Research Article

Periodicities of a System of Difference Equations

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We study the periodicities of a system of difference equations $x_{n+1} = \max\{1/x_n, A_n/y_{n-k}\}$, $y_{n+1} = \max\{1/y_n, B_n/x_{n-k}\}$, where initial values $(x_{-k}, y_{-k}), \dots, (x_0, y_0) \in (0, +\infty) \times (0, +\infty)$. We show that if $A_n, B_n \in (0, 1)$ are two periodic sequences, then every solution of the above system is eventually periodic with period 2. If k is even, there must be one in $\{x_n\}$ and $\{y_n\}$ converges to period two solution.

1. Introduction

Difference equations are powerful tool that describe the law of nature. Recently, the theory of difference equations has received extensive attention because it can be applied to many areas of science and technology, such as the fields of information and e-science. Moreover, people pay more attention to the dynamics of max-type difference equations; they are concerned about the research of the period character of solutions and the convergence of positive solutions. Please refer to [1–7].

Szalkai [8] studied the max-type difference equation

$$x_n = \max \left\{ \frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}, \dots, \frac{A_k}{x_{n-k}} \right\}, \quad n = 0, 1, 2, \dots, \quad (1)$$

where A_i ($i = 1, 2, \dots, k$) $\in \mathbb{R}$, and at least one of them is nonzero. The initial values $x_0, x_{-1}, \dots, x_{-k}$ are nonzero real numbers, so every solution of this equation is periodic solution. Chen [9] studied the dynamics of max-type difference equation

$$x_{n+1} = \max \left\{ \frac{1}{x_n}, \frac{A_n}{x_{n-1}} \right\}, \quad n = 0, 1, 2, \dots, \quad (2)$$

where $\{A_n\}_{n=0}^\infty = \{\dots, \alpha, \beta, \alpha, \beta, \dots\}$ is a two-cycle sequence, $\beta > \alpha > 1$. The initial values x_0, x_{-1} are nonzero real numbers; then every positive solution of this equation is bounded and persists, and $x_n \in [\sqrt{\alpha}/\beta, \beta\sqrt{\alpha}]$ for all $n \geq 0$.

Furthermore, every solution of this equation is periodic with period 4. Yang [10] studied the max-type difference equation

$$x_n = \max \left\{ \frac{1}{x_{n-1}^\alpha}, \frac{A}{x_{n-2}} \right\}, \quad n = 0, 1, 2, \dots, \quad (3)$$

where $0 < \alpha < 1$, $A > 0$; then every solution of this equation tends to be $\bar{x} = 1$ or is eventually periodic with period 4. Sun et al. [11] studied the dynamics of max-type difference equation

$$x_{n+1} = \max \left\{ \frac{1}{x_n}, \frac{A_n}{x_{n-1}} \right\}, \quad n = 0, 1, 2, \dots, \quad (4)$$

where $\{A_n\}_{n=0}^\infty$ is a k -cycle sequence with $k \geq 2$. The initial values x_0, x_{-1} are positive real numbers. Let $\limsup_{n \rightarrow \infty} x_n = S$; if there exist infinitely many n such that $x_n \geq S$ and $x_{n+1} \geq S$, then $\{x_n\}$ is eventually equal to 1. every positive solution of this equation is eventually periodic with period 2 provided $A_n \in (0, 1)$.

Motivated by the work of above max-type difference equations, the aim of this paper is to investigate the periodicities of a system of max-type difference equations

$$\begin{aligned} x_{n+1} &= \max \left\{ \frac{1}{x_n}, \frac{A_n}{y_{n-k}} \right\}, \\ y_{n+1} &= \max \left\{ \frac{1}{y_n}, \frac{B_n}{x_{n-k}} \right\}, \end{aligned} \quad (5)$$

where initial values $(x_{-k}, y_{-k}), (x_{-k+1}, y_{-k+1}), \dots, (x_0, y_0) \in (0, +\infty) \times (0, +\infty)$, $k \in \mathbb{N}^+$, and A_n, B_n are positive sequences. More precisely, we will prove that if $A_n, B_n \in (0, 1)$ are two periodic sequences, then every solution of (5) is eventually periodic with period 2. if k is even, then one of $\{x_n\}$ and $\{y_n\}$ converges to a period two solution.

2. Main Results

Before proceeding with the proof of our main results in this section, we shall need the following lemmas.

Lemma 1. Let $\{(x_n, y_n)\}_{n=-k}^{+\infty}$ be a solution of system (5); then $x_{n+1}x_n \geq 1$, $y_{n+1}y_n \geq 1$, and $x_{n+1} + x_n \geq 2$, $y_{n+1} + y_n \geq 2$ for all $n \geq 0$.

Proof. By system (5), we have

$$\begin{aligned} x_{n+1} &\geq \frac{1}{x_n}, \\ y_{n+1} &\geq \frac{1}{y_n} \end{aligned} \quad (6)$$

for all $n \geq 0$. Hence $x_{n+1}x_n \geq 1$, $y_{n+1}y_n \geq 1$, and $x_{n+1} + x_n \geq 2\sqrt{x_{n+1}x_n} \geq 2$, $y_{n+1} + y_n \geq 2\sqrt{y_{n+1}y_n} \geq 2$. This completes the proof of the lemma. $\square$

Lemma 2. Let $\{(x_n, y_n)\}_{n=-k}^{+\infty}$ be a solution of system (5). If $n > k$, then

$$\begin{aligned} x_{n+1} &\leq \max\{x_{n-1}, A_n y_{n-k-1}\} \leq \max\{x_{n-1}, y_{n-k-1}\}, \\ y_{n+1} &\leq \max\{y_{n-1}, B_n x_{n-k-1}\} \leq \max\{y_{n-1}, x_{n-k-1}\}. \end{aligned} \quad (7)$$

Proof. By Lemma 1 and $A_n < 1$, we obtain that, for any $n > k$,

$$\begin{aligned} x_{n+1} &= \max\left\{\frac{x_{n-1}}{x_n x_{n-1}}, \frac{A_n y_{n-k-1}}{y_{n-k} y_{n-k-1}}\right\} \\ &\leq \max\{x_{n-1}, A_n y_{n-k-1}\} \leq \max\{x_{n-1}, y_{n-k-1}\}. \end{aligned} \quad (8)$$

Similarly, we also obtain that, for any $n > k$,

$$\begin{aligned} y_{n+1} &= \max\left\{\frac{y_{n-1}}{y_n y_{n-1}}, \frac{B_n x_{n-k-1}}{x_{n-k} x_{n-k-1}}\right\} \\ &\leq \max\{y_{n-1}, B_n x_{n-k-1}\} \leq \max\{y_{n-1}, x_{n-k-1}\}. \end{aligned} \quad (9)$$

This completes the proof of the lemma. $\square$

Theorem 3. Let $\{(x_n, y_n)\}_{n=-k}^{+\infty}$ be a solution of system (5) and $r_{n+1} = \max\{x_n, x_{n-1}, \dots, x_{n-k-1}, y_n, y_{n-1}, \dots, y_{n-k-1}\}$. Then

- (1) $\{r_n\}$ is eventually monotonically decreasing. Moreover, if $T = \lim_{n \rightarrow \infty} r_n$, then $T \geq 1$;
- (2) $\max\{\limsup_{n \rightarrow \infty} x_n, \limsup_{n \rightarrow \infty} y_n\} = T$.

Proof. (1) By Lemma 2, if $n > k$, we have

$$\begin{aligned} x_{n+1} &\leq \max\{x_{n-1}, y_{n-k-1}\} \\ &\leq \max\{x_n, x_{n-1}, \dots, x_{n-k-1}, y_n, y_{n-1}, \dots, y_{n-k-1}\} \\ &= r_{n+1}. \end{aligned} \quad (10)$$

Similarly, we obtain $y_{n+1} \leq r_{n+1}$. Hence

$$\begin{aligned} r_{n+2} &= \max\{x_{n+1}, x_n, \dots, x_{n-k}, y_{n+1}, y_n, \dots, y_{n-k}\} \\ &\leq r_{n+1}, \end{aligned} \quad (11)$$

so $\{r_n\}$ is eventually monotonically decreasing. On the other hand, by Lemma 1, we know $x_{n+1}x_n \geq 1$, $y_{n+1}y_n \geq 1$, so $r_n \geq 1$. Hence $T \geq 1$.

(2) It is easy to know that there are infinitely many n such that $x_n = r_{n+1}$ or $y_n = r_{n+1}$, so $\limsup_{n \rightarrow \infty} x_n \geq T$, or $\limsup_{n \rightarrow \infty} y_n \geq T$. On the other hand, $x_n \leq r_{n+1}$ and $y_n \leq r_{n+1}$. Hence $\limsup_{n \rightarrow \infty} x_n \leq T$ and $\limsup_{n \rightarrow \infty} y_n \leq T$. Thus

$$\max\left\{\limsup_{n \rightarrow \infty} x_n, \limsup_{n \rightarrow \infty} y_n\right\} = T. \quad (12)$$

This completes the proof of the theorem. $\square$

Lemma 4. Let $a = \max\{A_n, B_n : n \in \mathbb{N}^+\}$ and $\eta = ((1-a)/2)T$. Then there exists $N \in \mathbb{N}^+$ such that

$$\begin{aligned} x_{n+1} &\leq \max\{x_{n-1}, T - \eta\}, \\ y_{n+1} &\leq \max\{y_{n-1}, T - \eta\}, \end{aligned} \quad (13)$$

for all $n \geq N$.

Proof. Let $\eta_1 = ((1-a)/2a)T$. By Theorem 3, there exists $N \in \mathbb{N}^+$ such that $n - k - 1 \geq N$:

$$y_{n-k-1} < T + \eta_1 = \frac{1+a}{2a}T. \quad (14)$$

Then $A_n y_{n-k-1} < a((1+a)/2a)T = T - ((1-a)/2)T = T - \eta$. By Lemma 2, it follows that

$$x_{n+1} \leq \max\{x_{n-1}, T - \eta\}. \quad (15)$$

Similarly, we obtain $y_{n+1} \leq \max\{y_{n-1}, T - \eta\}$. This completes the proof of the lemma. $\square$

Lemma 5. There exists a positive integer $N_1 \in \mathbb{N}^+$ such that, for all $t \geq 0$, $x_{N_1+2t} \geq x_{N_1+2t+2} \geq T$ (if $\limsup_{n \rightarrow \infty} x_n = T$), or $y_{N_1+2t} \geq y_{N_1+2t+2} \geq T$ (if $\limsup_{n \rightarrow \infty} y_n = T$).

Proof. Let N and η be as Lemma 4. By Theorem 3, if $\limsup_{n \rightarrow \infty} x_n = T$, then there exists a monotonically increasing sequence $n_t \rightarrow +\infty$ such that $n_{n_t} \geq T$. On the other hand, there exists a subsequence $\{n_{t_j}\}$ of $\{n_t\}$ such that $n_{t_j} = 2l_{t_j} + r$ ($r \in \{0, 1\}$). Without loss of generality, we may assume that $\exists N \in \mathbb{N}^+$, $n_t = 2l_t + r \geq N$ (for all n_t). By Lemma 4, we get

$$\begin{aligned} T &\leq x_{n_t} = x_{2l_t+r} \leq \max\{x_{2(l_t-1)+r}, T - \eta\} \\ &= x_{2(l_t-1)+r} \leq \max\{x_{2(l_t-2)+r}, T - \eta\} \\ &= x_{2(l_t-2)+r} \leq \max\{x_{2(l_t-3)+r}, T - \eta\} \\ &\dots = x_{2l_{t-1}+r}. \end{aligned} \quad (16)$$

Write $N_1 = 2l_1 + r$. Then $x_{N_1+2t} \geq x_{N_1+2t+2} \geq T$ for all $t \geq 0$.

Similarly, we obtain that if $\limsup_{n \rightarrow \infty} y_n = T$, then $y_{N_1+2t} \geq y_{N_1+2t+2} \geq T$. This completes the proof of the Lemma. $\square$

Lemma 6. (1) If $\limsup_{n \rightarrow \infty} x_n = T$, then there exists an integer $N \in \mathbb{N}^+$ such that both the number sequences $\{x_{N+2t}\}$ and $\{y_{N+2t+k+1}\}$ are monotonically decreasing, and $x_{N+2t}x_{N+2t-1} = 1$, $y_{N+2t+k}y_{N+2t+k+1} = 1$ for all $t \geq 0$.

(2) If $\limsup_{n \rightarrow \infty} y_n = T$, then there exists an integer $N \in \mathbb{N}^+$ such that both the number sequences $\{y_{N+2t}\}$ and $\{x_{N+2t+k+1}\}$ are monotonically decreasing, and $y_{N+2t}y_{N+2t-1} = 1$, $x_{N+2t+k}x_{N+2t+k+1} = 1$ for all $t \geq 0$.

Proof. Here, we only prove (1). By Lemma 5, if $\limsup_{n \rightarrow \infty} x_n = T$ and $\limsup_{n \rightarrow \infty} y_n = M \leq T$, then there exists $N \in \mathbb{N}^+$ such that $x_{N+2t} \geq x_{N+2t+2} \geq T$ for all $t \geq 0$. So $\{x_{N+2t}\}$ is monotonically decreasing for all $t \geq 0$. There exists a constant $\varepsilon > 0$ such that $ay_{N+2t-k} \leq T - \varepsilon$. Hence,

$$\begin{aligned} \frac{1}{x_{N+2t-1}} &\leq x_{N+2t} = \max \left\{ \frac{1}{x_{N+2t-1}}, \frac{A_{N+2t-1}}{y_{N+2t-k-1}} \right\} \\ &\leq \max \left\{ \frac{1}{x_{N+2t-1}}, by_{N+2t-1} \right\} \\ &\leq \max \left\{ \frac{1}{x_{N+2t-1}}, T - \varepsilon \right\} \leq \frac{1}{x_{N+2t-1}}. \end{aligned} \quad (17)$$

This implies $x_{N+2t}x_{N+2t-1} = 1$. Without loss of generality, we assume $y_{N+2t+k} < T/a$, so

$$\begin{aligned} \frac{1}{y_{N+2t+k}} &\leq y_{N+2t+k+1} = \max \left\{ \frac{1}{y_{N+2t+k}}, \frac{B_{N+2t+k}}{x_{N+2t}} \right\} \\ &= \frac{1}{y_{N+2t+k}} \leq y_{N+2t+k-1}. \end{aligned} \quad (18)$$

Hence, $y_{N+2t+k}y_{N+2t+k+1} = 1$, and $\{y_{N+2t+k+1}\}$ is monotonically decreasing for all $t \geq 0$. This completes the proof of the Lemma. $\square$

Theorem 7. Let $\{(x_n, y_n)\}_{n=-k}^{+\infty}$ be a solution of system (5). If k is even, then one of the $\{x_n\}$ and $\{y_n\}$ is eventually periodic with period 2.

Proof. By Lemma 6, if $\limsup_{n \rightarrow \infty} x_n = T$, then there exists $N \in \mathbb{N}^+$ such that both $\{x_{N+2t}\}$ and $\{y_{N+2t+k+1}\}$ are monotonically decreasing. We can easily get $x_{N+2t}x_{N+2t-1} = 1$ and $y_{N+2t+k}y_{N+2t+k+1} = 1$ for all $t \geq 0$.

We will prove that $\{y_n\}$ is eventually periodic with period 2. Otherwise, there will be infinitely many k_j such that

$$\begin{aligned} y_{N+2k_j+1} &= \max \left\{ \frac{1}{y_{N+2k_j}}, \frac{B_{N+2k_j}}{x_{N+2k_j-k}} \right\} = \frac{B_{N+2k_j}}{x_{N+2k_j-k}} \\ &> \frac{1}{y_{N+2k_j}}. \end{aligned} \quad (19)$$

We take a subsequence $\{B_{N+2k_j}\}$ and assume $B_{N+2k_j} \rightarrow B \leq a$, $y_{N+2k_j} \rightarrow L$. Since $-k$ is even, $x_{N+2k_j-k} \rightarrow T$. Take the limit on both sides of the inequality $B_{N+2k_j}/x_{N+2k_j-k} > 1/y_{N+2k_j}$; hence $BL > T$. Since $T \geq L$, $1 > a \geq B$, we have $aT \geq BL$, so

$T > aT \geq BL > T$. This is a contradiction. Therefore $\{y_n\}$ is eventually periodic with period 2.

Similarly, we obtain that if $\limsup_{n \rightarrow \infty} y_n = T$, then $\{x_n\}$ is eventually periodic with period 2. This completes the proof of the theorem. $\square$

Theorem 8. If both A_n and B_n are periodic sequences, then the solution $\{(x_n, y_n)\}_{n=-k}^{+\infty}$ of system (5) converges to a period two solution.

Proof. We will only prove the case of $\limsup_{n \rightarrow \infty} x_n = T$. By Lemma 6, we know that if $\limsup_{n \rightarrow \infty} x_n = T$, then there exists $N \in \mathbb{N}^+$ such that both $\{x_{N+2t}\}$ and $\{y_{N+2t+k+1}\}$ are monotonically decreasing. We can get $x_{N+2t}x_{N+2t-1} = 1$ and $y_{N+2t+k}y_{N+2t+k+1} = 1$ for all $t \geq 0$ easily. So both $\{x_{N+2t-1}\}$ and $\{y_{N+2t+k}\}$ are monotonically increasing.

Now, we prove that $\{x_n\}$ is eventually periodic with period 2. Assume that $\{x_n\}$ is not eventually periodic with period 2; then $\{x_{N+2t-1}\}$ is not an eventual constant, and there exist infinitely many $k_j \rightarrow +\infty$ such that

$$\begin{aligned} x_{N+2k_j-1} &= \max \left\{ \frac{1}{x_{N+2k_j-2}}, \frac{A_{N+2k_j-2}}{y_{N+2k_j-k-2}} \right\} \\ &= \frac{A_{N+2k_j-2}}{y_{N+2k_j-k-2}} > \frac{1}{x_{N+2k_j-2}}. \end{aligned} \quad (20)$$

Therefore

$$A_{N+2k_j-2} = x_{N+2k_j-1}y_{N+2k_j-k-2}. \quad (21)$$

By Lemma 6, we know that $\{x_{N+2k_j-1}y_{N+2k_j-k-2}\}$ is monotonically increasing, not an eventual constant, so $\{A_{N+2k_j-2}\}$ is monotonically increasing, not an eventual constant. This is a contradiction. So $\{x_n\}$ is eventually periodic with period 2.

Similarly, we obtain that $\{y_n\}$ is eventually periodic with period 2. So $\{(x_n, y_n)\}_{n=-k}^{+\infty}$ converges to a period two solution. This completes the proof of the theorem. $\square$

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors read and approved the final manuscript.

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