Research Article

The Existence of Solutions to Integral Boundary Value Problems of Fractional Differential Equations at Resonance

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This paper deals with the integral boundary value problems of fractional differential equations at resonance. By Mawhin's coincidence degree theory, we present some new results on the existence of solutions for a class of differential equations of fractional order with integral boundary conditions at resonance. An example is also included to illustrate the main results.

1. Introduction

In this paper, we are concerned with the following integral boundary value problem for nonlinear fractional differential equation:

$$\begin{aligned} -D_{0+}^p x(t) &= f(t, x(t), D_{0+}^{p-1} x(t), D_{0+}^{p-2} x(t)), \\ t &\in (0, 1), \\ x(0) &= x'(0) = 0, \\ x(1) &= \int_0^1 x(t) dA(t), \end{aligned} \quad (1)$$

where $2 < p < 3$, D_{0+}^p is the standard Riemann-Liouville differentiation, $f: [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$, and f satisfies Carathéodory conditions; $A(t)$ is right continuous on $[0, 1)$ and left continuous at $t = 1$; $\int_0^1 x(t) dA(t)$ denotes the Riemann-Stieltjes integrals of x with respect to A . Our problem are at resonance, in the sense that, under the integral boundary conditions, we study the linear equation $-D_{0+}^p x(t) = 0$, $t \in (0, 1)$, which has nontrivial solutions.

Recently, fractional differential equations have received considerable attentions not only because of a generalization of ordinary differential equations but also because they have

played a significant role in science, engineering, economy, and other fields; see, for example, [1–3].

When $A(t) \equiv 0$, problem (1) is nonresonant. In [4], the authors studied the existence of positive solutions for the nonresonant case by Krasnosel'skii's fixed point theorem. In [5], the author investigated the uniqueness of solutions for the nonresonant case by use of the u_0 -positive operator under a Lipschitz condition on f .

In present, many papers are devoted to the integral boundary value problem for fractional differential equation under nonresonance conditions; see [4–20]. On the other hand, there are some papers studying integral boundary value problem for differential equation under resonant conditions; we refer the reader to [21–29].

Motivated by the above results, in this paper, we consider the existence of solutions for the resonance integral boundary value problem (1) under nonlinear growth restriction of f . Our method is based upon the coincidence degree theorem of Mawhin.

Now, we recall the essentials of the coincidence degree theory. Let X and Y be real Banach spaces, and let $L: \text{dom } L \subset X \rightarrow Y$ be a Fredholm operator of index zero. If $P: X \rightarrow X$ and $Q: Y \rightarrow Y$ are continuous projectors such that $\text{Im } P = \text{Ker } L$, $\text{Ker } Q = \text{Im } L$, $X = \text{Ker } L \oplus \text{Im } P$, and $Y = \text{Im } L \oplus \text{Im } Q$, then the inverse operator of $L|_{\text{dom } L \cap \text{Ker } P}: \text{dom } L \cap \text{Ker } P \rightarrow \text{Im } L$

$\text{dom } L \cap \text{Ker } P \rightarrow \text{Im } L$ exists and is denoted by K_P (generalized inverse operator of L). If Ω is an open bounded subset of X such that $\text{dom } L \cap \Omega \neq \emptyset$, the mapping $N : X \rightarrow Y$ will be called L -compact on $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \rightarrow X$ is compact.

The abstract equation $Lx = Nx$ is shown to be solvable in view of Theorem IV.13 [30].

Theorem 1 (see [30]). *Let L be a Fredholm operator of index zero and let N be L -compact on $\overline{\Omega}$. Assume the following conditions are satisfied:*

- (i) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\text{dom } L \setminus \text{Ker } L) \cap \partial\Omega] \times (0, 1)$.
- (ii) $Nx \notin \text{Im } L$ for every $x \in \text{Ker } L \cap \partial\Omega$.
- (iii) $\deg(QN|_{\text{Ker } L}, \text{Ker } L \cap \Omega, 0) \neq 0$, where $Q : Z \rightarrow Z$ is a projector as above with $\text{Im } L = \text{Ker } Q$.

Then the equation $Lx = Nx$ has at least one solution in $\text{dom } L \cap \overline{\Omega}$.

Throughout this paper, we always suppose that

- (H₁) $\int_0^1 t^{p-1} dA(t) = 1, \kappa = \int_0^1 t^p dA(t) - 1 \neq 0$;
- (H₂) $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions; that is, $f(\cdot, u, v, w)$ is measurable for each fixed $(u, v, w) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $f(t, \cdot, \cdot, \cdot)$ is continuous for a.e. $t \in [0, 1]$, and for each $r > 0$, there exists $\Phi_r \in L^\infty[0, 1]$ such that $|f(t, u, v, w)| \leq \Phi_r(t)$ for all $|u|, |v|$, and $|w| \leq r$ and for a.e. $t \in [0, 1]$.

2. Preliminaries and Lemmas

In this section, first we provide recall some necessary basic definitions and lemmas of the fractional calculus theory, which will be used in this paper. For more details, we refer to books [1–3] for details.

Definition 2 (see [1]). The Riemann-Liouville fractional integral of order $p > 0$ of a function $f : (0, \infty) \rightarrow \mathbb{R}$ is given by

$$I_{0+}^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s) ds, \quad (2)$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 3 (see [1]). The Riemann-Liouville fractional derivative of order $p > 0$ of a continuous function $f : (0, \infty) \rightarrow \mathbb{R}$ is given by

$$D_{0+}^p f(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{f(s)}{(t-s)^{p-n+1}} ds, \quad (3)$$

where $n-1 \leq p < n$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 4 (see [1]). Assume that $u \in L^1[0, 1]$. If $\alpha, \beta, p > 0$, then

$$\begin{aligned} (I_{0+}^\alpha I_{0+}^\beta) u(t) &= (I_{0+}^{\alpha+\beta}) u(t), \\ (D_{0+}^p I_{0+}^p u)(t) &= u(t). \end{aligned} \quad (4)$$

Lemma 5 (see [1]). Assume that $I_{0+}^{N-p} u \in AC^{N-1}[0, 1]$, $p > 0$. Then one has

$$\begin{aligned} (I_{0+}^p D_{0+}^p u)(t) \\ = u(t) - \sum_{k=0}^{N-1} \frac{t^{p-k-1}}{\Gamma(p-k)} \left(\frac{d^{N-k-1}}{dt^{N-k-1}} I_{0+}^{n-p} u \right)(0), \end{aligned} \quad (5)$$

$t \in [0, 1],$

where $N = [p] + 1$.

Lemma 6 (see [4]). Let $\sigma \in C(0, 1) \cap L(0, 1)$ and $2 < p \leq 3$, then the unique solution of

$$\begin{aligned} D_{0+}^p x(t) + \sigma(t) &= 0, \quad t \in (0, 1), \\ x(0) = x'(0) &= 0, \\ x(1) &= 0 \end{aligned} \quad (6)$$

is given by

$$x(t) = \int_0^1 G(t, s) \sigma(s) ds, \quad (7)$$

where $G(t, s)$ is Green's function given by

$$\begin{aligned} G(t, s) \\ = \begin{cases} \frac{(1-s)^{p-1} t^{p-1} - (t-s)^{p-1}}{\Gamma(p)}, & 0 \leq s \leq t \leq 1, \\ \frac{(1-s)^{p-1} t^{p-1}}{\Gamma(p)}, & 0 \leq t \leq s \leq 1. \end{cases} \end{aligned} \quad (8)$$

Lemma 7 (see [4]). The function $G(t, s)$ defined by (8) satisfies $G(t, s) > 0$ and $t, s \in (0, 1)$.

We use the classical Banach space $C[0, 1]$ with the norm $\|x\|_\infty = \max_{t \in [0, 1]} |x(t)|$ and $L^1[0, 1]$ with the norm $\|x\|_1 = \int_0^1 |x(t)| dt$. We also use the Banach space X

$$X = \{x : [0, 1] \rightarrow \mathbb{R} \mid x, D_{0+}^{p-1} x, D_{0+}^{p-2} x \in C[0, 1]\} \quad (9)$$

with the norm $\|x\|_X = \max\{\|x\|_\infty, \|D_{0+}^{p-1} x\|_\infty, \|D_{0+}^{p-2} x\|_\infty\}$.

Define $L : \text{dom } L \subset X \rightarrow Y = L^1[0, 1]$ and $N : X \rightarrow Y$ as follows:

$$\begin{aligned} (Lx)(t) &= -D_{0+}^p x(t), \\ (Nx)(t) &= f(t, x(t), D_{0+}^{p-1} x(t), D_{0+}^{p-2} x(t)), \end{aligned} \quad (10)$$

$t \in [0, 1],$

where $\text{dom } L = \{x \in X \mid D_{0+}^p x \in Y, x(0) = x'(0) = 0, x(1) = \int_0^1 x(t) dA(t)\}$.

Then integral boundary value problems (1) can be rewritten as follows:

$$(Lx)(t) = (Nx)(t), \quad x \in \text{dom } L. \quad (11)$$

Lemma 8. *The operator L is a Fredholm operator of index zero.*

Proof. Firstly, we show that

$$\text{Ker } L = \{ct^{p-1} : c \in \mathbb{R}\}. \quad (12)$$

By Lemma 5, $u \in \text{Ker } L$ means that $u(t) = c_1 t^{p-1} + c_2 t^{p-2} + c_3 t^{p-3}$. It follows from $u(0) = u'(0) = 0$ that $c_2 = c_3 = 0$. That is, $\text{Ker } L = \{ct^{p-1} : c \in \mathbb{R}\}$.

Now we prove

$$\begin{aligned} \text{Im } L = \left\{ y \in Y : \int_0^1 (1-s)^{p-1} y(s) ds \right. \\ \left. - \int_0^1 \int_0^t (t-s)^{p-1} y(s) ds dA(t) = 0 \right\}. \end{aligned} \quad (13)$$

In fact, if $u \in \text{dom } L$ and $Lu = y$, then by Lemma 5,

$$u(t) = -I_{0+}^p y(t) + c_1 t^{p-1} + c_2 t^{p-2} + c_3 t^{p-3}. \quad (14)$$

By the boundary condition, we obtain $c_2 = c_3 = 0$,

$$\begin{aligned} \int_0^1 u(t) dA(t) &= -\frac{1}{\Gamma(p)} \int_0^1 \int_0^t (t-s)^{p-1} y(s) ds dA(t) \\ &\quad + c_1 \int_0^1 t^{p-1} dA(t) \\ &= -\frac{1}{\Gamma(p)} \int_0^1 \int_0^t (t-s)^{p-1} y(s) ds dA(t) + c_1, \\ u(1) &= -\frac{1}{\Gamma(p)} \int_0^1 (1-s)^{p-1} y(s) ds + c_1. \end{aligned} \quad (15)$$

The above equalities imply that

$$\begin{aligned} \int_0^1 (1-s)^{p-1} y(s) ds \\ = \int_0^1 \int_0^t (t-s)^{p-1} y(s) ds dA(t). \end{aligned} \quad (16)$$

On the other hand, if $y \in Y$ satisfies $\int_0^1 (1-s)^{p-1} y(s) ds = \int_0^1 \int_0^t (t-s)^{p-1} y(s) ds dA(t)$, let

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} y(s) ds \\ &\quad + \frac{t^{p-1}}{\Gamma(p)} \int_0^1 (1-s)^{p-1} y(s) ds. \end{aligned} \quad (17)$$

By a simple computation, we can obtain that $u \in \text{dom } L$ and $Lu = y$; that is, $y \in \text{Im } L$.

Clearly, $\dim \text{Ker } L = 1$ and $\text{Im } L$ is closed. It follows from $Y_1 = Y \setminus \text{Im } L$ that

$$\begin{aligned} Y_1 = \left\{ y_1 \in Y : y_1 = \frac{p}{\kappa} \left(\int_0^1 (1-s)^{p-1} y(s) ds \right. \right. \\ \left. \left. - \int_0^1 \int_0^t (t-s)^{p-1} y(s) ds dA(t) \right), y \in Y \right\}, \end{aligned} \quad (18)$$

where $\kappa = \int_0^1 t^p dA(t) - 1 \neq 0$. In fact, for each $y \in Y$, we have

$$\begin{aligned} \int_0^1 (1-s)^{p-1} (y(s) - y_1(s)) ds - \int_0^1 \int_0^t (t-s)^{p-1} \\ \cdot (y(s) - y_1(s)) ds dA(t) = \left(1 \right. \\ \left. - \frac{p}{\kappa} \left(\int_0^1 (1-s)^{p-1} ds \right. \right. \\ \left. \left. - \int_0^1 \int_0^t (t-s)^{p-1} ds dA(t) \right) \right) \cdot \left(\int_0^1 (1-s)^{p-1} \right. \\ \left. \cdot y(s) ds - \int_0^1 \int_0^t (t-s)^{p-1} y(s) ds dA(t) \right) = 0 \end{aligned} \quad (19)$$

which shows that $y - y_1 \in \text{Im } L$. This together with $Y_1 \cap \text{Im } L = \{\theta\}$ implies that $Y = Y_1 \oplus \text{Im } L$. Note that $\dim Y_1 = 1$ and thus $\text{codim } \text{Im } L = 1$. Therefore, L is a Fredholm operator of index zero. The proof is completed. $\square$

Next, define the projections $P : X \rightarrow X$ by

$$(Py)(t) = y(1) t^{p-1} \quad (20)$$

and $Q : Y \rightarrow Y$ by

$$\begin{aligned} (Qy)(t) &= \frac{p}{\kappa} \left(\int_0^1 (1-s)^{p-1} y(s) ds \right. \\ &\quad \left. - \int_0^1 \int_0^t (t-s)^{p-1} y(s) ds dA(t) \right). \end{aligned} \quad (21)$$

Clearly, $\text{Im } P = \text{Ker } L$ and $\text{Ker } Q = \text{Im } L$.

The generalized inverse operator of L , $K_p : \text{Im } L \rightarrow \text{dom } L \cap \text{Ker } P$ can be defined by

$$\begin{aligned} (K_p y)(t) &= \int_0^1 G(t, s) y(s) ds \\ &= \frac{t^{p-1}}{\Gamma(p)} \int_0^1 (1-s)^{p-1} y(s) ds \\ &\quad - \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} y(s) ds. \end{aligned} \quad (22)$$

In fact, if $y \in \text{Im } L$, then

$$\begin{aligned} LK_p y &= -D_{0+}^p \left(-\frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} y(s) ds \right. \\ &\quad \left. + \frac{t^{p-1}}{\Gamma(p)} \int_0^1 (1-s)^{p-1} y(s) ds \right) = y. \end{aligned} \quad (23)$$

For $x \in \text{dom } L \cap \text{Ker } P$, $Lx = y$, we have

$$\begin{aligned} -D_{0+}^p x(t) &= y(t), \quad t \in (0, 1), \\ x(0) &= x'(0) = 0, \\ x(1) &= 0. \end{aligned} \quad (24)$$

Then by Lemma 6, we obtain $K_p Lx = x$ whenever $x \in \text{dom } L \cap \text{Ker } P$.

Using (21) and (22), we write

$$\begin{aligned} K_p(I - Q)Ny(t) &= \int_0^1 G(t, s) Ny(s) ds \\ &- \frac{t^{p-1}(1-t)}{\kappa \Gamma(p)} \cdot \left(\int_0^1 (1-s)^{p-1} Ny(s) ds \right. \\ &\left. - \int_0^1 \int_0^t (t-s)^{p-1} Ny(s) ds dA(t) \right). \end{aligned} \quad (25)$$

By a standard method, we obtain the following lemma.

Lemma 9. $K_p(I - Q)N : Y \rightarrow Y$ is completely continuous.

Lemma 10. For $y \in Y$,

$$\begin{aligned} \|K_p y\|_\infty &\leq \frac{1}{\Gamma(p)} \|y\|_1, \\ \|D_{0+}^{p-1} K_p y\|_\infty &\leq 2 \|y\|_1, \\ \|D_{0+}^{p-2} K_p y\|_\infty &\leq 2 \|y\|_1. \end{aligned} \quad (26)$$

Moreover,

$$\|K_p y\|_X \leq 2 \|y\|_1. \quad (27)$$

Proof. It is easy to see that

$$\begin{aligned} (K_p y)(t) &= \frac{t^{p-1}}{\Gamma(p)} \int_0^1 (1-s)^{p-1} y(s) ds \\ &- \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} y(s) ds, \\ D_{0+}^{p-1} (K_p y)(t) &= \int_0^1 (1-s)^{p-1} y(s) ds - \int_0^t y(s) ds, \\ D_{0+}^{p-2} (K_p y)(t) &= t \int_0^1 (1-s)^{p-1} y(s) ds \\ &- \int_0^t (t-s) y(s) ds. \end{aligned} \quad (28)$$

Then by Lemma 7, we obtain

$$\begin{aligned} \|K_p y\|_\infty &\leq \frac{1}{\Gamma(p)} \max_{t,s \in [0,1]} G(t, s) \|y\|_1 \\ &= \frac{1}{\Gamma(p)} \|y\|_1 \end{aligned} \quad (29)$$

$$\|D_{0+}^{p-1} K_p y\|_\infty \leq 2 \|y\|_1,$$

$$\|D_{0+}^{p-2} K_p y\|_\infty \leq 2 \|y\|_1.$$

It follows that

$$\|K_p y\|_X \leq \max \left\{ \frac{1}{\Gamma(p)}, 2 \right\} \|y\|_1 = 2 \|y\|_1. \quad (30)$$

The proof is completed. $\square$

3. Main Results

In this section, we will use Theorem 1 to prove the existence of solutions to IBVP (1). To obtain our main theorem, we need the following conditions:

(H₃) There exist functions $\rho, \alpha, \beta, \gamma \in L^1[0, 1]$ such that, for all $(u, v, w) \in \mathbb{R}^3, t \in [0, 1]$,

$$|f(t, u, v, w)| \leq \rho(t) + \alpha(t)|u| + \beta(t)|v| + \gamma(t)|w|. \quad (31)$$

(H₄) There exists a constant $B > 0$ such that either for each $c \in \mathbb{R} : |c| > B$

$$cQN(ct^{p-1}) > 0 \quad (32)$$

or for each $c \in \mathbb{R} : |c| > B$,

$$cQN(ct^{p-1}) < 0. \quad (33)$$

(H₅) There exists a constant $M > 0$ such that if $|D_{0+}^{p-1} x(t)| > M$ for all $t \in [0, 1]$, then $QNx \neq 0$.

(H₆) There exists a constant $M > 0$ such that if $|D_{0+}^{p-1} x(t)| + |D_{0+}^{p-2} x(t)| > M$ for all $t \in [0, 1]$, then $QNx \neq 0$.

Theorem 11. Suppose (H₁)–(H₅) hold. Then IBVP (1) has at least one solution in Y , provided

$$\|\alpha\|_1 + \|\beta\|_1 + \|\gamma\|_1 < \frac{1}{5}. \quad (34)$$

Proof. Set

$$\begin{aligned} \Omega_1 &= \{x \in \text{dom } L \setminus \text{Ker } L : Lx = \lambda Nx \text{ for some } \lambda \\ &\in [0, 1]\}. \end{aligned} \quad (35)$$

Take $x \in \Omega_1$. Since $Lx = \lambda Nx$, so $\lambda \neq 0$, $Nx \in \text{Im } L = \text{Ker } Q$, and hence

$$QNx = 0. \quad (36)$$

Thus, from (H_5) , there exists $t_0 \in [0, 1]$ such that

$$\left| D_{0+}^{p-1} x(t_0) \right| < M. \quad (37)$$

Noticing that

$$D_{0+}^{p-1} x(t) = D_{0+}^{p-1} x(t_0) + \int_{t_0}^t D_{0+}^p x(s) ds, \quad (38)$$

we obtain

$$\left| D_{0+}^{p-1} x(t) \right| \leq M + \int_{t_0}^t |Nx(s)| ds \leq M + \|Nx\|_1. \quad (39)$$

Observe that $(I - P)x \in \text{dom } L \cap \text{Ker } P$ for all $x \in \Omega_1$. Then by Lemma 10,

$$\begin{aligned} \|(I - P)x\|_X &= \|K_p L(I - P)x\|_X = \|K_p Lx\|_X \\ &\leq 2\|Lx\|_1 \leq 2\|Nx\|_1, \end{aligned} \quad (40)$$

$$\begin{aligned} \|D_{0+}^{p-1}(I - P)x\|_\infty &\leq \|D_{0+}^{p-1} K_p Lx\|_\infty \leq 2\|Lx\|_1 \\ &\leq 2\|Nx\|_1. \end{aligned} \quad (41)$$

Using (39) and (41), we have

$$\begin{aligned} \Gamma(p) |x(1)| &= \left| D_{0+}^{p-1} (x(1) t^{p-1}) \right| = \left| D_{0+}^{p-1} Px(t) \right| \\ &= \left| D_{0+}^{p-1} (x(t) - ((I - P)x)(t)) \right| \\ &\leq \left| D_{0+}^{p-1} x(t) \right| + \left| D_{0+}^{p-1} ((I - P)x)(t) \right| \\ &\leq M + 3\|Nx\|_1, \end{aligned} \quad (42)$$

so that

$$\begin{aligned} \|x\|_X &\leq \|(I - P)x\|_X + \|Px\|_X \\ &\leq \frac{1}{\Gamma(p)} \|Nx\|_1 + |x(1)| \cdot \|t^{p-1}\|_X \\ &\leq 2\|Nx\|_1 + \frac{1}{\Gamma(p)} (M + 3\|Nx\|_1) \Gamma(p) \\ &= M + 5\|Nx\|_1; \end{aligned} \quad (43)$$

that is, for all $x \in \Omega_1$,

$$\|x\|_X \leq M + 5\|Nx\|_1. \quad (44)$$

Applying (H_3) , we have

$$\begin{aligned} \|x\|_X &\leq M + 5(\|\rho\|_1 + \|\alpha\|_1 \|x\|_\infty + \|\beta\|_1 \|D_{0+}^{p-1} x\|_\infty \\ &\quad + \|\gamma\|_1 \|D_{0+}^{p-2} x\|_\infty) \leq M + 5\|\rho\|_1 + 5(\|\alpha\|_1 + \|\beta\|_1 \\ &\quad + \|\gamma\|_1) \|x\|_X. \end{aligned} \quad (45)$$

Therefore, Ω_1 is bounded.

Let

$$\Omega_2 = \{x \in \text{Ker } L : Nx \in \text{Im } L\}. \quad (46)$$

Then for $x \in \Omega_2$, $x = ct^{p-1}$ for some $c \in \mathbb{R}$. So, $QNx(t) = 0$. By (H_4) , we have $|c| \leq B$. Therefore, Ω_2 is bounded.

We define the isomorphism $J : \text{Ker } L \rightarrow \text{Im } Q$ by

$$J(ct^{p-1}) = c. \quad (47)$$

If (32) holds, then let

$$\begin{aligned} \Omega_3 &= \{x \in \text{Ker } L : \lambda Jx + (1 - \lambda) QNx = 0, \lambda \in [0, 1]\}. \end{aligned} \quad (48)$$

For $x = ct^{p-1} \in \Omega_3$,

$$\lambda c = -(1 - \lambda) QN(ct^{p-1}). \quad (49)$$

If $\lambda = 1$, then $c = 0$. Otherwise, if $|c| > B$, in view of (H_4) , one has

$$c(1 - \lambda) QN(ct^{p-1}) > 0 \quad (50)$$

which contradicts $\lambda c^2 \geq 0$. Thus Ω_3 is bounded.

If (33) holds, then define the set

$$\begin{aligned} \Omega_3 &= \{x \in \text{Ker } L : -\lambda Jx + (1 - \lambda) QNx = 0, \lambda \in [0, 1]\}, \end{aligned} \quad (51)$$

where J is as above. Similar to the above argument, we can show that Ω_3 is bounded too.

Next, we will prove that all the assumptions of Theorem 1 are satisfied. Let Ω be given any bounded open subset of Y such that $\bigcup_{i=1}^3 \overline{\Omega_i} \subset \Omega$. By Lemma 9, $K_p(I - Q)N : \Omega \rightarrow Y$ is compact; thus N is L -compact on $\overline{\Omega}$. Clearly, assumptions (i) and (ii) of Theorem 1 are fulfilled.

At last, we will prove that (iii) of Theorem 1 is satisfied. Let $H(x, \lambda) = \pm \lambda Jx + (1 - \lambda) QNx$. According to above argument, we know

$$H(x, \lambda) \neq 0 \quad \text{for } x \in \text{Ker } L \cap \partial\Omega. \quad (52)$$

Thus, by the homotopy property of degree

$$\begin{aligned} \deg(QN|_{\text{Ker } L}, \text{Ker } L \cap \Omega, 0) &= \deg(H(\cdot, 0), \text{Ker } L \cap \Omega, 0) \\ &= \deg(H(\cdot, 1), \text{Ker } L \cap \Omega, 0) \\ &= \deg(\pm J, \text{Ker } L \cap \Omega, 0) \neq 0. \end{aligned} \quad (53)$$

Then by Theorem 1, $Lx = Nx$ has at least one solution in $\text{dom } L \cap \overline{\Omega}$, so that IBVP (1) has a solution. The proof is completed. $\square$

Theorem 12. Suppose (H_1) – (H_4) and (H_6) hold. Then IBVP (1) has at least one solution in Y , provided

$$\|\alpha\|_1 + \|\beta\|_1 + \|\gamma\|_1 < \Gamma(p - 1). \quad (54)$$

Proof. As in the proof of Theorem 11, $x \in \Omega_1$, implies from (H_6) , there exist $t_0 \in [0, 1]$ such that

$$\begin{aligned} |D_{0+}^{p-1} x(t_0)| &< M, \\ |D_{0+}^{p-2} x(t_0)| &< M. \end{aligned} \quad (55)$$

Noticing that

$$\begin{aligned} D_{0+}^{p-1} x(t) &= D_{0+}^{p-1} x(t_0) + \int_{t_0}^t D_{0+}^p x(s) ds, \\ D_{0+}^{p-2} x(t) &= D_{0+}^{p-2} x(t_0) + \int_{t_0}^t D_{0+}^{p-1} x(s) ds, \end{aligned} \quad (56)$$

we obtain

$$|D_{0+}^{p-1} x(t)| \leq M + \int_{t_0}^t |Nx(s)| ds \leq M + \|Nx\|_1. \quad (57)$$

Hence, we have

$$\begin{aligned} |D_{0+}^{p-2} x(t)| &\leq |D_{0+}^{p-2} x(t_0)| + \int_{t_0}^t |D_{0+}^{p-1} x(s)| ds \\ &\leq 2M + \|Nx\|_1. \end{aligned} \quad (58)$$

In view of $x(0) = 0$, $I^{3-p} x(0) = 0$. From this together with Lemma 5, for $x \in \text{dom } L$, we have

$$\begin{aligned} x(t) &= (I_{0+}^{p-2} D_{0+}^{p-2} x)(t) + \frac{I^{3-p} x(0)}{\Gamma(p-2)} t^{p-3} \\ &= (I_{0+}^{p-2} D_{0+}^{p-2} x)(t) \\ &= \frac{1}{\Gamma(p-2)} \int_0^t (t-s)^{p-3} D_{0+}^{p-2} x(s) ds. \end{aligned} \quad (59)$$

Thus, by (58),

$$\begin{aligned} |x(t)| &\leq \frac{1}{\Gamma(p-2)} \int_0^t (t-s)^{p-3} |D_{0+}^{p-2} x(s)| ds \\ &\leq \frac{1}{\Gamma(p-1)} (2M + \|Nx\|_1). \end{aligned} \quad (60)$$

Considering (57), (58), and (60), and applying (H_3) , we get

$$\begin{aligned} \|x\|_X &= \max \{ \|x\|_\infty, \|D_{0+}^{p-1} x\|_\infty, \|D_{0+}^{p-2} x\|_\infty \} \leq \max \left\{ M \right. \\ &\quad \left. + \|Nx\|_1, 2M + \|Nx\|_1, \frac{1}{\Gamma(p-1)} (2M + \|Nx\|_1) \right\} \\ &\leq \frac{1}{\Gamma(p-1)} \|Nx\|_1 + \frac{2M}{\Gamma(p-1)} \\ &\leq \frac{\|\rho\|_1 + \|\alpha\|_1 \|x\|_\infty + \|\beta\|_1 \|D_{0+}^{p-1} x\|_\infty + \|\gamma\|_1 \|D_{0+}^{p-2} x\|_\infty}{\Gamma(p-1)} \\ &\quad + \frac{2M}{\Gamma(p-1)} \leq \frac{1}{\Gamma(p-1)} (\|\alpha\|_1 + \|\beta\|_1 + \|\gamma\|_1) \|x\|_X \end{aligned}$$

$$+ \frac{\|\rho\|_1 + 2M}{\Gamma(p-1)},$$

$$\|x\|_X \leq \frac{\|\rho\|_1 + 2M}{\|\alpha\|_1 + \|\beta\|_1 + \|\gamma\|_1 - \Gamma(p-1)}. \quad (61)$$

Therefore, Ω_1 is bounded. The rest of the proof repeats that of Theorem 11. $\square$

Example 13. Consider the IBVP

$$\begin{aligned} -D_{0+}^{5/2} x(t) &= t + \frac{1}{24} \sin x(t) + \frac{1}{12} D_{0+}^{p-1} x(t) \\ &\quad + \frac{1}{24} \arctan D_{0+}^{p-2} x(t), \end{aligned} \quad (62)$$

$$x(0) = x'(0) = 0,$$

$$x(1) = \frac{5}{2} \int_0^1 x(t) dt.$$

Let $p = 5/2$ and $A(t) = (5/2)t$, then $(5/2) \int_0^1 x^{3/2} dt = 1$ and $(5/2) \int_0^1 x^{5/2} dt = 5/7$; thus (H_1) is satisfied. Let

$$f(t, u, v, w) = t + \frac{1}{24} \sin u + \frac{1}{12} v + \frac{1}{24} \arctan w, \quad (63)$$

so that $\|\alpha\|_1 = \|\gamma\|_1 = 1/24$, $\|\beta\|_1 = 1/12$, and $\|\alpha\|_1 + \|\beta\|_1 + \|\gamma\|_1 = 1/6 < 1/5$, which verifies (H_2) and (34).

Taking $M = 15$, we have $|D_{0+}^{p-1} x(t)| > 15$,

$$\begin{aligned} f(t, x(t), D_{0+}^{p-1} x(t), D_{0+}^{p-2} x(t)) &\geq \frac{1}{12} D_{0+}^{p-1} x(t) \\ &\quad - \frac{2 + \pi}{48} > 0, \quad \text{if } D_{0+}^{p-1} x(t) > 15; \end{aligned} \quad (64)$$

$$\begin{aligned} f(t, x(t), D_{0+}^{p-1} x(t), D_{0+}^{p-2} x(t)) &\leq \frac{1}{12} D_{0+}^{p-1} x(t) \\ &\quad + \frac{50 + \pi}{48} < 0, \quad \text{if } D_{0+}^{p-1} x(t) < -15, \end{aligned}$$

$$\begin{aligned} QNx(t) &= \frac{p}{\kappa} \left(\int_0^1 (1-s)^{p-1} Nx(s) ds \right. \\ &\quad \left. - \int_0^1 \int_0^t (t-s)^{p-1} Nx(s) ds dA(t) \right) = \frac{p}{\kappa} \\ &\quad \cdot \int_0^1 (1-s)^{p-1} sNx(s) ds = \frac{p}{\kappa} \int_0^1 (1-s)^{p-1} \\ &\quad \cdot sf(s, x(s), D_{0+}^{p-1} x(s), D_{0+}^{p-2} x(s)) ds \neq 0. \end{aligned} \quad (65)$$

Hence (H_5) holds. Finally, taking $B = 10$, when $|c| > 10$,

$$\begin{aligned} cf(t, ct^{p-1}, cD_{0+}^{p-1} t^{p-1}, cD_{0+}^{p-2} t^{p-1}) \\ \geq \frac{c^2}{12} \Gamma(p) - |c| \frac{50 + \pi}{48} > 0, \end{aligned} \quad (66)$$

then we obtain $cQN(ct^{p-1}) > 0$; that is, condition (H_4) is satisfied. It follows from Theorem 11 that IBVP (62) has at least one solution.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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