Research Article

On a Fourth-Order Boundary Value Problem at Resonance

Man Xu and Ruyun Ma

Department of Mathematics, Northwest Normal University, Lanzhou 730070, China

Correspondence should be addressed to Ruyun Ma; mary@nwnu.edu.cn

Received 13 April 2017; Revised 17 May 2017; Accepted 18 May 2017; Published 11 June 2017

Academic Editor: Gennaro Infante

Copyright © 2017 Man Xu and Ruyun Ma. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We investigate the spectrum structure of the eigenvalue problem

\[ u^{(4)}(x) = \lambda u(x), \quad x \in (0,1); \quad u(0) = u(1) = u'(0) = u'(1) = 0 \]

As for the application of the spectrum structure, we show the existence of solutions of the fourth-order boundary value problem at resonance

\[ -u^{(4)}(x) + \lambda_1 u(x) + g(x,u(x)) = h(x), \quad x \in (0,1); \quad u(0) = u(1) = u'(0) = u'(1) = 0 \]

which models a statically elastic beam with both end-points being cantilevered or fixed, where \( \lambda_1 \) is the first eigenvalue of the corresponding eigenvalue problem and nonlinearity \( g \) may be unbounded.

1. Introduction

Starting from the seminal paper of Landesman and Lazer [1], the existence and multiplicity of solutions of nonlinear second-order boundary value problem at resonance,

\[ u''(x) + \pi^2 u(x) + g(x, u(x)) = e(x), \quad x \in (0,1), \quad u(0) = u(1) = 0, \quad (1) \]

and its general case have been extensively studied; see Gupta [2, 3], Iannacci and Nkashama [4, 5], Costa and Goncalves [6], Ambrosetti and Mancini [7], Fonda and Habets [8], Cárc [9], and Ahmad [10] and the references therein. Because of the linear operator \( \mathcal{L} : D(\mathcal{L}) \rightarrow L^2(0,1) \),

\[ \mathcal{L}u = u'' + \pi^2 u, \quad u, u' \in D(\mathcal{L}) = \{ y \in L^2(0,1) : u(0) = u(1) = 0 \} \quad (2) \]

is not reversible; this kind of problems as (1) is of problems at resonance.

In the past twenty years, the existence and multiplicity of solutions (or positive solutions) of nonlinear fourth-order boundary value problems at nonresonance case have been investigated by many authors. Especially, many works address the nonlinear fourth-order differential equation of the following form:

\[ u^{(4)}(x) = g\left(x, u(x), u'(x), u''(x), u'''(x)\right), \quad x \in (0,1), \quad (3) \]

with one of the following sets of boundary conditions:

(i) Both end-points simply supported conditions:

\[ u(0) = u(1) = u''(0) = u''(1) = 0 \quad (\text{Navier boundary condition}). \quad (4) \]

(ii) Both end-points cantilevered or fixed conditions:

\[ u(0) = u(1) = u'(0) = u'(1) = 0 \quad (\text{Dirichlet boundary condition}). \quad (5) \]

(iii) One end simply supported and the other end sliding clamped conditions:

\[ u(0) = u''(0) = u''(1) = u'''(1) = 0. \quad (6) \]

See Rynne [11], Korman [12], Ma et al. [13–15], Cabada et al. [16, 17], Vrabel [18], Schröder [19], Drábek and Holubová.
The purpose of this paper is to show the existence of solutions for problem (7). Finally, needed to apply Leray-Schauder continuation method to problem (8). In Section 3, we give some preliminary results that are needed to apply Leray-Schauder continuation method to obtain the existence of solutions for problem (7). Finally, Section 4 is devoted to stating and proving our main result.

2. The Eigenvalue Problem

In this section, we consider the linear eigenvalue problem:

\begin{align}
    u^{(4)}(x) &= \lambda u(x), \quad x \in (0,1), \\
    u(0) = u(1) = u'(0) = u'(1) = 0.
\end{align}

Lemma 1. The equation

\begin{equation}
    \cos m \cosh m - 1 = 0, \quad m \in \mathbb{R}^+
\end{equation}

has infinitely many simple roots

\begin{equation}
    0 < m_1 < m_2 < m_3 \cdots \rightarrow +\infty.
\end{equation}

Moreover,

\begin{align}
    m_{2k-1} &= \left( \left( 2k - \frac{1}{2} \right) \pi, 2k\pi \right), \\
    m_{2k} &= \left( 2k\pi, \left( 2k + \frac{1}{2} \right) \pi \right)
\end{align}

for \( k \in \mathbb{N} \).

Proof. Let

\begin{equation}
    \gamma(m) = \cos m \cosh m - 1, \quad m \in \mathbb{R}^+.
\end{equation}

It is easy to check that, for \( k \in \mathbb{N} \),

\begin{align}
    \gamma(2k-1)\pi) &< 0, \\
    \gamma(2k\pi) &> 0.
\end{align}

We claim that \( \gamma(m) \) has exactly one root \( m_j \in [j\pi, (j+1)\pi] \); moreover, for any \( j \in \mathbb{N}, m_j \) is simple. Assume that the claim is not true. Then, the following two cases must occur.

Case 1. There are three zeros in \((j_0\pi, (j_0+1)\pi)\) for some \( j_0 \in \mathbb{N} \). In this case, we may find \( \tau \in (j_0\pi, (j_0+1)\pi) \) such that

\begin{equation}
    \gamma''(\tau) = 0.
\end{equation}

However, this contradicts the fact that

\begin{equation}
    \gamma''(m) = -2\sin m \sinh m.
\end{equation}

Case 2. There is a double zero \( \hat{\tau} \in (j_0\pi, (j_0+1)\pi) \) for some \( j_0 \in \mathbb{N} \). In this case, we only deal with case \( j_0 \) being odd. Case \( j_0 \) is even and can be treated similarly.

Since

\begin{align}
    \gamma(j_0\pi) &< 0, \\
    \gamma((j_0 + 1)\pi) &> 0,
\end{align}

we may assume that there exists \( \tilde{\tau} \in (j_0\pi, (j_0+1)\pi) \) such that

\begin{align}
    \gamma(m) &< 0, \quad m \in (j\pi, \tilde{\tau}), \\
    \gamma(\tilde{\tau}) &= \gamma'(\tilde{\tau}) = 0.
\end{align}

Combining this with the fact \( \gamma''(\tilde{\tau}) > 0 \), it concludes that \( \gamma(m) > 0 \) in some left neighborhood of \( \tilde{\tau} \). However, this is a contradiction.
Lemma 2. The linear eigenvalue problem (11) has infinitely many eigenvalues:
\[ \lambda_j = m_j^4 \quad j \in \mathbb{N}, \]  
and the eigenfunction corresponding to \( \lambda_j \) is given by
\[ \varphi_j(x) = \sin m_j x - \sinh m_j x \]
\[ + \frac{\sin m_j - \sinh m_j}{\cos m_j - \cosh m_j} \left( \cosh m_j x - \cos m_j x \right). \]  
Moreover, \( \varphi_j \in S_{k,+} \), where \( S_{k,+} \) denote the set of \( u \in C^2(0,1) \) such that
(i) \( u \) has only simple zeros in \( (0,1) \) and has exactly \( k - 1 \) such zeros;
(ii) \( u''(0) > 0 \) and \( u''(1) \neq 0 \).

Proof. By [11, P. 308], we know problem (11) has a sequence of eigenvalues \( 0 < \lambda_1 < \lambda_2 < \cdots \) with \( \lim_{k \to \infty} \lambda_k = +\infty \). For any given \( k \in \mathbb{N} \), each eigenvalue \( \lambda_k \) is simple and has a corresponding eigenfunction \( \varphi_k \) satisfying (i) and (ii) by a direct calculation, we have (21) and (22). \( \square \)

3. Preliminaries

Throughout the paper, we assume that

(H0) \( p \in L^\infty(0,1) \) such that, for a.e. \( x \in (0,1) \), \( 0 \leq p(x) \leq m_1^4 - m_2^4 \); moreover \( p(x) < m_2^4 - m_1^4 \) on a subset of \( (0,1) \) of positive measure.

Define a linear operator \( L : D(L) \subset L^2(0,1) \to L^2(0,1) \) by
\[ L(u) = u'' + \lambda_1 u, \]  
where \( D(L) = H^4(0,1) \cap H_0^2(0,1) \) is the linear self-adjoint operator, and thus \( L^* \) admits the orthogonal direct sum decomposition \( L^2(0,1) = N \oplus R \), where \( N \) is the one-dimensional null space of \( L \) and \( R \) is the range space of \( L \), namely,
\[ N = \left\{ y \in L^2(0,1) \mid y = s \varphi_1 \text{ for some } s \in \mathbb{R} \right\}, \]
\[ R = \left\{ y \in L^2(0,1) \mid \int_0^1 y(x) \varphi_1(x) \, dx = 0 \right\}. \]  
Therefore each \( u \in H_0^2(0,1) \subset L^2(0,1) \) has a unique decomposition:
\[ u = s \varphi_1 + w = \overline{u} + \tilde{u}, \]  
where \( s \in \mathbb{R} \), \( w \in R \), so that, with obvious notations, \( H_0^2(0,1) = H_0^2(0,1) \).

Since \( u \in H_0^2(0,1) \subset L^2(0,1) \), it follows that \( u \) has the Fourier series expansion:
\[ u(x) = \sum_{j=1}^{\infty} s_j \varphi_j(x), \]  
\[ s_j = \int_0^1 u(x) \varphi_j(x) \, dx. \]

By (25), we observe that
\[ \overline{u}(x) = s_1 \varphi_1(x), \]
\[ \tilde{u}(x) = \sum_{j=2}^{\infty} s_j \varphi_j(x). \]  

Lemma 3. Assume that \( p \) satisfies (H0). Let \( \sigma > 0 \) and \( q \in L^\infty(0,1) \) satisfy
\[ 0 \leq q(x) \leq p(x) + \sigma \quad \text{for a.e. } x \in (0,1). \]  
Then there exists a constant \( \delta = \delta(p) > 0 \) such that, for all \( u \in H \), we have
\[ \int_0^1 \left[ -u''(x) + \lambda_1 u(x) + q(x) u(x) \right] \cdot (\overline{u}(x) - \tilde{u}(x)) \, dx \geq \delta \| \overline{u} \|_{H^2}. \]  

Proof. We will divide the proof into three steps.

Step 1. It follows from Lemma 2 that, for all \( u \in H \),
\[ \overline{u}''(x) = \lambda_1 \overline{u}(x), \quad x \in (0,1), \]
\[ u(0) = u(1) = u'(0) = u'(1) = 0. \]  
Multiplying both sides of the equation in (30) by \( \overline{u} \) and integrating from 0 to 1, we get that
\[ \int_0^1 (\overline{u}''(x))^2 \, dx - \lambda_1 \int_0^1 (\overline{u}(x))^2 \, dx = 0. \]  
This together with the orthogonality of \( \overline{u} \) and \( \tilde{u} \) in \( L^2(0,1) \) implies that
\[ \int_0^1 \left[ -u''(x) + \lambda_1 u(x) + q(x) u(x) \right] \cdot (\overline{u}(x) - \tilde{u}(x)) \, dx \geq \int_0^1 (\overline{u}(x))^2 \, dx \]
\[ + \int_0^1 (\tilde{u}''(x))^2 \, dx - \int_0^1 (q(x) + \lambda_1) (\tilde{u}(x))^2 \, dx \geq \int_0^1 (\tilde{u}''(x))^2 \, dx - \int_0^1 (q(x) + \lambda_1) (\tilde{u}(x))^2 \, dx \]
\[ \equiv D_q (\tilde{u}). \]

Subsequently, by (28), we have
\[ D_q (\tilde{u}) \geq D_p (\overline{u}) - \sigma \int_0^1 (\tilde{u}(x))^2 \, dx. \]  

Step 2. We show that if \( u \in H \), then there exists a constant \( \delta = \delta(p) > 0 \) satisfying
\[ D_p (\overline{u}) \geq \delta \| \overline{u} \|_{H^2}. \]
Firstly, by (27), we observe that
\[ \tilde{u}''(x) = \sum_{j=2}^{\infty} s_j \phi_j''(x). \]  
(35)

By Lemma 2, \((m_j^4, \phi_j)\) is a solution of (11). So that, substituting into (11) and multiplying both sides of the equation by \(\phi_j\) and integrating from 0 to 1, we get that, for \(x \in (0, 1)\),
\[ \int_0^1 (\phi_j''(x))^2 \, dx = m_j^4 \int_0^1 (\phi_j(x))^2 \, dx. \]  
(36)

This fact together with (35) and using Parseval identity yields that
\[ \int_0^1 (\tilde{u}'(x))^2 \, dx = \sum_{j=2}^{\infty} s_j^2, \]  
(37)

\[ \int_0^1 (\tilde{u}''(x))^2 \, dx = \sum_{j=2}^{\infty} m_j^4 s_j^2. \]  
(38)

Therefore, by (H0), we find that
\[ D_p (\tilde{u}) \geq \sum_{j=2}^{\infty} s_j^2 (m_j^4 - m_j^2) \geq 0, \]  
with equality if and only if \(s_j (m_j^4 - m_j^2) = 0\) for all \(j \in \mathbb{N}\) and \(j \geq 2\). Therefore, for \(j > 2\), one has \(s_j = 0\), and, by using the series expansion, \(\tilde{u}(x)\) reduces to \(\tilde{u}(x) = s_2 \phi_2(x)\). But then, we have
\[ D_p (\tilde{u}) = \int_0^1 \left( \left( \tilde{u}''(x) \right)^2 - m_2^4 (\tilde{u}(x))^2 \right) \, dx = 0. \]  
(39)

It follows from (H0) that \(s_2 = 0\), and hence \(\tilde{u} = 0\).

Next we will prove that (34) is true. Suppose, on the contrary, that there exists a sequence \(\{\tilde{u}_n\} \subset H^2_0(0, 1)\) and \(\tilde{u} \in H^2_0(0, 1)\) such that
\[ \lim_{n \to \infty} D_p (\tilde{u}_n) = 0 \quad \text{with} \quad \|\tilde{u}_n\|_{H^2} = 1. \]  
(40)

It follows from the compact embedding of \(H^2_0(0, 1)\) into \(C^1[0, 1]\) that
\[ \tilde{u}_n \rightharpoonup \tilde{u} \quad \text{in} \quad C^1[0, 1], \]  
(41)

\[ \tilde{u}_n \to \tilde{u} \quad \text{in} \quad H^2_0(0, 1). \]

Now (41) implies that
\[ \|\tilde{u}\|_{H^2}^2 \leq \liminf_{n \to \infty} \|\tilde{u}_n\|_{H^2}^2. \]  
(42)

At the same time by (40) and (41), we obtain
\[ \lim_{n \to \infty} \|\tilde{u}_n\|_{H^2}^2 = \int_0^1 (p(x) + \lambda_1) (\tilde{u}(x))^2 \, dx. \]  
(43)

This together with (42) implies that
\[ \|\tilde{u}\|_{H^2}^2 \leq \int_0^1 (p(x) + \lambda_1) (\tilde{u}(x))^2 \, dx, \]  
(44)

that is, \(D_p (\tilde{u}) \leq 0\). By the fact that \(D_p (\tilde{u}) \geq 0\) with equality if \(\tilde{u} = 0\), we know \(\tilde{u} = 0\); this contradicts the fact that \(\|\tilde{u}\|_{H^2} = 1\).

Step 3. By a direct observation of (33) and (34), we obtain the desired results.

**Lemma 4.** Let \(\xi \in (0, m_4^4 - m_4^2)\) be fixed constant. Define a linear operator \(E : H \to L^2(0, 1)\) by
\[ E(u) = u^{(4)} + \lambda_1 u + \xi u. \]  
(45)

Then \(E^{-1} : L^2(0, 1) \to H\) is completely continuous.

**Proof.** By the theory of linear fourth-order differential equations, the operator \(E : H \to L^2(0, 1)\) defined by
\[ E(u) = u^{(4)} + \lambda_1 u + \xi u \]  
(46)

is one-to-one and continuous obviously. It follows that \(E^{-1} : L^2(0, 1) \to H\) is completely continuous.

**4. The Main Result and the Proof**

The main result of the paper addresses the existence of solutions of fourth-order problem (7), when the nonlinearity is unbounded. For the sake of simplicity, we assume the following:

(H1) \(g : (0, 1) \times \mathbb{R} \to \mathbb{R}\) is a \(L^2\)-Carathéodory function; namely, \(g(\cdot, u)\) is measurable on \((0, 1)\) for every \(u \in \mathbb{R}\), \(g(x, \cdot)\) is continuous on \(\mathbb{R}\) for a.e. \(x \in (0, 1)\), for any constant \(r > 0\), and there exists a function \(\Gamma_r \in L^1(0, 1)\) such that
\[ |g(x, u)| \leq \Gamma_r(x) \]  
(47)

for a.e. \(x \in (0, 1)\) and all \(u \in \mathbb{R}\) with \(|u| \leq r\).

(H2) \(ug(x, u) \geq 0\) for a.e. \(x \in (0, 1)\) and all \(u \in \mathbb{R}\).

(H3) For all constant \(\sigma > 0\), there exist a constant \(R = R(\sigma) > 0\) and a function \(b = b(\sigma) \in L^\infty(0, 1)\) such that
\[ |g(x, u)| \leq (p(x) + \sigma) |u| + b(x) \]  
(48)

for a.e. \(x \in (0, 1)\) and all \(u \in \mathbb{R}\) with \(|u| \geq R\), where \(p \in L^\infty(0, 1)\) has been given by (H0).

**Theorem 5.** Assume that (H0)–(H3) hold. Then problem (7) has at least one solution for any \(h \in L^2(0, 1)\) provided:
\[ \int_0^1 h(x) \varphi_1(x) \, dx = 0. \]  
(49)
Proof. Let $\delta > 0$ be associated with function $p$ and $\xi \in (0, m_2^4 - m_1^4)$ be a fixed constant with $\xi < \delta/2$. To study problem (7) using Leray-Schauder continuation method, we prove that each of the possible solutions of the homotopy

$$-u^{(4)} + \lambda_1 u + (1 - \lambda) \xi u + \lambda g(x, u) = \lambda h$$

$$x \in (0, 1), \quad (50)$$

$$u(0) = u(1) = u'(0) = u'(1) = 0$$

has a priori bound. Therefore, we claim that if $u \in H$ is a solution of (50), then there exists a constant $\rho > 0$ independently of $\lambda \in [0, 1)$ such that

$$\|u\|_H < \rho.$$  \hspace{1cm} (51)

If we assume on the contrary that there exists a sequence $\{\lambda_n\} \subset (0, 1)$ and a sequence $\{u_n\} \subset H$ with $\|u_n\|_H \geq n$ for all $n \in \mathbb{N}$ such that

$$-u^{(4)}_n + \lambda_1 u_n + (1 - \lambda_n) \xi u_n + \lambda_n g(x, u_n) = \lambda_n h,$$

$$u_n(0) = u_n(1) = u'_n(0) = u'_n(1) = 0.$$ \hspace{1cm} (52)

Let $v_n = u_n/\|u_n\|_H$. Then

$$-v^{(4)}_n + \lambda v_n + \xi v_n = \lambda \frac{h}{\|u_n\|_H}, \quad x \in (0, 1),$$

$$v_n(0) = v_n(1) = v'_n(0) = v'_n(1) = 0.$$ \hspace{1cm} (53)

Obviously, by Lemma 4, (53) is equivalent to

$$v_n = \frac{1}{\lambda}\left[ \lambda v_n + \xi v_n - \frac{h}{\|u_n\|_H} \right].$$ \hspace{1cm} (54)

(47) together with (48) yields that there exists a function $c \in L^\infty(0, 1)$ depending only on $R = R(\delta)$ such that

$$|g(x, u)| \leq \left( p(x) + \frac{\delta}{2} \right)|u| + b(x) + c(x)$$

for a.e. $x \in (0, 1)$ and all $u \in \mathbb{R}$.

Subsequently, the right-hand member of (54) is bounded in $L^2(0, 1)$ independently of $n$. By Lemma 4, there exists $v \in H$ such that $\lim_{n \to \infty} v_n = v$ in $H$. Moreover, $\|v\|_H = 1$.

On the other hand, (H3) yields that there exist $R = R(\delta) > 0$ and $b = b(\delta) \in L^\infty(0, 1)$ such that

$$|g(x, u)| \leq \left( p(x) + \frac{\delta}{4} \right)|u| + b(x)$$ \hspace{1cm} (56)

for a.e. $x \in (0, 1)$ and all $u \in \mathbb{R}$ with $|u| \geq R$, where $R$ is chosen such that $b(x)/|u| < \delta/4$. Let us define a function $\tilde{p} : (0, 1) \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{p}(x, u) = \begin{cases} 
\frac{g(x, u)}{u}, & \text{for } |u| \geq R, \\
\frac{g(x, R)}{R} u + \left( 1 - \frac{u}{R} \right) p(x), & \text{for } 0 \leq u \leq R, \\
\frac{g(x, -R)}{R} u + \left( 1 + \frac{u}{R} \right) p(x), & \text{for } -R \leq u \leq 0.
\end{cases}$$ \hspace{1cm} (57)

Then, this together with (H2) and (56) yields that

$$0 \leq \tilde{p}(x, u) \leq p(x) + \frac{\delta}{2}$$ \hspace{1cm} (58)

for a.e. $x \in (0, 1)$ and all $u \in \mathbb{R}$.

Moreover, $\tilde{p}(x, u)u$ is a $L^2$-Carathéodory function. Define $f : (0, 1) \times \mathbb{R} \to \mathbb{R}$ by

$$f(x, u) = g(x, u) - \tilde{p}(x, u)u.$$ \hspace{1cm} (59)

By (H1), it yields that, for a.e. $x \in (0, 1)$ and all $u \in \mathbb{R}$, there exists $v \in L^2(0, 1)$, such that

$$\|f(x, u)\| \leq v(x).$$ \hspace{1cm} (60)

Observe that $v$ depend only on $\Gamma$ and $\gamma_R$.

Thus, problem (50) is equivalent to

$$-u^{(4)}(x) + \lambda_1 u(x) + (1 - \lambda) \xi u(x)$$

$$+ \lambda \tilde{p}(x, u(x)) u(x) + \lambda f(x, u(x)) = \lambda h(x),$$

$$u(0) = u(1) = u'(0) = u'(1) = 0.$$ \hspace{1cm} (61)

The fact $\xi \in (0, m_2^4 - m_1^4)$ with $\xi < \delta/2$ together with (58) yields that

$$0 \leq (1 - \lambda) \xi + \lambda \tilde{p}(x, u) \leq p(x) + \frac{\delta}{2}$$ \hspace{1cm} (62)

for a.e. $x \in (0, 1)$ and all $u \in \mathbb{R}$.

Therefore, by Lemma 3, (60), and the compact embedding of $H^2(0, 1)$ into $L^2(0, 1)$, we have

$$0 = \int_0^1 \left[ -u^{(4)}(x) + \lambda_1 u(x) + (1 - \lambda) \xi u(x) \\
+ \lambda \tilde{p}(x, u(x)) u(x) \right] (\pi(x) - \tilde{u}(x)) \, dx$$

$$+ \int_0^1 \left( \lambda f(x, u(x)) - \lambda h(x) \right) (\pi(x) - \tilde{u}(x)) \, dx \geq \frac{\delta}{2} \|\pi\|_{L^2}^2 - (\|\pi\|_{L^2} + \|\tilde{u}\|_{L^2}) \left( \|f\|_{L^2} + \|h\|_{L^2} \right)$$

$$\geq \frac{\delta}{2} \|\pi\|_{L^2}^2 - C (\|\pi\|_{L^2} + \|\tilde{u}\|_{L^2})$$

for some constant $C > 0$. 
By (63), we deduce immediately that \( \lim_{n \to \infty} \overline{v}_n = 0 \) in \( H^2(0, 1) \). Therefore we can write \( v_n = \overline{v}_n \). Since \( \|v\|_{H^2} = 1 \), we shall suppose that

\[
\nu(x) = c \psi_1(x) \quad \text{for some } c > 0.
\]

(64)

Now, using Lemma 2, we can get that there exists \( N \) such that, for \( n \geq N \), \( v_n(x) > 0 \) on \( (0, 1) \). So that, for \( n \geq N \),

\[
u_n(0) = u_n(1) = u_n'(0) = u_n'(1) = 0, \quad v_n(x) > 0.\]

(65)

Multiplying both sides of the equation in (53) by \( \overline{v}_n \) and integrating from 0 to 1, by (31), (49), and the fact \( \lambda_n \in (0, 1) \), we have

\[
(1 - \lambda_n) \int_0^1 (\overline{v}_n)^2 \, dx = -\frac{\lambda_n}{\|u_n\|_{H^2}} \int_0^1 g(x, u_n) \overline{v}_n \, dx.
\]

(66)

So that \( \int_0^1 \overline{g}(x, u_n) \overline{v}_n \, dx < 0 \). By (H2) and (65), we conclude a contradiction. \( \square \)

Conflicts of Interest

All of the authors of this article declare that there are no conflicts of interest regarding the publication of this paper.

Authors’ Contributions

Man Xu and Ruyun Ma completed the main study together and Man Xu wrote the manuscript; Ruyun Ma checked the proofs process and verified the calculation. Moreover, all the authors read and approved the last version of the manuscript.

Acknowledgments

This work was supported by NSFC (no. 11671322) and NSFC (no. 11361054).

References


