 16. Let $s \geq 1$. We say that a b-metric space (M, d_b, s) is *regular* if $\{p_n\}$ is a sequence in M such that $\alpha(p_n, p_{n+1}) \geq 1$ for each n and $p_n \to x \in M$ as $n \to \infty$; then there is a subsequence $\{p_{n(k)}\}$ of $\{p_n\}$ such that $\alpha(p_{n(k)}, x) \geq 1$ for each k.

By removing the continuity condition from the main result, Theorem 15 is possible. But, in this case, we should add the "regularity" condition which is mentioned in Definition 16.

Theorem 17. *Let M be a nonempty set, $s \geq 1$, and $\alpha : M \times M \to \mathbb{R}_0^+$ be a function. Suppose that (M, d_b, s) is regular and* *a self-mapping T on a complete b-metric space (M, d_b, s) is α-orbital admissible. Suppose also the mapping T forms an $(\alpha-\phi)$-type $\mathcal{S}$-contraction with respect to σ. If there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geq 1$, then there exists $p \in M$ such that $Tp = p$.*

Proof. By repeating the steps in the proof of Theorem 15, we guarantee that the sequence $\{p_n\}$ defined by $p_{n+1} = Tp_n$ for each $n \geq 0$ converges for some $p \in M$. From (17) and regularity of the metric, there exists a subsequence $\{p_{n(k)}\}$ of $\{p_n\}$ such that $\alpha(p_{n(k)}, p) \geq 1$ for each k. By implementing (16), for each k, we get that

$$\begin{aligned} 0 &\leq \sigma\left(\alpha\left(p_{n(k)}, p\right) d_b\left(Tp_{n(k)}, Tp\right), \phi\left(d_b\left(p_{n(k)}, p\right)\right)\right) \\ &= \sigma\left(\alpha\left(p_{n(k)}, p\right) d_b\left(p_{n(k)+1}, Tp\right), \phi\left(d_b\left(p_{n(k)}, p\right)\right)\right) \\ &< \phi\left(d_b\left(p_{n(k)}, p\right)\right) - \alpha\left(p_{n(k)}, p\right) d_b\left(p_{n(k)+1}, Tp\right), \end{aligned} \tag{31}$$

which leads to

$$\begin{aligned} d_b\left(p_{n(k)+1}, Tp\right) &= d_b\left(Tp_{n(k)}, Tp\right) \\ &\leq \alpha\left(p_{n(k)}, p\right) d_b\left(Tp_{n(k)}, Tp\right) \\ &\leq \phi\left(d_b\left(p_{n(k)}, p\right)\right) < d_b\left(p_{n(k)}, p\right). \end{aligned} \tag{32}$$

Taking $k \to \infty$ in inequality (32), we deduce that $d_b(u, Tp) = 0$; that is, $p = Tp$. □

Note that, in Theorems 15 and 17, we observe only the existence of the fixed point of the given operator. As a next step, we shall investigate the uniqueness of the obtained fixed point. Let $\mathrm{Fix}(T)$ represent the set of all fixed points of operator T. For this purpose, we need the following additional condition:

$(\mathcal{U})$ $\alpha(p, q) \geq 1$ for each $p, q \in \mathrm{Fix}(T)$.

Theorem 18. *Under the assumption of additional condition $(\mathcal{U})$, the obtained fixed point p of the operator T defined in Theorem 15 (resp., Theorem 17) turns to be unique fixed point.*

Proof. Let T be an α-orbital admissible $\mathcal{S}$-*contraction* with respect to σ. Regarding Theorem 15 or Theorem 17, we guarantee the existence of a fixed point of the mapping T; namely, $p = Tp$. Suppose p is not the unique fixed point of T; thus, there exists q with $p \neq q$. So, we have $d_b(p, q) > 0$. Regarding the condition $(\mathcal{U})$, the definition of T yields that

$$\sigma\left(\alpha\left(p, q\right) d_b\left(Tp, Tq\right), \phi\left(d_b\left(p, q\right)\right)\right) \geq 0. \tag{33}$$

Due to definition of the auxiliary function σ, the inequality above implies that

$$\begin{aligned} d_b\left(p, q\right) &= d_b\left(Tp, Tq\right) \\ &\leq \alpha\left(p, q\right) d_b\left(Tp, Tq\right), \phi\left(d_b\left(p, q\right)\right) \\ &< d_b\left(p, q\right), \end{aligned} \tag{34}$$

which is a contradiction. Thus, p is the unique fixed point of T. □

3. Consequences

3.1. Consequences in the Setting of b-Metric Space. Consider a mapping $\phi_1 \in \phi$ related with an α-orbital admissible $\mathcal{S}$-*contraction* with respect to σ, namely T; that is, $\sigma_E(\alpha(p, q)d_b(Tp, Tq), \phi_1(d_b(p, q))$. For a function $\psi \in \Psi$, we set

$$\sigma_E(t, s) = \psi(s) - t \quad \text{for each } s, t \in \mathbb{R}_0^+. \tag{35}$$

It is straightforward that σ_{BW} is a simulation function. Combining the observations above, we have

$$\begin{aligned} &\sigma_E \alpha\left(p, q\right) d_b\left(Tp, Tq\right), \phi_1\left(d_b\left(p, q\right)\right) \\ &\quad = \psi\left(\phi_1\left(d_b\left(p, q\right)\right)\right) - \alpha\left(p, q\right) d_b\left(Tp, Tq\right) \\ &\qquad \text{for each } s, t \in \mathbb{R}_0^+, \end{aligned} \tag{36}$$

which is equivalent to

$$\alpha\left(p, q\right) d_b\left(Tp, Tq\right) \leq \phi\left(d_b\left(p, q\right)\right), \quad \text{for each } p, q \in M, \tag{37}$$

where

$$\phi(t) \coloneqq \psi\left(\phi_1(t)\right) \in \Phi. \tag{38}$$

Thus, the above sample of simulation function together with Theorem 18 yields the following result.

Theorem 19. *Let (M, d) be a complete b-metric space and let $T : M \to M$ be defined as*

$$\alpha\left(p, q\right) d_b\left(Tp, Tq\right) \leq \phi\left(d_b\left(p, q\right)\right), \quad \text{for each } p, q \in M, \tag{39}$$

where $\alpha : M \times M \to \mathbb{R}_0^+$ and $\phi \in \Phi$. Suppose that

(i) *T is α-orbital admissible;*

(ii) *there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geq 1$;*

(iii) *either T is continuous or (M, d) is regular.*

Then there exists $p \in M$ such that $Tp = p$. Moreover, if the condition $(\mathcal{U})$ is fulfilled, then we guarantee that the obtained fixed point p of T is unique.

Letting $\phi(t) = kt$ with $k \in [0, 1)$, Theorem 19 implies the following.

Theorem 20. *Let (M, d) be a complete b-metric space and let $T : M \to M$ be defined as*

$$\alpha\left(p, q\right) d_b\left(Tp, Tq\right) \leq k d_b\left(p, q\right), \quad \text{for each } p, q \in M, \tag{40}$$

where $k \in [0, 1)$, $\alpha : M \times M \to \mathbb{R}_0^+$, and $\phi \in \Phi$. Suppose that

(i) *T is α-orbital admissible;*

(ii) *there exists $p_0 \in M$ such that $\alpha(p_0, Tp_0) \geq 1$;*

(iii) *either T is continuous or (M, d) is regular.*

Then there exists $p \in M$ such that $Tp = p$. Moreover, if the condition $(\mathcal{U})$ is fulfilled, then we guarantee that the obtained fixed point p of T is unique.

By letting $\alpha(p, q) = 1$ in Theorems 19 and 20 we get the main results of Theorems 1 and 2 of Czerwik [5]. Notice that, in this case, conditions (i)–(iii) of Theorems 19 and 20 are fulfilled trivially.

3.2. Consequences in the Setting of Standard Metric Space. In this section, we consider the results in the setting of standard metric. Thus, we consider $s = 1$ throughout this section. We shall show that a number of existing fixed point results in the literature are the simple consequence of our main results. In particular, by taking Example 2 into consideration, we can list many well-known results as a consequence of our main results.

If $\psi \in \Psi$ and we define

$$\sigma_E(t, s) = \psi(s) - t \quad \text{for each } s, t \in \mathbb{R}_0^+, \tag{41}$$

then σ_{BW} is a simulation function (cf. Example 2 (v)).

First, we derive the very interesting recent results of Samet et al. [13] as a corollary of Theorem 18.

Theorem 21. *Theorems 2.1 and 2.2 in [13] are consequences of the following.*

Proof. Taking $\sigma_E(r, s) = \psi(s) - r$ for each $s, r \in [0, \infty)$ in Theorem 18, we derive that

$$\alpha(p, q) d_b(Tp, Tq) \leq \psi(d_b(p, q)), \quad \text{for each } p, q \in M. \tag{42}$$

We skip the details. □

As is well known, the main theorem in [13] covers several fixed point results, including the pioneer fixed point theorem of Banach. Moreover, as is shown in [13, 16], several fixed point theorems in different settings (in the sense of partially ordered set, in the sense of cyclic mapping, etc.) can be concluded from Theorems 2.1 and 2.2 in [13] by setting $\alpha(p, q)$ in a proper way.

Notice also that one can express the main result of Khojasteh et al. [1] as a straight consequence of our main result.

Theorem 22. *Theorem 18 yields Theorem 7.*

Proof. It is sufficient to take $\alpha(p, q) = 1$ for each $p, q \in M$ in Theorem 21. □

It is obvious that all presented results in [1] follow from our main result.

4. Conclusion

It is very easy to see that one can list a further outcome of our main results by letting the mappings $\sigma, \alpha, \psi, \phi$ in a suitable way like in Example 2. More precisely, by following the techniques in [13, 16] one can easily derive a number of well-known fixed point results in the distinct settings (such as in the frame of *cyclic contraction* and in the setting of *partially ordered set endowed with a metric*). We prefer not to list all consequences due to our concerns on the length of the paper. This paper can be also considered as a continuation of the recent paper [17].

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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