Research Article

Restriction of Toeplitz Operators on Their Reducing Subspaces

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We study the restrictions of analytic Toeplitz operator on its minimal reducing subspaces for the unit disc and construct their models on slit domains. Furthermore, it is shown that T_{z^n} is similar to the sum of n copies of the Bergman shift.

1. Introduction

Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$; a subspace $\mathfrak{M}$ of $\mathcal{H}$ is called an *invariant subspace* of T if $T(\mathfrak{M}) \subseteq \mathfrak{M}$ and a *reducing subspace* of T if $\mathfrak{M}$ is an invariant subspace of T and T^* . A reducing subspace $\mathfrak{M}$ of T is called *minimal* if for every reducing subspace $\tilde{\mathfrak{M}}$ of T such that $\mathfrak{M} \subseteq \tilde{\mathfrak{M}}$ then either $\tilde{\mathfrak{M}} = \mathfrak{M}$ or $\tilde{\mathfrak{M}} = 0$. For a concrete bounded operator T on a separable Hilbert space $\mathcal{H}$, it is important to determine invariant subspaces and reducing subspaces for T .

Let $\mathbb{D}$ be the unit open disc in the complex plane and $dA(z)$ be the normalized area measure on $\mathbb{D}$. The Bergman space $L_a^2(\mathbb{D})$ consists of all analytic functions in the Lebesgue space $L^2(dA(z))$. It is clear that $L_a^2(\mathbb{D})$ is a closed subspace of $L^2(dA(z))$, and let P denote the projection from $L^2(dA(z))$ onto $L_a^2(\mathbb{D})$. The Toeplitz operator T_φ on $L_a^2(\mathbb{D})$ with symbol $\varphi \in L^\infty(dA(z))$ is defined by $(T_\varphi f)(z) = P(\varphi f)(z)$; it is called an analytic Toeplitz operator if $\varphi \in H^\infty(\mathbb{D})$.

An n th-order Blaschke product B is the analytic function on $\mathbb{D}$ given by

$$B(z) = e^{i\theta} \prod_{i=1}^n \frac{z - a_i}{1 - \bar{a}_i z}, \quad (1)$$

where θ is a real number and $a_i \in \mathbb{D}$ for $1 \leq i \leq n$. A Blaschke product is very important in the theory of Hardy space.

Characterization of reducing subspaces of an analytic Toeplitz operator T_φ on Bergman space has been of great interest for last two decades. Thomson [1, 2] showed that it

suffices to study reducing subspace of T_B for a finite Blaschke product in the case of Hardy space. It can be generalized to Bergman spaces easily.

Zhu studied the reducing subspaces of T_B for a Blaschke product of order 2 firstly and showed that T_B has exactly two distinct minimal reducing subspaces (cf. [3]). Motivated by this fact, Zhu conjectured that the number of minimal reducing subspaces of T_B equals the order of B (cf. [3]). Guo et al. showed that in general this is not true (cf. [4]), and they found that the number of minimal reducing subspaces of T_B equals the number of connected components of the Riemann surface of $B(z) = B(w)$ when the order of B is 3, 4, 6. Then they conjectured that the number of minimal reducing subspaces of T_B equals the number of connected components of the Riemann surface of $B(z) = B(w)$ for any finite Blaschke product (called the refined Zhu's conjecture, cf. [4]). Douglas et al. confirmed the conjecture in [5, 6] by using local inverses of Blaschke products [7]. Tikaradze [8] generalized a part of results in [5] to bounded smooth pseudoconvex domains in $\mathbb{C}^N$. Douglas and Kim [9] studied reducing subspace T_{z^n} on Bergman space of the annulus; the case of Hardy space was summarized in [10].

In [11], Douglas et al. generalized the bundle shift [12] to the case of Bergman spaces, constructed a vector bundle model for analytic Toeplitz operator T_φ on the Bergman space $L_a^2(\mathbb{D})$, and tried to build vector bundle models for restrictions of T_φ to its minimal reducing subspaces, but it is not completed. Douglas [13] studied unitary equivalence of the restrictions by computing their curvatures of corresponding

geometric models. Hu et al. [14] showed that, for T_φ , there is a distinguished reducing subspace $\mathcal{M}_0$ such that the restriction of T_φ on $\mathcal{M}_0$ is the Bergman shift. In this paper, we analyze the concrete examples to see what are the possible models for these restrictions for further research.

2. Unitary Equivalence and Similarity of Weighted Shifts

Let S be an operator on a separable infinite-dimensional Hilbert space $\mathcal{H}$. S is called a one-side weighted shift if there exist an orthonormal basis $\{e_i\}$, $i = 0, 1, 2, \dots$ for $\mathcal{H}$ and a bounded sequence $\{\alpha_i\}$ of complex numbers such that $Se_i = \alpha_i e_{i+1}$. Similarly, S is called a two-side weighted shift if there exist an orthonormal basis $\{e_i\}$, $i = \dots, -2, -1, 0, 1, 2, \dots$ for $\mathcal{H}$ and a bounded sequence $\{\alpha_i\}$ of complex numbers such that $Se_i = \alpha_i e_{i+1}$ for all $i \in \mathbb{Z}$.

Lemma 1 (see [15]). *Suppose that S and T are two injective one-side weighted shifts with weights $\{\alpha_i\}$ and $\{\beta_i\}$ respectively; then S is unitarily equivalent to T if and only if $|\alpha_i| = |\beta_i|$ for all i .*

Lemma 2 (see [15]). *Suppose that S and T are two injective one-side weighted shifts with weights $\{\alpha_i\}$ and $\{\beta_i\}$, respectively; then S is similar to T if and only if there exist two constants M and N such that*

$$0 < M \leq \prod_{j=0}^i \left| \frac{\beta_j}{\alpha_j} \right| \leq N < \infty \quad (2)$$

for all i .

3. Models for Restriction of Toeplitz Operators on Their Minimal Reducing Subspaces

3.1. The Bergman Spaces on the Slit Disc. The domain $G = \mathbb{D} \setminus [0, 1)$ is called the slit disk. Let dA denote the normalized area measure on G . $L_a^2(G)$ is the set of analytic functions in the Lebesgue space $L^2(G, dA)$. For a nonnegative measurable function $g(w)$ on G , we can define the weighted Bergman space with respect to $g(w)dA(w)$ to be the set of all analytic functions in the Lebesgue space $L^2(G, g(w)dA)$. Ross studied invariant subspaces of Bergman spaces on slit domains in [16]. Aleman et al. defined and studied the Hardy space of a slit domain and in particular they studied the invariant subspace of the slit disk; one can consult [17] for details.

3.2. Slit Disc Models for M_{z^2} . It is easy to check that $1/4|w|$ is a measurable function on G and that $(1/4|w|)dA(w)$ is a probability measure on G .

Lemma 3. *The Bergman space $L_a^2(G, (1/4|w|)dA(w))$ contains constants and power functions $\{w^n \mid n \in \mathbb{N}\}$. Furthermore, $\{\sqrt{2(2n+1)}w^n; n \in \mathbb{N}\}$ and $\{2\sqrt{n+1}w^n; n \in \mathbb{N}\}$ are orthonormal basis of $L_a^2(G, (1/4|w|)dA(w))$ and $L_a^2(G, (1/4|w|)dA(w))$, respectively.*

Proof. For the Bergman space $L_a^2(G, (1/4|w|)dA(w))$ and $n \in \mathbb{N}$,

$$\begin{aligned} \|w^n\|^2 &= \int_G |w^n|^2 \frac{1}{4|w|} dA(w) = \frac{1}{4\pi} \int_0^1 r^{2n} dr \int_0^{2\pi} d\theta \\ &= \frac{1}{2(2n+1)}, \end{aligned} \quad (3)$$

so $\|w^n\| = 1/\sqrt{2(2n+1)}$.

For the Bergman space $L_a^2(G, (1/4)dA(w))$ and $n \in \mathbb{N}$,

$$\begin{aligned} \|w^n\|^2 &= \int_G |w^n|^2 \frac{1}{4} dA(w) = \frac{1}{4\pi} \int_0^1 r^{2n+1} dr \int_0^{2\pi} d\theta \\ &= \frac{1}{4(n+1)}, \end{aligned} \quad (4)$$

so $\|w^n\| = 1/2\sqrt{(n+1)}$. $\square$

Lemma 4. *The multiplication operator T_w is a bounded operator on $L_a^2(G, (1/4|w|)dA(w))$ and $L_a^2(G, (1/4)dA(w))$.*

Proof. For the Bergman space $L_a^2(G, (1/4|w|)dA(w))$, it is easy to show that

$$\begin{aligned} T_w(\sqrt{2(2n+1)}w^n) &= \sqrt{2(2n+1)}w^{n+1} \\ &= \sqrt{\frac{2n+1}{2n+3}}(\sqrt{2(2n+3)}w^{n+1}). \end{aligned} \quad (5)$$

So T_w is a weighted shift with weight $\{\sqrt{(2n+1)/(2n+3)}\}_{n=0}^\infty$ and $\|T_w\| \leq 1$. Similarly, on the Bergman space $L_a^2(G, (1/4)dA(w))$, T_w is a weighted shift with weight $\{\sqrt{(n+1)/(n+2)}\}_{n=0}^\infty$ and so bounded. $\square$

We know that

$$\begin{aligned} \mathcal{M}_0 &= \bigvee \{z^k; k \equiv 0 \pmod{2}\}, \\ \mathcal{M}_1 &= \bigvee \{z^k; k \equiv 1 \pmod{2}\} \end{aligned} \quad (6)$$

are the only two minimal reducing subspaces of T_{z^2} on $L_a^2(\mathbb{D})$.

Theorem 5. *$T_{z^2}|_{\mathcal{M}_0}$, $T_{z^2}|_{\mathcal{M}_1}$ are unitarily equivalent to the multiplication operators T_w on the following two spaces, respectively:*

$$\begin{aligned} L_a^2\left(G, \frac{1}{4|w|}dA(w)\right), \\ L_a^2\left(G, \frac{1}{4}dA(w)\right). \end{aligned} \quad (7)$$

Proof. Define the following maps:

$$\begin{aligned} L_a^2\left(G, \frac{1}{4}dA(w)\right) &\longrightarrow \mathcal{M}_1 \\ V_1: f(w) &\longmapsto zf(z^2) \\ L_a^2\left(G, \frac{1}{4|w|}dA(w)\right) &\longrightarrow \mathcal{M}_0 \\ V_0: f(w) &\longmapsto f(z^2). \end{aligned} \quad (8)$$

Then V_1, V_0 are two isometries on $\mathcal{M}_1, \mathcal{M}_0$, respectively. As a fact, by changing of variable $z^2 = w$, $|dz/dw| = |1/2\sqrt{w}|$, we have

$$\begin{aligned}\|V_1 f(w)\|^2 &= \|zf(z^2)\|^2 = \int_{\mathbb{D}} |zf(z^2)|^2 dA(z) \\ &= \int_G |\sqrt{w}f(w)|^2 \left| \frac{dz}{dw} \right|^2 dA(w) \\ &= \int_G |\sqrt{w}f(w)|^2 \left| \frac{1}{2\sqrt{w}} \right|^2 dA(w) \\ &= \int_G |f(w)|^2 \frac{1}{4} dA(w) = \|f(w)\|^2,\end{aligned}\quad (9)$$

so V_1 is an isometry. And

$$\begin{aligned}\|V_0 f(w)\|^2 &= \|f(z^2)\|^2 = \int_{\mathbb{D}} |f(z^2)|^2 dA(z) \\ &= \int_G |f(w)|^2 \left| \frac{dz}{dw} \right|^2 dA(w) \\ &= \int_G |f(w)|^2 \left| \frac{1}{2\sqrt{w}} \right|^2 dA(w) \\ &= \int_G |f(w)|^2 \frac{1}{4|w|} dA(w) = \|f(w)\|^2;\end{aligned}\quad (10)$$

it shows that V_0 is an isometry. The next is to show that $T_{z^2}V_1 = V_1T_w$ on $L_a^2(G, (1/4)dA(w))$. For every function $f(w) \in L_a^2(G, (1/4)dA(w))$, we have

$$\begin{aligned}T_{z^2}V_1 f(w) &= T_{z^2}zf(z^2) = z^3 f(z^2) = z(z^2 f(z^2)) \\ &= V_1(uf(w)) = V_1T_w f(w).\end{aligned}\quad (11)$$

One can show that $T_{z^2}V_0 = V_0T_w$ on $L_a^2(G, (1/4w)dA(w))$ similarly. $\square$

Suppose that φ is a Blaschke product $\varphi = ((a - z)/(1 - \bar{a}z))((b - z)/(1 - \bar{b}z)) = \varphi_a(z)\varphi_b(z)$ with two different zeroes a and b ; let $k_\lambda(z) = (1 - |\lambda|^2)/(1 - \bar{\lambda}z)^2$ be the normalized reproducing kernel at λ .

Lemma 6 (see [3]). *Let $\varphi = ((a - z)/(1 - \bar{a}z))((b - z)/(1 - \bar{b}z))$ with two different zeroes $a, b \in \mathbb{D}$. Let m be the geodesic midpoint between a and b . Then the operator*

$$M_\varphi : L_a^2(\mathbb{D}) \longrightarrow L_a^2(\mathbb{D}) \quad (12)$$

has

$$\begin{aligned}X_e &= \{f \circ \varphi_m k_m; f \in L_a^2(\mathbb{D}), f(-z) = f(z)\}, \\ X_o &= \{f \circ \varphi_m k_m; f \in L_a^2(\mathbb{D}), f(-z) = -f(z)\},\end{aligned}\quad (13)$$

as its only two proper reducing subspaces, where $\varphi_m = (m - z)/(1 - \bar{m}z)$.

Theorem 7. *The restrictions of T_φ on X_e and X_o are unitarily equivalent to $T_{\psi(\sqrt{w})}$ on $L_a^2(G, (1/4|w|)dA(w))$ and $L_a^2(G, (1/4)dA(w))$, respectively, where $\psi = \varphi \circ \varphi_m$.*

Proof. Now we define the following operators:

$$\begin{aligned}L_a^2\left(G, \frac{1}{4}dA(w)\right) &\longrightarrow X_o \\ V_o : f(w) &\longmapsto (uf(u^2)) \circ \varphi_m(z) k_m(z), \\ L_a^2\left(G, \frac{1}{4|w|}dA(w)\right) &\longrightarrow X_e \\ V_e : f(w) &\longmapsto f(u^2) \circ \varphi_m(z) k_m(z).\end{aligned}\quad (14)$$

Then V_o, V_e are two isometries on X_o, X_e , respectively. As a fact, by changing of variables two times, ($u = \varphi_m(z)$, $\varphi'_m(z) = (1 - |m|^2)/(1 - \bar{m}z)^2 = k_m(z)$ for the first time; $z^2 = w$, $|dz/dw| = |1/2\sqrt{w}|$ for the second time), we have

$$\begin{aligned}\|V_o f(w)\|^2 &= \|(uf(u^2)) \circ \varphi_m(z) k_m(z)\|^2 \\ &= \int_{\mathbb{D}} |(uf(u^2)) \circ \varphi_m(z) k_m(z)|^2 dA(z) \\ &= \int_{\mathbb{D}} |uf(u^2)|^2 dA(u) \\ &= \int_{\mathbb{D}} |\sqrt{w}f(w)|^2 \left| \frac{dz}{dw} \right|^2 dA(w) \\ &= \int_{\mathbb{D}} |\sqrt{w}f(w)|^2 \left| \frac{1}{2\sqrt{w}} \right|^2 dA(w) \\ &= \int_{\mathbb{D}} |f(w)|^2 \frac{1}{4} dA(w) = \|f(w)\|^2,\end{aligned}\quad (15)$$

so V_o is a isometry. And

$$\begin{aligned}\|V_e f(w)\|^2 &= \|(f(u^2)) \circ \varphi_m(z) k_m(z)\|^2 \\ &= \int_{\mathbb{D}} |(f(u^2)) \circ \varphi_m(z) k_m(z)|^2 dA(z) \\ &= \int_{\mathbb{D}} |f(u^2)|^2 dA(u) \\ &= \int_{\mathbb{D}} |f(w)|^2 \left| \frac{dz}{dw} \right|^2 dA(w) \\ &= \int_{\mathbb{D}} |f(w)|^2 \left| \frac{1}{2\sqrt{w}} \right|^2 dA(w) \\ &= \int_{\mathbb{D}} |f(w)|^2 \frac{1}{4|w|} dA(w) = \|f(w)\|^2;\end{aligned}\quad (16)$$

it shows that V_e is a isometry. Moreover, for any $f(w) \in L_a^2(G, (1/4|w|)dA(w))$, we have

$$\begin{aligned}T_\varphi V_e f(w) &= \varphi(z) f(\varphi_m^2(z)) k_m(z) \\ &= \psi(\varphi_m(z)) f(\varphi_m^2(z)) k_m(z),\end{aligned}\quad (17)$$

where $\psi(z) = \varphi(\varphi_m(z))$ and m is the midpoint between a and b . On the other hand,

$$\begin{aligned} V_e T_{\psi(\sqrt{w})} f(w) &= V_e (\psi(\sqrt{w}) f(w)) \\ &= \psi(\varphi_m(z)) f(\varphi_m^2(z)) k_m(z). \end{aligned} \quad (18)$$

Note that $\psi(\sqrt{w})$ is a function in $L_a^2(G, (1/4|w|)dA(w))$. For $f(w) \in L_a^2(G, (1/4)dA(w))$, one can prove similarly that

$$T_\varphi V_o f(w) = V_o T_{\psi(\sqrt{w})} f(w). \quad (19)$$

□

3.3. Weighted Shift Models

Proposition 8. *The restrictions of T_{z^2} on its minimal reducing subspaces are one-side weighted shifts, and they are not unitarily equivalent to each other.*

Proof. We know that

$$\begin{aligned} \mathcal{M}_0 &= \left\{ \sum_{k=0}^{\infty} a_k z^{2k}; \sum_{k=0}^{\infty} (2k+1) |a_k|^2 < \infty \right\}, \\ \mathcal{M}_1 &= \left\{ \sum_{k=0}^{\infty} b_k z^{2k+1}; \sum_{k=0}^{\infty} (2k+2) |b_k|^2 < \infty \right\} \end{aligned} \quad (20)$$

are the two minimal reducing subspaces of T_{z^2} on the Bergman space $L_a^2(\mathbb{D})$. It is clear that

$$\begin{aligned} T_{z^2}(\sqrt{2k+1} z^{2k}) &= \sqrt{\frac{2k+1}{2k+3}} (\sqrt{2k+3} z^{2(k+1)}), \\ T_{z^2}(\sqrt{2k+2} z^{2k+1}) &= \sqrt{\frac{k+1}{k+2}} (\sqrt{2k+4} z^{2(k+1)+1}), \end{aligned} \quad (21)$$

so $T_{z^2}|_{\mathcal{M}_0}$ and $T_{z^2}|_{\mathcal{M}_1}$ are weighted shifts with the weight sequences $\{\sqrt{(2k+1)/(2k+3)}\}$ and $\{\sqrt{(k+1)/(k+2)}\}$, respectively. It is clear that $T_{z^2}|_{\mathcal{M}_0}$ is not unitarily equivalent to $T_{z^2}|_{\mathcal{M}_1}$ by Lemma 1. □

Proposition 9. *The restrictions of T_{z^2} on its minimal reducing subspaces are one-side weighted shifts, and they are similar to each other.*

Proof. Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be the minimal reducing subspaces of T_φ ; we know that $T_\varphi|_{\mathcal{M}_0}$ is the weighted shift with weight $\{\sqrt{(2k+1)/(2k+3)}\}_{k=0}^{\infty}$ and $T_\varphi|_{\mathcal{M}_1}$ is the weighted shift with weight $\{\sqrt{(k+1)/(k+2)}\}_{k=0}^{\infty}$. By a direct computation, we have

$$\begin{aligned} \prod_{k=0}^n \frac{\sqrt{(2k+1)/(2k+3)}}{\sqrt{(k+1)/(k+2)}} &= \frac{\prod_{k=0}^n \sqrt{(2k+1)/(2k+3)}}{\prod_{k=0}^n \sqrt{(k+1)/(k+2)}} \\ &= \sqrt{\frac{n+2}{2n+3}}. \end{aligned} \quad (22)$$

So for any n , we have

$$\sqrt{\frac{1}{2}} \leq \prod_{k=0}^n \frac{\sqrt{(2k+1)/(2k+3)}}{\sqrt{(k+1)/(k+2)}} \leq \sqrt{\frac{2}{3}}, \quad (23)$$

since $\sqrt{(n+2)/(2n+3)}$ is decreasing as n increases to infinity. It implies that $T_\varphi|_{\mathcal{M}_0}$ is similar to $T_\varphi|_{\mathcal{M}_1}$ by Lemma 2. □

Remark 10. $T_\varphi|_{\mathcal{M}_1}$ is just the Bergman shift, so we know that T_φ is similar to the direct sum of two copies of the Bergman shift.

3.4. For Reducing Subspace of M_{z^n}

Theorem 11. *The slit disc models for restrictions of T_{z^n} are the T_w on the following n spaces:*

$$L_a^2\left(G, \frac{1}{n^2 |w|^{2(n-k-1)/n}} dA(w)\right), \quad k = 0, 1, \dots, n-1. \quad (24)$$

Proof. Let

$$\mathcal{M}_k^n = \bigvee \{z^m; m \equiv k \pmod n\}, \quad k = 0, 1, \dots, n-1. \quad (25)$$

We define the following n operators, for $k = 0, 1, \dots, n-1$:

$$L_a^2\left(G, \frac{1}{n^2 |w|^{2(n-k-1)/n}} dA(w)\right) \longrightarrow \mathcal{M}_k^n \quad (26)$$

$$V_k^n: f(w) \mapsto z^k f(z^n),$$

and then we can show that V_k^n are isometries; as a fact, by changing of variable $w = z^n$, $|dz/dw| = |1/n \sqrt[n]{w^{n-1}}|$, so

$$\begin{aligned} \|V_k^n f(w)\|^2 &= \|z^k f(z^n)\|^2 = \int_{\mathbb{D}} |z^k f(z^n)|^2 dA(z) \\ &= \int_{\mathbb{D}} |w^{k/n} f(w)|^2 \left| \frac{dz}{dw} \right|^2 dA(w) \\ &= \int_{\mathbb{D}} |w|^{2k/n} |f(w)|^2 \left| \frac{1}{n \sqrt[n]{w^{n-1}}} \right|^2 dA(w) \\ &= \int_{\mathbb{D}} |f(w)|^2 \frac{1}{n^2 |w|^{2(n-k-1)/n}} dA(w) \\ &= \|f(w)\|^2. \end{aligned} \quad (27)$$

Moreover, for any $f(w) \in L_a^2(G, (1/n^2 |w|^{2(n-k-1)/n})dA(w))$, we have

$$\begin{aligned} T_{z^n} V_k^n f(w) &= T_{z^n} (z^k f(z^n)) = z^{n+k} f(z^n) \\ &= z^k (z^n f(z^n)) = V_k^n T_w f(w). \end{aligned} \quad (28)$$

□

Remark 12. $r^\alpha dA(z)$ is a finite measure on G for $\alpha > -2$, so $(1/n^2 |w|^{2(n-k-1)/n})dA(w)$ is a finite measure on G .

Proposition 13. *The restrictions of T_{z^n} on its minimal reducing subspaces are one-side weighted shifts, and they are not unitarily equivalent to each other.*

Proof. For the Bergman space $L_a^2(G, (1/n^2 |w|^{2(n-k-1)/n}) dA(w))$ and any $j \geq 0$, we have

$$\begin{aligned} \|w^j\|^2 &= \int_G |w^j|^2 \frac{1}{n^2 |w|^{2(n-k-1)/n}} dA(w) \\ &= \frac{1}{n^2 \pi} \int_0^{2\pi} d\theta \int_0^1 r^{2j+1-2(n-k-1)/n} dr \\ &= \frac{1}{n(nj+k+1)}, \end{aligned} \quad (29)$$

that is, $\|w\| = \sqrt{1/n(nj+k+1)}$. It implies that

$$\begin{aligned} T_w \left(\sqrt{n(nj+k+1)} w^j \right) \\ = \sqrt{\frac{nj+k+1}{n(j+1)+k+1}} \left(\sqrt{n(n(j+1)+k+1)} w^{j+1} \right), \end{aligned} \quad (30)$$

that is, T_w , and so $T_\varphi|_{M_k^n}$ is the weighted shift with weight $\{\sqrt{(nj+k+1)/(n(j+1)+k+1)}\}$, and so T_w is bounded and $T_\varphi|_{M_k^n}$ is not unitarily equivalent to $T_\varphi|_{M_j^n}$ when $k \neq j$. $\square$

Proposition 14. *The restrictions of T_{z^n} on its minimal reducing subspaces are one-side weighted shifts, and they are similar to each other.*

Proof. $T_\varphi|_{M_k^n}$ and $T_\varphi|_{M_\ell^n}$ are the one-side weighted shift with weights $\{\sqrt{(nj+k+1)/(n(j+1)+k+1)}\}$ and $\{\sqrt{(nj+\ell+1)/(n(j+1)+\ell+1)}\}$, where $0 \leq k < \ell \leq n-1$. So, for any m ,

$$\begin{aligned} \prod_{j=0}^m \frac{\sqrt{(nj+k+1)/(n(j+1)+k+1)}}{\sqrt{(nj+\ell+1)/(n(j+1)+\ell+1)}} \\ = \frac{\prod_{j=0}^m \sqrt{(nj+k+1)/(n(j+1)+k+1)}}{\prod_{j=0}^m \sqrt{(nj+\ell+1)/(n(j+1)+\ell+1)}} \\ = \sqrt{\frac{k+1}{\ell+1}} \sqrt{\frac{n(m+1)+\ell+1}{n(m+1)+k+1}}. \end{aligned} \quad (31)$$

Then,

$$\begin{aligned} 0 < \sqrt{\frac{k+1}{\ell+1}} \leq \prod_{j=0}^m \frac{\sqrt{(nj+k+1)/(n(j+1)+k+1)}}{\sqrt{(nj+\ell+1)/(n(j+1)+\ell+1)}} \\ \leq \sqrt{\frac{k+1}{\ell+1}} \sqrt{\frac{n+\ell+1}{n+k+1}}, \end{aligned} \quad (32)$$

since $(n(m+1)+\ell+1)/(n(m+1)+k+1)$ is decreasing as m increases to infinity. It implies that $T_{z^n}|_{M_k^n}$ is similar to $T_{z^n}|_{M_\ell^n}$ for any $k \neq \ell$. $\square$

Remark 15. For any $0 \leq \ell \leq n-1$, $T_{z^n}|_{M_\ell^n}$ is similar to $T_{z^n}|_{M_0^n}$ which is the Bergman shift, and so T_{z^n} is similar to the direct sum of n copies of the Bergman shift.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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