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Research Article

On Harmonically (p, h, m)-Preinvex Functions

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We define a new generalized class of harmonically preinvex functions named harmonically (p, h, m)-preinvex functions, which includes harmonic (p, h)-preinvex functions, harmonic p-preinvex functions, harmonic h-preinvex functions, and m-convex functions as special cases. We also investigate the properties and characterizations of harmonically (p, h, m)-preinvex functions. Finally, we establish some integral inequalities to show the applications of harmonically (p, h, m)-preinvex functions.

1. Introduction

Theory of convex functions had not only stimulated new and deep results in many branches of mathematical and engineering sciences, but also provided us with a unified and general framework to study a wide class of unrelated problems. For applications, generalizations, and other aspects of convex functions, see [1–10] and the references therein.

It is well known that a function is a convex function on an interval I, if and only if, the function f satisfies the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

which is known as the Hermite-Hadamard inequality. For the applications in various fields of pure and applied sciences, we refer the reader to [6, 7, 11–17].

The concepts of the convex sets and convex functions have been extended and generalized in several directions by using innovative ideas and techniques. Hanson [13] introduced the invex functions in mathematical programming. Note that the invex functions are not convex functions. Ben-Israel and Mond [18] introduced the concept of invex sets and peinvex functions. They have shown that the differentiable preinvex functions are invex functions, but the inverse may not be true. It is well known that the preinvex functions may not be convex functions (see [19–21]). For example, the function $f(x) = -|x|$ is not a convex function, but it is a preinvex function with respect to η, where

$$\eta(v,u) = \begin{cases} v-u, & \text{if } v \leq 0,\ u \leq 0,\ v \geq 0,\ u \geq 0, \\ u-v, & \text{otherwise.} \end{cases} \tag{2}$$

Pitea and Postolache [16, 22, 23] investigated the properties of quasi-invexity in theoretical mechanics and nonlinear optimization. This shows that the preinvexity and its variant generalizations play an important and significant role in the developments of various fields of pure and applied sciences. Noor [19] has shown that the following inequality

$$\begin{aligned} f\left(\frac{2a+\eta(b,a)}{2}\right) &\leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \\ &\leq \frac{f(a)+f(b)}{2} \end{aligned} \tag{3}$$

holds for preinvex function f. For more integral inequalities of type (3), see [20].

In recent years, harmonic convex functions, which can be viewed as an important and significant extension of the convex functions, are being used to develop several iterative methods for solving nonlinear equations. Anderson et al. [1] and İşcan [5] have investigated various properties of harmonic convex functions. It has been shown that if f is

a harmonic convex function, then the following inequality holds:

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}, \quad (4)$$

which is also called Hermite-Hadamard inequality for harmonic convex function.

Later on, Noor et al. [24] introduced a new unified class of convex functions, the harmonic h-preinvex function, which includes two classes of convex functions (harmonic preinvex functions and harmonic s-preinvex functions) as special cases.

Recently, the functions of p-preinvexity and harmonic p-convexity have attracted the interest of many researchers. Motivated by the idea of Noor et al. [19, 20, 24, 25], in an earlier investigation, we have introduced a more general unified class of convex functions; this class of convex functions is called harmonic (p,h)-preinvex functions. One can easily show that harmonic (p,h)-preinvex functions include harmonic preinvex, harmonic p-convex, and preinvex functions. Following this way, the aim of this paper is to introduce a new generalized class of harmonically preinvex functions which is called harmonically (p,h,m)-preinvex functions; also we will discuss the properties of these classes of functions. Our results include several previous results as special cases. We hope that the interested readers may discover new and innovative applications of these harmonic (p,h,m)-preinvex functions.

2. Preliminaries

We begin with recalling some basic concepts and notions in the convex analysis. For more details, we refer the reader to [24, 26–28] and the references therein.

Definition 1. A set K in $\mathbb{R}^n$ is said to be a convex set, if

$$(1-t)x+ty \in K, \quad \forall x,y \in K,\ t \in [0,1]. \quad (5)$$

Definition 2. A function f on the convex set K is said to be a convex function, if

$$f((1-t)x+ty) \leq (1-t)f(x)+tf(y), \quad \forall x,y \in K,\ t \in [0,1]. \quad (6)$$

Definition 3. A set $K_\eta \subseteq \mathbb{R}$ is said to be invex set with respect to the bifunction $\eta(\cdot,\cdot)$, if

$$x+t\eta(y,x) \in K_\eta, \quad \forall x,y \in K_\eta,\ t \in [0,1]. \quad (7)$$

Definition 4. The function f on the invex set K_η is said to be preinvex with respect to the bifunction $\eta(\cdot,\cdot)$, if

$$f(x+t\eta(y,x)) \leq (1-t)f(x)+tf(y) \quad \forall x,y \in K_\eta,\ t \in [0,1]. \quad (8)$$

The function f is said to be preconcave if $-f$ is preinvex.

The invex set K_η is also called η-connected set. Note that if $\eta(b,a)=b-a$, then invex set becomes the convex set. Clearly, every convex set is an invex set but converse is not true in general.

Definition 5. Let K_η be an invex set in $\mathbb{R}$, and let $h:[0,1]\to\mathbb{R}$ be a nonnegative function. Then, a function $f:\mathbb{R}\to\mathbb{R}$ is said to be h-preinvex function with respect to the bifunction $\eta(\cdot,\cdot)$, if

$$f(x+t\eta(y,x)) \leq h(1-t)f(x)+h(t)f(y) \quad \forall x,y \in K_\eta,\ t \in [0,1]. \quad (9)$$

Definition 6. A set $K_h \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ is said to be a harmonically p-convex set, if

$$\frac{xy}{\left[ty^p+(1-t)x^p\right]^{1/p}} \in K_h, \quad \forall x,y \in K_h,\ t \in [0,1]. \quad (10)$$

Definition 7. Let $h:[0,1]\to\mathbb{R}$ be a nonnegative function. A function $f:K_h \subseteq \mathbb{R}/\{0\}\to\mathbb{R}$ is said to be harmonically (p,h)-convex function, if

$$f\left(\frac{xy}{\left[ty^p+(1-t)x^p\right]^{1/p}}\right) \leq h(t)f(x)+h(1-t)f(y) \quad \forall x,y \in K_h,\ t \in [0,1]. \quad (11)$$

Definition 8. A set $I=[a,a+\eta(b,a)] \subseteq \mathbb{R}/\{0\}$ is said to be a harmonic invex set with respect to the bifunction $\eta(\cdot,\cdot)$, if

$$\frac{x(x+\eta(y,x))}{(1-t)x+t(x+\eta(y,x))} \in I, \quad \forall x,y \in I,\ t \in [0,1]. \quad (12)$$

Definition 9. Let $h:[0,1]\to\mathbb{R}$ be a nonnegative function. A function $f:I\to[a,a+\eta(b,a)] \subseteq \mathbb{R}/\{0\}\to\mathbb{R}$ is said to be harmonically h-preinvex function with respect to an arbitrary bifunction $\eta(\cdot,\cdot)$, if

$$f\left(\frac{x(x+\eta(y,x))}{(1-t)x+t(x+\eta(y,x))}\right) \leq h(t)f(x)+h(1-t)f(y), \quad \forall x,y \in I,\ t \in [0,1]. \quad (13)$$

Definition 10. Let $p \in \mathbb{R}/\{0\}$. The function f on the p-invex set $A_{\eta,p}$ is said to be p-preinvex function with respect to η, where $p \in \mathbb{R}/\{0\}$, if

$$f\left(\left[(1-t)x^p+t(x+\eta(y,x))^p\right]^{1/p}\right) \leq tf(x)+(1-t)f(y), \quad (14)$$

for all $x,y \in A_{\eta,p}$ and $t \in [0,1]$.

Definition 11. The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0, 1]$, if we have

$$f(tx + (1-t)my) \le tf(x) + (1-t)mf(y) \tag{15}$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is m-concave if $-f$ is m-convex.

3. Definitions and Properties of Harmonically (p, h, m)-Preinvex Functions

In this section, we introduce some definitions and properties related to harmonically (p, h, m)-preinvex functions, which is a component part of our main results.

Definition 12. A function $f : [a, a + \eta(b, a)] \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ is said to be harmonically p-preinvex functions with respect to the bifunction $\eta(\cdot, \cdot)$ where $p \in \mathbb{R}/\{0\}$ if

$$\begin{aligned} &f\left(\frac{x(x+\eta(y,x))}{\left[(1-t)x^p + t(x+\eta(y,x))^p\right]^{1/p}}\right) \\ &\quad = f\left(\left(\frac{t}{x^p} + \frac{1-t}{(x+\eta(y,x))^p}\right)^{-1/p}\right) \\ &\quad \le tf(x) + (1-t)f(y) \end{aligned} \tag{16}$$

for all $x, y \in [a, a + \eta(b, a)]$ and $t \in [0, 1]$.

Remark 13. Notice that for $p = 1$, harmonic p-preinvexity reduces to harmonic preinvexity; for $\eta(b, a) = b-a$, harmonic p-preinvexity reduces to harmonic p-convexity; for $p = -1$, harmonic p-preinvexity reduces to preinvexity.

Definition 14. Two functions f, g are said to be similarly ordered (f is g-monotone) on interval I, if and only if,

$$(f(x) - f(y))(g(x) - g(y)) \ge 0, \quad \forall x, y \in I. \tag{17}$$

Proposition 15. *If $f, g : [a, a + \eta(b, a)] \subseteq \mathbb{R}/\{0\} \to \mathbb{R}_+$ are two similarly ordered harmonic p-preinvex functions, then the product fg is also a harmonic p-preinvex function.*

Proof. Since f, g are two positive harmonic p-preinvex functions, by using Definition 12, one has

$$\begin{aligned} &f\left(\frac{x(x+\eta(y,x))}{\left[(1-t)x^p + t(x+\eta(y,x))^p\right]^{1/p}}\right) \\ &\quad \cdot g\left(\frac{x(x+\eta(y,x))}{\left[(1-t)x^p + t(x+\eta(y,x))^p\right]^{1/p}}\right) \\ &\quad \le [tf(x) + (1-t)f(y)][tg(x) + (1-t)g(y)] \\ &\quad = t^2 f(x)g(x) + t(1-t)[f(x)g(y) \\ &\quad + f(y)g(x)] + (1-t)^2 f(y)g(y) = tf(x)g(x) \\ &\quad + (1-t)f(y)g(y) + t^2 f(x)g(x) + t(1-t) \\ &\quad \cdot [f(x)g(y) + f(y)g(x)] + (1-t)^2 f(y)g(y) \\ &\quad - tf(x)g(x) - (1-t)f(y)g(y) = tf(x)g(x) \\ &\quad + (1-t)f(y)g(y) - t(1-t)[f(x)g(x) \\ &\quad + f(y)g(y) - f(x)g(y) - f(y)g(x)] = tf(x) \\ &\quad \cdot g(x) + (1-t)f(y)g(y) - t(1-t)(f(x) \\ &\quad - f(y))(g(x) - g(y)) \le tf(x)g(x) + (1-t) \\ &\quad \cdot f(y)g(y), \end{aligned} \tag{18}$$

where, we have used the fact that two harmonic p-preinvex functions are similarly ordered functions. This shows that product of two similarly ordered harmonic p-preinvex functions is a harmonic p-preinvex function. □

Definition 16. Let $h : [0, 1] \to \mathbb{R}$ be a nonnegative function. A function $f : [a, a + \eta(b, a)] \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ is said to be harmonically (p, h, m)-preinvex functions with respect to the bifunction $\eta(\cdot, \cdot)$ or belongs to the class $hpx(p, h, m)$, where $p \in \mathbb{R}/\{0\}$, if

$$\begin{aligned} &f\left(\frac{x(x+\eta(my,x))}{\left[(1-t)x^p + t(x+\eta(my,x))^p\right]^{1/p}}\right) \\ &\quad = f\left(\left(\frac{t}{x^p} + \frac{1-t}{(x+\eta(my,x))^p}\right)^{-1/p}\right) \\ &\quad \le h(t)f(x) + mh(1-t)f(y) \end{aligned} \tag{19}$$

for all $x, y \in [a, a + \eta(b, a)]$, $t \in [0, 1]$, and $m \in (0, 1]$.

Let $h : [0, 1] \to \mathbb{R}$ be a nonnegative function. A function $f : [a, a + \eta(b, a)] \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ is said to be harmonically (p, h, m)-preconcave functions with respect to the bifunction $\eta(\cdot, \cdot)$ or belongs to the class $hpv(p, h, m)$, where $p \in \mathbb{R}/\{0\}$, if

$$\begin{aligned} &f\left(\frac{x(x+\eta(my,x))}{\left[(1-t)x^p + t(x+\eta(my,x))^p\right]^{1/p}}\right) \\ &\quad = f\left(\left(\frac{t}{x^p} + \frac{1-t}{(x+\eta(my,x))^p}\right)^{-1/p}\right) \\ &\quad \ge h(t)f(x) + mh(1-t)f(y) \end{aligned} \tag{20}$$

for all $x, y \in [a, a + \eta(b, a)]$, $t \in [0, 1]$, and $m \in (0, 1]$.

Remark 17. Note that for $m = 1$ and $h(t) = t$ harmonically (p, h, m)-preinvex functions reduce to the harmonically p-preinvex functions; for $m = 1$ and $p = 1$ harmonically

(p,h,m)-preinvex functions reduce to the harmonically h-preinvex functions; for $m = 1$ and $\eta(my,x) = my - x$ harmonically (p,h,m)-preinvex functions reduce to the harmonically (p,h)-convex functions; for $p = -1, h(t) = t$ and $\eta(my,x) = my - x$, harmonically (p,h,m)-preinvex functions reduce to the m-convex functions.

Now, we discuss some interesting results of harmonically (p,h,m)-preinvex (preconcave) functions, which includes linearity, product, composition properties, and the order property of h and p.

Proposition 18. *If $f,g \in hpx(p,h,m)$ and $\lambda > 0$, then $f + g, \lambda f \in hpx(p,h,m)$. Similarly, if $f,g \in hpv(p,h,m)$ and $\lambda > 0$, then $f + g$, $\lambda f \in hpv(p,h,m)$.*

Proof. The proof follows immediately from the definitions of the classes $hpx(p,h,m)$ and $hpv(p,h,m)$. □

Proposition 19. *(a) Let $f \in hpx(p_1,h,m)$ and $0 < a < a + \eta(b,a)$. If f is monotone decreasing (monotone increasing) and $p_1 \geq p_2$ ($p_1 \leq p_2$), then $f \in hpx(p_2,h,m)$.*

(b) Let $f \in hpv(p_1,h,m)$ and $0 < a < a + \eta(b,a)$. If f is monotone decreasing (monotone increasing) and $p_2 \geq p_1$ ($p_1 \geq p_2$), then $f \in hpv(p_2,h,m)$.

Proof. Let us recall a well-known result on the monotonicity of weighted mean (see [29]).

Let

$$M_n^{[p]}(x;\lambda) = \left(\frac{\lambda_1 x_1^p + \lambda_2 x_2^p + \cdots + \lambda_n x_n^p}{\lambda_1 + \lambda_2 + \cdots + \lambda_n}\right)^{1/p}, \tag{21}$$

where $\lambda_i > 0$, $x_i > 0$ $(i = 1,2,\ldots,n)$ and

$$M_n^{[0]}(x;\lambda) = \left(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}\right)^{1/(\lambda_1+\lambda_2+\cdots+\lambda_n)}. \tag{22}$$

In [29], it is proved that $M_n^{[p]}(x;\lambda)$ is a strictly increasing function for $p \in (-\infty,+\infty)$.

Let

$$g(p) = \frac{x(x+\eta(my,x))}{\left[t(x+\eta(my,x))^p + (1-t)x^p\right]^{1/p}}. \tag{23}$$

By using the monotonicity of weighted mean for $n = 2$, we conclude that $g(p)$ is a strictly decreasing function for $p \in (-\infty,+\infty)$. Thus, for $p_1 \geq p_2$ and $x, x+\eta(my,x) \in [a, a+\eta(b,a)]$, we have $g(p_2) \geq g(p_1)$. Since f is monotone decreasing and $f \in hpx(p_1,h,m)$, we then obtain

$$\begin{aligned} f(g(p_2)) &\leq f(g(p_1)) \\ &\leq h(t) f(x) + mh(1-t) f(y). \end{aligned} \tag{24}$$

Therefore, we get $f \in hpx(p_2,h,m)$. The result of (b) follows by the similar arguments as discussed above. □

Proposition 20. *Let f and g be nonnegative functions, and let $h_2 \leq h_1$ in their domains $[0,1]$. If $f \in hpx(p,h_2,m)$, then $f \in hpx(p,h_1,m)$. Similarly, if $f \in hpv(p,h_1,m)$, then $f \in hpv(p,h_2,m)$.*

Proof. Since $f \in hpx(p,h_2,m)$ form the definition of the classes $hpx(p,h,m)$ and the constraint conditions for f, g, h_1, and h_2, we have

$$\begin{aligned} &f\left(\left(\frac{t}{x^p} + \frac{1-t}{(x+\eta(my,x))^p}\right)^{-1/p}\right) \\ &\quad \leq h_2(t) f(x) + mh_2(1-t) f(y) \\ &\quad \leq h_1(t) f(x) + mh_1(1-t) f(y), \end{aligned} \tag{25}$$

which implies that $f \in hpx(p,h_1,m)$. The result in the second part of Proposition 20 can be deduced in a similar way as above. □

Proposition 21. *(a) Let f,g be similarly ordered positive functions on $[a, a+\eta(b,a)]$, that is,*

$$(f(x) - f(y))(g(x) - g(y)) \geq 0 \tag{26}$$

for all $x,y \in [a, a+\eta(b,a)]$. If $f \in hpx(p,h_1,m)$, $g \in hpx(p,h_2,m)$, and $h(\alpha)+mh(1-\alpha) \leq c$ for all $\alpha \in [0,1]$, where $h(t) = \max(h_1(t),h_2(t))$ and c is a fixed positive number, then the product fg belongs to $hpx(p,ch,m)$.

(b) Let f and g be opposite ordered positive functions, that is,

$$(f(x) - f(y))(g(x) - g(y)) \leq 0 \tag{27}$$

for all $x,y \in [a, a+\eta(b,a)]$. If $f \in hpv(p,h_1,m)$, $g \in hpv(p,h_2,m)$, and $h(\alpha)+mh(1-\alpha) \geq c$ for all $\alpha \in [0,1]$, where $h(t) = \min(h_1(t),h_2(t))$ and c is a fixed positive number, then the product fg belongs to $hpv(p,ch,m)$.

Proof. We only give a proof of the first part and the result of the second part of this proposition followed by a similar argument. Since f and g are similarly ordered, we have

$$f(x)g(x) + f(y)g(y) \geq f(x)g(y) + f(y)g(x). \tag{28}$$

Let α and β be positive numbers such that $\alpha + \beta = 1$. We obtain

$$\begin{aligned} &f\left(\frac{x(x+\eta(my,x))}{\left[\alpha(x+\eta(my,x))^p + \beta x^p\right]^{1/p}}\right) \\ &\quad \cdot g\left(\frac{x(x+\eta(my,x))}{\left[\alpha(x+\eta(my,x))^p + \beta x^p\right]^{1/p}}\right) \\ &\quad \leq (h_1(\alpha) f(x) + mh_1(\beta) f(y)) \\ &\quad \cdot (h_2(\alpha) g(x) + mh_2(\beta) g(y)) \leq h^2(\alpha) f(x) \\ &\quad \cdot g(x) + mh(\alpha) h(\beta) f(x) g(y) + mh(\alpha) h(\beta) \\ &\quad \cdot f(y) g(x) + m^2 h^2(\beta) f(y) g(y) \leq h^2(\alpha) f(x) \\ &\quad \cdot g(x) + mh(\alpha) h(\beta) f(x) g(x) + mh(\alpha) h(\beta) \\ &\quad \cdot f(y) g(y) + m^2 h^2(\beta) f(y) g(y) \end{aligned}$$

$$= (h(\alpha) + mh(\beta)) \cdot (h(\alpha) f(x) g(x) + mh(\beta) f(y) g(y)) \leq ch(\alpha) \cdot f(x) g(x) + cmh(\beta) f(y) g(y). \tag{29}$$

This completes the proof of Proposition 21. □

Proposition 22. *Let $f : I_1 \to (0, \infty)$ and $g : I_2 \to (0, \infty)$ be functions with $g(I_2) \subseteq I_1$.*

(a) If the function f is m-convex and increasing (decreasing) and $g \in hpx(p, h, m)$ ($g \in hpv(p, h, m)$) with $h(\alpha)+h(1-\alpha) = 1$ for $\alpha \in [0, 1]$, then $f \circ g$ belongs to $hpx(p, h, m)$.

(b) If the function f is m-concave and increasing (decreasing) and $g \in hpv(p, h, m)$ ($g \in hpx(p, h, m)$) with $h(\alpha)+h(1-\alpha) = 1$ for $\alpha \in [0, 1]$, then $f \circ g$ belongs to $hpv(p, h, m)$.

Proof. In view of $g \in hpx(p, h, m)$ and f is an increasing function, we have

$$f \circ g\left(\frac{x(x+\eta(my,x))}{\left[\alpha(x+\eta(my,x))^p + (1-\alpha)x^p\right]^{1/p}}\right) \leq f(h(\alpha) g(x) + mh(1-\alpha) g(y)). \tag{30}$$

Since, $h(\alpha) + h(1 - \alpha) = 1$ and f is m-convex, we obtain

$$f(h(\alpha) g(x) + mh(1-\alpha) g(y)) \leq h(\alpha) f \circ g(x) + mh(1-\alpha) f \circ g(y), \tag{31}$$

which leads to

$$f \circ g\left(\frac{x(x+\eta(my,x))}{\left[\alpha(x+\eta(my,x))^p + (1-\alpha)x^p\right]^{1/p}}\right) \leq h(\alpha) f \circ g(x) + mh(1-\alpha) f \circ g(y). \tag{32}$$

We conclude that $f \circ g \in hpx(p, h, m)$. The rest of the proof of Proposition 22 can be dealt with in the same way as described above. □

4. Integral Inequalities for Harmonically (p, h, m)-Preinvex Functions

We illustrate the applications of the definitions and properties of harmonically (p, h, m)-preinvex functions stated in foregoing section; we will establish some integral inequalities via harmonically (p, h, m)-preinvex functions. Our results include, as special cases, some related inequalities for harmonic p-preinvex functions, harmonic h-preinvex functions, and m-convex functions.

Theorem 23. *Let $f : [a, a + \eta(mb, a)] \to (0, \infty)$ be an integrable harmonic (p, h, m)-preinvex function, and let h be a concave function on $[0, 1]$. Then we have the following inequality:*

$$f\left(\frac{a(a+\eta(mb,a))}{\left[a^p + (a+\eta(mb,a))^p - x^p\right]^{1/p}}\right) \leq 2(f(a) + mf(b)) h\left(\frac{1}{2}\right) - f\left(\frac{a(a+\eta(mb,a))}{x}\right). \tag{33}$$

Proof. Let $x = [(1-t)a^p + t(a+\eta(mb,a))^p]^{1/p}$ and $t \in [0, 1]$. Using the definition of harmonic (p, h, m)-preinvex function along with a simple computation gives

$$\begin{aligned} f\left(\frac{a(a+\eta(mb,a))}{\left[a^p + (a+\eta(mb,a))^p - x^p\right]^{1/p}}\right) &= f\left(\frac{a(a+\eta(mb,a))}{\left[ta^p + (1-t)(a+\eta(mb,a))^p\right]^{1/p}}\right) \\ &\leq h(1-t) f(a) + mh(t) f(b) \\ &= [f(a) + mf(b)][h(1-t) + h(t)] - [h(t) f(a) + mh(1-t) f(b)] \\ &\leq [f(a) + mf(b)][h(1-t) + h(t)] - f\left(\frac{a(a+\eta(mb,a))}{\left[(1-t)a^p + t(a+\eta(mb,a))^p\right]^{1/p}}\right) \\ &= [f(a) + mf(b)][h(1-t) + h(t)] - f\left(\frac{a(a+\eta(mb,a))}{x}\right) \\ &\leq 2(f(a) + mf(b)) h\left(\frac{1}{2}\right) - f\left(\frac{a(a+\eta(mb,a))}{x}\right), \end{aligned} \tag{34}$$

where the last inequality follows from the assumption that h is a concave function on $[0, 1]$. This completes the proof of Theorem 23. □

Choosing $m = 1$, $h(t) = t$ in Theorem 23, we get the following.

Corollary 24. *Let $f : [a, a+\eta(b,a)] \to (0,\infty)$ be an integrable harmonic p-preinvex function. Then we have the inequality*

$$f\left(\frac{a(a+\eta(b,a))}{\left[a^p+(a+\eta(b,a))^p-x^p\right]^{1/p}}\right) \le f(a)+f(b)-f\left(\frac{a(a+(a+\eta(b,a)))}{x}\right). \tag{35}$$

Theorem 25. *Let $f : [a, a+\eta(mb,a)] \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ be a harmonic (p,h,m)-preinvex function with $0 < a < a+\eta(mb,a)$. If f, h are integrable, then*

$$\frac{p(a(a+\eta(mb,a)))^p}{(a+\eta(mb,a))^p-a^p}\int_a^{a+\eta(mb,a)}\frac{f(x)}{x^{p+1}}dx \le (f(a)+mf(b))\int_0^1 h(t)\,dt. \tag{36}$$

Proof. Since f is harmonic (p,h,m)-preinvex function, we have

$$f\left(\frac{a(a+\eta(mb,a))}{\left[(1-t)a^p+t(a+\eta(mb,a))^p\right]^{1/p}}\right) \le h(t)f(a)+mh(1-t)f(b). \tag{37}$$

Integrating both sides of the above inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}\int_0^1 f\left(\frac{a(a+\eta(mb,a))}{\left[(1-t)a^p+t(a+\eta(mb,a))^p\right]^{1/p}}\right)dt &\le \int_0^1 (h(t)f(a)+mh(1-t)f(b))\,dt\\ &= f(a)\int_0^1 h(t)\,dt+mf(b)\int_0^1 h(1-t)\,dt\\ &= f(a)\int_0^1 h(t)\,dt+mf(b)\int_0^1 h(t)\,dt\\ &= (f(a)+mf(b))\int_0^1 h(t)\,dt.\end{aligned} \tag{38}$$

On the other hand, by setting

$$x=\frac{a(a+\eta(mb,a))}{\left[(1-t)a^p+t(a+\eta(mb,a))^p\right]^{1/p}},\quad t\in[0,1], \tag{39}$$

then the left-hand side of integral expression can be simplified to

$$\int_0^1 f\left(\frac{a(a+\eta(mb,a))}{\left[(1-t)a^p+t(a+\eta(mb,a))^p\right]^{1/p}}\right)dt = \frac{p(a(a+\eta(mb,a)))^p}{(a+\eta(mb,a))^p-a^p}\int_a^{a+\eta(mb,a)}\frac{f(x)}{x^{p+1}}dx. \tag{40}$$

Thus

$$\frac{p(a(a+\eta(mb,a)))^p}{(a+\eta(mb,a))^p-a^p}\int_a^{a+\eta(mb,a)}\frac{f(x)}{x^{p+1}}dx \le (f(a)+mf(b))\int_0^1 h(t)\,dt. \tag{41}$$

The proof of Theorem 25 is complete. □

Taking $m = 1$, $h(t) = t$ in Theorem 25, we obtain the following.

Corollary 26. *Let $f : [a, a+\eta(b,a)] \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ be a harmonic p-preinvex function with $0 < a < a+\eta(b,a)$. If f is integrable, then*

$$\frac{p(a(a+\eta(b,a)))^p}{(a+\eta(b,a))^p-a^p}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^{p+1}}dx \le \frac{f(a)+f(b)}{2}. \tag{42}$$

Theorem 27. *Let $f, g : [a, a+\eta(mb,a)] \subseteq \mathbb{R}/\{0\} \to (0,\infty)$ be harmonic (p,h,m)-preinvex functions with $0 < a < a+\eta(mb,a)$. If f, g, h are integrable, then the following inequality holds:*

$$\begin{aligned}\frac{p(a(a+\eta(mb,a)))^p}{(a+\eta(mb,a))^p-a^p}\int_a^{a+\eta(mb,a)}\frac{f(x)g(x)}{x^{p+1}}dx &\le M(a,b,m)\int_0^1 h^2(x)\,dx\\ &\quad + N(a,b,m)\int_0^1 h(x)h(1-x)\,dx,\end{aligned} \tag{43}$$

where $M(a,b,m) = f(a)g(a)+m^2 f(b)g(b)$ and $N(a,b,m) = m(f(a)g(b)+f(b)g(a))$.

Proof. Since f, g are harmonic (p,h,m)-preinvex functions, it follows that

$$\begin{aligned}&\int_0^1 f\left(\frac{a(a+\eta(mb,a))}{\left[(1-t)a^p+t(a+\eta(mb,a))^p\right]^{1/p}}\right)\cdot g\left(\frac{a(a+\eta(mb,a))}{\left[(1-t)a^p+t(a+\eta(mb,a))^p\right]^{1/p}}\right)dt\\ &\le \int_0^1 [h(t)f(a)+mh(1-t)f(b)][h(t)g(a)+mh(1-t)g(b)]\,dt\\ &= \int_0^1 \left((h(t))^2 f(a)g(a)+m^2(h(1-t))^2 f(b)g(b)\right)dt+\int_0^1 mh(t)\end{aligned}$$

$$\cdot h(1-t)\left(f(a)g(b)+f(b)g(a)\right)dt = f(a) \cdot g(a)\int_0^1 (h(t))^2\,dt + m^2 f(b)g(b) \cdot \int_0^1 (h(1-t))^2\,dt + m\left(f(a)g(b)+f(b) \cdot g(a)\right)\int_0^1 h(t)h(1-t)\,dt = \left(f(a)g(a) + m^2 f(b)g(b)\right)\int_0^1 (h(t))^2\,dt + m\left(f(a)g(b) + f(b)g(a)\right)\int_0^1 h(t)h(1-t)\,dt = M(a,b,m) \cdot \int_0^1 h^2(x)\,dx + N(a,b,m)\int_0^1 h(x)h(1-x)\,dx. \tag{44}$$

On the other hand, by using a substitution,

$$x = \frac{a(a+\eta(mb,a))}{\left[(1-t)a^p + t(a+\eta(mb,a))^p\right]^{1/p}}, \quad t\in[0,1], \tag{45}$$

and then a straightforward calculation gives

$$\int_0^1 f\left(\frac{a(a+\eta(mb,a))}{\left[(1-t)a^p + t(a+\eta(mb,a))^p\right]^{1/p}}\right) \cdot g\left(\frac{a(a+\eta(mb,a))}{\left[(1-t)a^p + t(a+\eta(mb,a))^p\right]^{1/p}}\right)dt = \frac{p(a(a+\eta(mb,a)))^p}{(a+\eta(mb,a))^p - a^p}\int_a^{a+\eta(mb,a)} \frac{f(x)g(x)}{x^{p+1}}dx. \tag{46}$$

The proof of Theorem 27 is completed. □

Putting $m = 1$, $h(t) = t$ into Theorem 27, we have the following.

Corollary 28. *Let $f, g : [a, a+\eta(b,a)] \subseteq \mathbb{R}/\{0\} \to (0,\infty)$ be harmonic p-preinvex functions with $0 < a < a+\eta(b,a)$. If f, g are integrable, then the following inequality holds:*

$$\frac{p(a(a+\eta(b,a)))^p}{(a+\eta(b,a))^p - a^p}\int_a^{a+\eta(b,a)} \frac{f(x)g(x)}{x^{p+1}}dx \le \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b), \tag{47}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$ and $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 29. *Let $f : [a, a+\eta(mb,a)] \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ be harmonic (p,h,m)-preinvex functions with $0 < a < a+\eta(mb,a)$. If f, h, w are integrable, then the following inequality holds:*

$$\int_a^{a+\eta(mb,a)} \frac{f(x)w(x)}{x^{p+1}}dx \le \frac{M}{2}\int_a^{a+\eta(mb,a)} \frac{w(x)}{x^{p+1}}dx, \tag{48}$$

where $M = \sup_{t\in[0,1]}\{(f(a)+mf(b))(h(t)+h(1-t))\}$, and $w(x)$ is nonnegative and satisfies

$$w(x) = w\left(\frac{ax(a+\eta(mb,a))}{\left[\left(a^p + (a+\eta(mb,a))^p\right)x^p - (a(a+\eta(mb,a)))^p\right]^{1/p}}\right). \tag{49}$$

Proof. Note that f is harmonic (p,h,m)-preinvex function; we have

$$f\left(\frac{a(a+\eta(mb,a))}{\left[(1-t)a^p + t(a+\eta(mb,a))^p\right]^{1/p}}\right) + f\left(\frac{a(a+\eta(mb,a))}{\left[ta^p + (1-t)(a+\eta(mb,a))^p\right]^{1/p}}\right) \le h(t)f(a) + mh(1-t)f(b) + h(1-t)f(a) + mh(t)f(b) = f(a)(h(t)+h(1-t)) + mf(b)(h(t)+h(1-t)) = \left(f(a)+mf(b)\right)(h(t)+h(1-t)). \tag{50}$$

Using a substitution

$$x = \frac{a(a+\eta(mb,a))}{\left[(1-t)a^p + t(a+\eta(mb,a))^p\right]^{1/p}}, \quad t\in[0,1], \tag{51}$$

we get

$$\int_0^1 f\left(\frac{a(a+\eta(mb,a))}{\left[(1-t)a^p + t(a+\eta(mb,a))^p\right]^{1/p}}\right) \cdot w\left(\frac{a(a+\eta(mb,a))}{\left[(1-t)a^p + t(a+\eta(mb,a))^p\right]^{1/p}}\right)dt = \frac{p(a(a+\eta(mb,a)))^p}{(a+\eta(mb,a))^p - a^p}\int_a^{a+\eta(mb,a)} \frac{f(x)w(x)}{x^{p+1}}dx. \tag{52}$$

Similarly, we have

$$\int_0^1 f\left(\frac{a(a+\eta(mb,a))}{\left[ta^p + (1-t)(a+\eta(mb,a))^p\right]^{1/p}}\right) \cdot w\left(\frac{a(a+\eta(mb,a))}{\left[ta^p + (1-t)(a+\eta(mb,a))^p\right]^{1/p}}\right)dt = \frac{p(a(a+\eta(mb,a)))^p}{(a+\eta(mb,a))^p - a^p}\int_a^{a+\eta(mb,a)} \frac{f(x)w(x)}{x^{p+1}}dx. \tag{53}$$

From the properties of expression $w(x)$, it is easy to find that

$$w\left(\frac{a\left(a+\eta\left(mb,a\right)\right)}{\left[\left(1-t\right)a^{p}+t\left(a+\eta\left(mb,a\right)\right)^{p}\right]^{1/p}}\right) = w\left(\frac{a\left(a+\eta\left(mb,a\right)\right)}{\left[ta^{p}+\left(1-t\right)\left(a+\eta\left(mb,a\right)\right)^{p}\right]^{1/p}}\right). \tag{54}$$

By utilizing the above inequality and identities, we deduce that

$$\begin{aligned}
&\frac{2p\left(a\left(a+\eta\left(mb,a\right)\right)\right)^{p}}{\left(a+\eta\left(mb,a\right)\right)^{p}-a^{p}}\int_{a}^{a+\eta(mb,a)}\frac{f\left(x\right)w\left(x\right)}{x^{p+1}}dx \\
&\quad=\int_{0}^{1}f\left(\frac{a\left(a+\eta\left(mb,a\right)\right)}{\left[\left(1-t\right)a^{p}+t\left(a+\eta\left(mb,a\right)\right)^{p}\right]^{1/p}}\right) \\
&\quad\cdot w\left(\frac{a\left(a+\eta\left(mb,a\right)\right)}{\left[\left(1-t\right)a^{p}+t\left(a+\eta\left(mb,a\right)\right)^{p}\right]^{1/p}}\right)dt \\
&\quad+\int_{0}^{1}f\left(\frac{a\left(a+\eta\left(mb,a\right)\right)}{\left[ta^{p}+\left(1-t\right)\left(a+\eta\left(mb,a\right)\right)^{p}\right]^{1/p}}\right) \\
&\quad\cdot w\left(\frac{a\left(a+\eta\left(mb,a\right)\right)}{\left[ta^{p}+\left(1-t\right)\left(a+\eta\left(mb,a\right)\right)^{p}\right]^{1/p}}\right)dt \\
&\quad=\int_{0}^{1}\left(f\left(\frac{a\left(a+\eta\left(mb,a\right)\right)}{\left[\left(1-t\right)a^{p}+t\left(a+\eta\left(mb,a\right)\right)^{p}\right]^{1/p}}\right)\right. \\
&\quad\left.+\int_{0}^{1}f\left(\frac{a\left(a+\eta\left(mb,a\right)\right)}{\left[ta^{p}+\left(1-t\right)\left(a+\eta\left(mb,a\right)\right)^{p}\right]^{1/p}}\right)\right) \\
&\quad\cdot w\left(\frac{a\left(a+\eta\left(mb,a\right)\right)}{\left[\left(1-t\right)a^{p}+t\left(a+\eta\left(mb,a\right)\right)^{p}\right]^{1/p}}\right)dt \\
&\quad\le\int_{0}^{1}\left(f\left(a\right)+mf\left(b\right)\right)\left(h\left(t\right)+h\left(1-t\right)\right) \\
&\quad\cdot w\left(\frac{a\left(a+\eta\left(mb,a\right)\right)}{\left[ta^{p}+\left(1-t\right)\left(a+\eta\left(mb,a\right)\right)^{p}\right]^{1/p}}\right)dt \\
&\quad\le M\int_{0}^{1}w\left(\frac{a\left(a+\eta\left(mb,a\right)\right)}{\left[ta^{p}+\left(1-t\right)\left(a+\eta\left(mb,a\right)\right)^{p}\right]^{1/p}}\right)dt \\
&\quad=\frac{Mp\left(a\left(a+\eta\left(mb,a\right)\right)\right)^{p}}{\left(a+\eta\left(mb,a\right)\right)^{p}-a^{p}}\int_{a}^{a+\eta(mb,a)}\frac{w\left(x\right)}{x^{p+1}}dx,
\end{aligned} \tag{55}$$

where $M=\sup_{t\in[0,1]}\{(f(a)+mf(b))(h(t)+h(1-t))\}$. The proof of Theorem 29 is completed. □

Putting $m=1$, $h(t)=t$ into Theorem 29, we have the following.

Corollary 30. *Let $f:[a,a+\eta(b,a)]\subseteq\mathbb{R}/\{0\}\to\mathbb{R}$ be harmonic p-preinvex functions with $0<a<a+\eta(b,a)$. If f,w are integrable, then we have the following inequality:*

$$\int_{a}^{a+\eta(b,a)}\frac{f\left(x\right)w\left(x\right)}{x^{p+1}}dx \le \frac{f\left(a\right)+f\left(b\right)}{2}\int_{a}^{a+\eta(b,a)}\frac{w\left(x\right)}{x^{p+1}}dx, \tag{56}$$

where $w(x)$ is nonnegative and satisfies

$$w\left(x\right) = w\left(\frac{ax\left(a+\eta\left(b,a\right)\right)}{\left[\left(a^{p}+\left(a+\eta\left(b,a\right)\right)^{p}\right)x^{p}-\left(a\left(a+\eta\left(b,a\right)\right)\right)^{p}\right]^{1/p}}\right). \tag{57}$$

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors read and approved the final manuscript.

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References

[1] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, "Generalized convexity and inequalities," *Journal of Mathematical Analysis and Applications*, vol. 335, no. 2, pp. 1294–1308, 2007.

[2] F. Chen and S. Wu, "Fejér and Hermite-Hadamard type inequalities for harmonically convex functions," *Journal of Applied Mathematics*, vol. 2014, Article ID 386806, 6 pages, 2014.

[3] S. S. Dragomir, "Some new inequalities of Hermite-Hadamard type for GA-convex functions," *RGMIA Research Report Collection*, vol. 18, article 30, 2015.

[4] S. S. Dragomir, "Inequalities of Hermite-Hadamard type for HA-convex functions," *RGMIA Research Report Collection*, vol. 18, article 38, 2015.

[5] İ. İşcan, "Hermite-Hadamard type inequalities for harmonically convex functions," *Hacettepe Journal of Mathematics and Statistics*, vol. 43, no. 6, pp. 935–942, 2014.

[6] C. P. Niculescu and L.-E. Persson, *Convex Functions and Their Applications*, CMS Books in Mathematics, Springer, New York, NY, USA, 2006.

[7] R. Pini, "Invexity and generalized convexity," *Optimization*, vol. 22, no. 4, pp. 513–525, 1991.

[8] H.-N. Shi and J. Zhang, "Some new judgement theorems of Schur geometric and Schur harmonic convexities for a class of symmetric functions," *Journal of Inequalities and Applications*, vol. 2013, article 527, 2013.

[9] S. Varošanec, "On h-convexity," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 303–311, 2007.

[10] T. Weir and B. Mond, "Pre-invex functions in multiple objective optimization," *Journal of Mathematical Analysis and Applications*, vol. 136, no. 1, pp. 29–38, 1988.

[11] G. Cristescu and L. Lupşa, *Non-Connected Convexities and Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.

[12] J. Hadamard, "Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann," *Journal de Mathématiques Pures et Appliquées*, vol. 58, pp. 171–216, 1893.

[13] M. A. Hanson, "On sufficiency of the Kuhn-Tucker conditions," *Journal of Mathematical Analysis and Applications*, vol. 80, no. 2, pp. 545–550, 1981.

[14] B. G. Pachpatte, "On some integral inequalities involving convex functions," *RGMIA Research Report Collection*, vol. 3, no. 3, pp. 487–492, 2000.

[15] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex functions, Partial ordering and Statistical Applications*, vol. 187 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1992.

[16] A. Pitea and M. Postolache, "Minimization of vectors of curvilinear functionals on the second order jet bundle: sufficient efficiency conditions," *Optimization Letters*, vol. 6, no. 8, pp. 1657–1669, 2012.

[17] M. Tunç, "On some new inequalities for convex functions," *Turkish Journal of Mathematics*, vol. 36, no. 2, pp. 245–251, 2012.

[18] A. Ben-Israel and B. Mond, "What is invexity?" *Australian Mathematical Society. Journal. Series B. Applied Mathematics*, vol. 28, no. 1, pp. 1–9, 1986.

[19] M. A. Noor, "Hermite-Hadamard integral inequalities for log-preinvex functions," *Journal of Mathematical Analysis and Approximation Theory*, vol. 2, no. 2, pp. 126–131, 2007.

[20] M. A. Noor, "Hadamard integral ineqaualities for product of two preinvex functions," *Nonlinear Analysis Forum*, vol. 14, pp. 167–173, 2009.

[21] X. M. Yang and D. Li, "On properties of preinvex functions," *Journal of Mathematical Analysis and Applications*, vol. 256, no. 1, pp. 229–241, 2001.

[22] A. Pitea and M. Postolache, "Duality theorem for a new class of multitime multiobjective variational problems," *Journal of Global Optimization*, vol. 54, no. 1, pp. 47–58, 2012.

[23] A. Pitea and M. Postolache, "Minimization of vectors of curvilinear functionals on the second order jet bundle," *Optimization Letters*, vol. 6, no. 3, pp. 459–470, 2012.

[24] M. A. Noor, K. I. Noor, and S. Iftikhar, "Fractional ostrowski inequalities for harmonic h-preinvex functions," *Facta Universitatis. Series: Mathematics and Informatics*, vol. 31, no. 2, pp. 417–445, 2016.

[25] M. A. Noor, K. I. Noor, M. U. Awan, and S. Khan, "Hermite-Hadamard inequalities for s-Godunova-Levin preinvex functions," *Journal of Advanced Mathematical Studies*, vol. 7, no. 2, pp. 12–19, 2014.

[26] M. K. Bakula, M. E. Özdemir, and J. Pecaric, "Hadamard type inequalities for m-convex and (α, m)-convex functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 4, article 96, 2008.

[27] S. R. Mohan and S. K. Neogy, "On invex sets and preinvex functions," *Journal of Mathematical Analysis and Applications*, vol. 189, no. 3, pp. 901–908, 1995.

[28] M. A. Noor, K. I. Noor, and S. Iftikhar, "Hermite-Hadamard inequalities for harmonic preinvex functions," *Saussurea*, vol. 6, no. 1, pp. 34–53, 2016.

[29] D. S. Mitrinović and P. M. Vasić, *Analytic Inequalities*, Springer, New York, NY, USA, 1970.