Global Hölder Estimates via Morrey Norms for Hypoelliptic Operators with Drift

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1. Introduction and the Main Results

Let \( G \) be a homogeneous group on \( \mathbb{R}^N \) and let \( X_0, X_1, \ldots, X_m \) be left invariant real vector fields on \( G \), where \( X_0 \) is homogeneous of degree two and \( X_1, \ldots, X_m \) homogeneous of degree one satisfying Hörmander’s condition

\[ \text{rank} \mathcal{L}(X_0, X_1, \ldots, X_m)(x) = N, \quad x \in G, \quad (1) \]

\( \mathcal{L}(X_0, X_1, \ldots, X_m) \) denotes the Lie algebra generated by \( X_0, X_1, \ldots, X_m \). The purpose of this paper is to study the following hypoelliptic operator with drift:

\[ L = \sum_{i,j=1}^{m} a_{ij} X_i X_j + a_0 X_0, \quad (2) \]

where \( a_0 \neq 0 \) and \( (a_{ij})_{i,j=1}^{m} \) is a constant coefficients matrix and there exists a constant \( \mu > 0 \) such that

\[ \mu^{-1} |\xi|^2 \leq \sum_{i,j=1}^{m} a_{ij} \xi_i \xi_j \leq \mu |\xi|^2, \quad \xi \in \mathbb{R}^m \quad (3) \]

\[ \mu^{-1} \leq a_0 \leq \mu. \]

Since Hörmander put forward the operator of sum of squares in [1], many authors paid attention to regularity of hypoelliptic operators constructed by Hörmander’s vector fields. Folland [2] concluded that any left invariant homogeneous differential operator of second order possesses a unique homogeneous fundamental solution. Bramanti and Brandolini [3] investigated further the related properties of the fundamental solutions. Recently, the a priori estimates for the operator \( L \) in (2) have been considered by several researchers. A priori \( L^p \) estimates, \( C^\alpha \) estimates, and Sobolev-Morrey estimates for \( L \) especially were proved in [3–5], respectively. We mention that the operator \( L \) contains Laplacian and parabolic operators in the Euclidean space. When \( X_0 = \sum_{i,j=1}^{m} b_{ij} x_i \partial_{x_j} - \partial_t, X_i = \partial_{x_i}, \quad i = 1, \ldots, m, \quad m < N \), the operator \( L \) becomes

\[ L_1 u = \sum_{i,j=1}^{m} a_{ij} \partial_{x_i}^2 u + \sum_{i,j=1}^{m} b_{ij} x_i \partial_{x_j} u - \partial_t u, \quad (4) \]

where \( (x, t) \in \mathbb{R}^{m+1} \), \( (a_{ij})_{i,j=1}^{m} \) is a positive definite matrix in \( \mathbb{R}^m \), and \( (b_{ij}) \) is a constant coefficients matrix with a suitable upper triangular structure. Clearly, \( L_1 \) is a class of Kolmogorov-Fokker-Planck ultraparabolic operators and appears in many research ranges, for example, stochastic processes and kinetic models [6, 7] and mathematical finance theory [8, 9]. After the previous study on \( L_1 \) in [10, 11], the authors of [12–14] established an invariant Harnack inequality for the nonnegative solution of \( L_1 u = 0 \) by using
the mean value formula. Based on the theory of singular integral, Poldor and Ragusa in [15] demonstrated Morrey-type imbedding results and gave a local Hölder continuity of the solution.

Komori and Shirai in [16] defined weighted Morrey spaces in the Euclidean space, which are the extension of Morrey spaces (see [17]) and showed the boundedness in these spaces of some important operators in harmonic analysis. The authors of [18] established the boundedness of commutators of fractional integral operators with BMO functions on Morrey spaces with two weights. In the framework of homogeneous groups, we proved in [19] similar results on Morrey spaces with two weights. In this paper, we try to study the global Hölder estimates for $L$ on Morrey spaces with two weights.

Before stating the main results, we first introduce $A_{p,q}$ classes and the Morrey space with two weights on the homogeneous group $G$. Let us recall that a weight is a nonnegative locally integrable function on $G$. Given a weight $w$ and a measurable set $E \subset G$, we set

$$w(E) = \int_E w(y) \, dy.$$  \hfill (5)

For $1 < p < q < \infty$ and the weight $w$, if there exists $c > 1$ such that, for any ball $B$ in $G$,

$$\left( \frac{1}{|B|} \int_B w(x)^q \, dx \right)^{1/q} \left( \frac{1}{|B|} \int_B w(x)^{-p'} \, dx \right)^{1/p'} \leq c,$$  \hfill (6)

where $1/p + 1/p' = 1$, then $w$ is said to be in the class $A_{p,q}$. The infimum of these constants is denoted by $[w]_{A_{p,q}}$.

For $p \in (1, \infty)$ and $\kappa \in (0, 1)$, the Morrey space with two weights $\mu$ and $\nu$ on $G$ is defined by

$$L^{p,\kappa}(\mu, \nu, G) = \left\{ g \in L^{1,\kappa}_{loc}(\nu, G) : \|g\|_{L^{p,\kappa}(\mu, \nu, G)} < \infty \right\},$$  \hfill (7)

where

$$\|g\|_{L^{p,\kappa}(\mu, \nu, G)} = \sup_B \left( \frac{1}{\nu(B)^{\kappa}} \int_B |g(y)|^p \, \mu(y) \, dy \right)^{1/p},$$  \hfill (8)

and the supremum is taken over all balls $B$ in $G$.

Observe that if $\mu = \nu = 1$ and $\kappa = \lambda/Q$ in (7), $0 < \lambda < Q$, $Q$ is the homogeneous dimension of $G$, thus $L^{p,\lambda}(G)$ which is the usual Morrey space.

We next state the main results of this paper.

**Theorem 1.** If $1 < p < q < \infty$, $1/Q < 1/p - 1/q < 2/Q$, $w \in A_{p,q}$, and $Lu \in L^{p,p'/q}(w^p, w^{p'/q}, G)$, then there exists $c > 0$ such that, for any test function $u$ and every $x, z \in G, x \neq z$,

$$\frac{|u(x) - u(z)|}{\|x - z\|^\theta} \leq c \|Lu\|_{L^{p,p'/q}(w^p, w^{p'/q}, G)},$$  \hfill (9)

where $\theta = 2 - Q(1/p - 1/q)$.

**Theorem 2.** If $1 < p < q < \infty$, $0 < 1/p - 1/q < 1/Q$, $w \in A_{p,q}$, and $Lu \in L^{p,p'/q}(w^p, w^{p'/q}, G)$, then there exists $c > 0$ such that, for any test function $u$ and every $x, z \in G, x \neq z$,

$$\frac{|X_i u(x) - X_i u(z)|}{\|x - z\|^\delta} \leq c \|Lu\|_{L^{p,p'/q}(w^p, w^{p'/q}, G)},$$  \hfill (10)

where $\delta = 1 - Q(1/p - 1/q)$.

**Remark 3.** Authors in [19] have proved Morrey estimates with two weights for $L$: if $1 < p < \infty$, $1/Q = 1/p - 1/\kappa$, $0 < \kappa < p/q$, and $w \in A_{p,\kappa}$, there exists a constant $c > 0$ such that, for every $Lu \in L^{p,\kappa}(w^p, w^{\kappa}, G)$, we have

$$\|X_i u\|_{L^{p,\kappa}(w^p, w^{\kappa}, G)} \leq c \|Lu\|_{L^{p,\kappa}(w^p, w^{\kappa}, G)},$$  \hfill (11)

where $i = 1, 2, \ldots, m$.

Our results reflect the relations between the weighted Morrey norms of $Lu$ and Hölder exponents of $u$ and $X_i u$. These statements are new even to elliptic operators, parabolic operators, and some ultraparabolic operators.

Since the second and higher order vector fields derivatives of a test function $u$ are determined by Calderón-Zygmund operators (see [3]), we cannot use the method here to give Hölder estimates for higher order derivatives of $u$.

The paper is organized as follows. In Section 2 we present some preliminaries about homogeneous groups and fundamental solutions of $L$. Furthermore, we establish pointwise estimates for the two integral operators on the Morrey spaces with two weights. Section 3 is devoted to the proofs of the main results.

### 2. Preliminaries and Two Integral Operators

Given two smooth mappings

$$[(x, y) \mapsto x \circ y] : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}^N;$$  \hfill (12)

the space $\mathbb{R}^N$ with these mappings forms a group and the identity is the origin. If there exist $0 < \omega_1 \leq \omega_2 \leq \cdots \leq \omega_N$, such that the dilations

$$D(\lambda) : (x_1, \ldots, x_N) \mapsto (\lambda^{\omega_1} x_1, \ldots, \lambda^{\omega_N} x_N),$$  \hfill (13)

$\lambda > 0$

are group automorphisms, then the group with this structure is called a homogeneous group denoted by $G$. Homogeneous groups include the Euclidean space, the Heisenberg group, and the Carnot group; see [20, 21].

**Definition 4.** A homogeneous norm $\| \cdot \|$ on $G$ is defined as follows: for any $x \in G, x \neq 0$,

$$\|x\| = \rho \Longleftrightarrow \rho \left( \frac{1}{\rho} \right) x = 1,$$  \hfill (14)

where $\rho$ is the Euclidean norm. Also, define $\|0\| = 0$. 
It is not difficult to verify that the following properties hold for the homogeneous norm:

1. \( \|r(\lambda)x\| = \lambda \|x\|, \ x \in G, \ \lambda > 0; \)
2. there exists \( c_1, c_2 \geq 1 \), such that, for \( x, y \in G, \)
   \[
   \|x^{-1}\| \leq c_1 \|x\|, \\
   \|x \circ y\| \leq c_2 (\|x\| + \|y\|). 
   \]

By virtue of these properties, it is natural to define the quasi-distance \( d \) by

\[
d(x, y) = \|y^{-1} \circ x\|. 
\]

Furthermore, the ball in \( G \) with respect to \( d \) is defined by

\[
B(x, r) \equiv B_r(x) = \{ y \in G : d(x, y) < r \}. 
\]

Observe that \( B(0, r) = \{ r \} \equiv B(0, 1) \); then

\[
|B(x, r)| = r^Q |B(0, 1)|, \ \ x \in G, \ r > 0, 
\]

where

\[
Q = \omega_1 + \cdots + \omega_N 
\]
is the homogeneous dimension of \( G \). By (18) the doubling condition holds on \( G \); that is,

\[
|B(x, 2r)| \leq c |B(x, r)|, \ \text{for} \ x \in G, \ r > 0, 
\]

where \( c \) is a positive constant, and therefore \((G, d, x, d)\) is a space of homogeneous type.

**Definition 5.** A differential operator \( Y \) on \( G \) is said to be homogeneous operator of degree \( \beta \) \( (\beta > 0) \), if, for every test function \( \varphi \),

\[
Y \left( \varphi \left( D \left( \lambda \right) x \right) \right) = \lambda^\beta \left( Y \varphi \right) \left( D \left( \lambda \right) x \right), \ \lambda > 0, \ x \in G; \tag{21}
\]
a function \( f \) is said to be homogeneous operator of degree \( \alpha \), if

\[
f \left( \left( D \left( \lambda \right) x \right) \right) = \lambda^\alpha f \left( x \right), \ \lambda > 0, \ x \in G. \tag{22}
\]

Obviously, if \( Y \) is a homogeneous differential operator of degree \( \beta \) and \( f \) is a homogeneous function of degree \( \alpha \), then \( Yf \) is homogeneous operator of degree \( \alpha - \beta \).

**Lemma 6** (see [3]). The operator \( L \) is a homogeneous left invariant differential operator of degree two on \( G \) and there is a unique fundamental solution \( \Gamma(\cdot) \) such that, for any test function \( u \) and every \( x \in G, \)

(1) \( \Gamma(\cdot) \in C^\infty(G \setminus \{0\}); \)

(2) \( \Gamma(\cdot) \) is homogeneous operator of degree \( 2 - Q; \)

(3) \( u(x) = (Lu \ast \Gamma)(x) = \int_G \Gamma(y^{-1} \circ x) Lu(y) dy; \)

(4) \( X_{\mu}(x) = \int_G X_\mu \Gamma(y^{-1} \circ x) Lu(y) dy. \)

If we set \( \Gamma_i = X_{\mu} \Gamma, i = 1, \ldots, m, \) then it is obvious from Definition 5 that \( \Gamma_i(\cdot) \) is homogeneous of degree \( 1 - Q. \)

**Lemma 7** (see [22]). For any \( x, y, z \in G, \) the following hold:

1. there exists a constant \( c > 0, \) such that
   \[
   \Gamma \left( y^{-1} \circ x \right) \leq \frac{c}{\|y^{-1} \circ x\|^{Q-1}}, \\
   \Gamma_i \left( y^{-1} \circ x \right) \leq \frac{c}{\|y^{-1} \circ x\|^{Q-1}}; 
   \]

2. there exist two constants \( c > 0 \) and \( M > 1, \) such that if
   \[
   \|y^{-1} \circ x\| \geq M \|x^{-1} \circ z\|, \text{then} \\
   \|\Gamma \left( y^{-1} \circ x \right) - \Gamma \left( y^{-1} \circ z \right)\| \leq \frac{c}{\|y^{-1} \circ x\|^{Q-1}}, \\
   \|\Gamma_i \left( y^{-1} \circ x \right) - \Gamma_i \left( y^{-1} \circ z \right)\| \leq \frac{c}{\|y^{-1} \circ x\|^{Q-1}}. 
   \]

Now we introduce two integral operators. For \( 1 < p < q < \infty, \) \( w \in A_{p,q}, \) \( x, z \in G, \) we define for every \( g \in L^{p,\alpha}(w^p, w^\alpha, G), \)

\[
T_{\alpha} g(x) = \int_{\Gamma(x) \geq \sigma \|x^{-1} \circ z\|} \frac{g(y)}{\|y^{-1} \circ x\|^{Q-\alpha}} dy, \\
T_{\beta} g(x) = \int_{\Gamma(x) \geq \sigma \|x^{-1} \circ z\|} \frac{g(y)}{\|y^{-1} \circ x\|^{Q-\beta}} dy, \\
\alpha \in [0, Q), \\
\beta \in (0, Q). 
\]

**Lemma 8.** For \( 1 < p < q < \infty, \) \( w \in A_{p,q}, x, z \in G, \) and \( x \neq z, \) if \( a(\sigma) < 1/p - 1/q, \) then there exists \( c > 0 \) such that

\[
|T_{\alpha} g(x)| \leq c \|T\|_{L^{p,q}(w^p, w^\alpha, G)} \|x^{-1} \circ x\|^{Q-\alpha(1/p-1/q)}; \\
|T_{\beta} g(x)| \leq c \|T\|_{L^{p,q}(w^p, w^\alpha, G)} \|x^{-1} \circ x\|^{Q-\beta(1/p-1/q)}. 
\]

*Proof.* By decomposing the domain of integration and applying the Hölder inequality, it is shown that

\[
|T_{\alpha} g(x)| \\
\leq \sum_{k=1}^\infty \int_{\Gamma(x) \geq \sigma \|x^{-1} \circ z\|} \frac{|g(y)|}{\|y^{-1} \circ x\|^{Q-\alpha}} dy \\
\leq \sum_{k=1}^\infty \frac{1}{(2k-1)^\alpha \|x^{-1} \circ x\|^{Q-\alpha}} \int_{B_{\Gamma(x) \geq \sigma \|x^{-1} \circ z\|}} |g(y)| |dy| \\
\leq \sum_{k=1}^\infty \frac{1}{(2k-1)^\alpha \|x^{-1} \circ x\|^{Q-\alpha}} \left( \int_{B_{\Gamma(x) \geq \sigma \|x^{-1} \circ z\|}} |g(y)|^p \right)^{1/p} \\
\cdot (\int_{B_{\Gamma(x) \geq \sigma \|x^{-1} \circ z\|}} |w(y)|^p \cdot dy)^{1/p'}.
\]
Due to $w \in A_{p,q}$, we get
\[
\left( \int_B w(x)^{-p'} dx \right)^{1/p'} \leq c \frac{\|B\|^{1/p+1/q}}{w^q(B)^{1/q}},
\] (29)
then
\[
\left| T_\alpha g(x) \right|
\leq c \sum_{k=1}^\infty \frac{1}{\omega^q \left( B_{2^k \alpha x} \right)} \int_{B_{2^k \alpha x}} |g(y)|^p \frac{w(y)^p dy}{\omega^q \left( B_{2^k \alpha x} \right)^{1/q}}.
\] (30)

It follows from (29) that
\[
\left| T_\beta g(x) \right|
\leq c \sum_{k=1}^\infty \int_{B_{2^k \beta x}} \frac{|g(y)|^p}{\omega^q \left( B_{2^k \beta x} \right)^{1/q}} \frac{w(y)^p dy}{\omega^q \left( B_{2^k \beta x} \right)^{1/q}}.
\]

If $\alpha/Q < 1/p - 1/q$, then the series in the right hand side in (31) is convergent, and (26) is proved. Analogously, it yields
\[
\left| T_\alpha g(x) \right|
\leq c \sum_{k=1}^\infty \int_{B_{2^k \alpha x}} \frac{|g(y)|^p}{\omega^q \left( B_{2^k \alpha x} \right)^{1/q}} \frac{w(y)^p dy}{\omega^q \left( B_{2^k \alpha x} \right)^{1/q}}.
\] (31)

3. Proof of the Main Theorems

Proof of Theorem 1. For any test function $u$ and every $x, z \in G$, $x \neq z$, applying Lemma 6, there exists $M > 1$ such that
\[
|u(x) - u(z)| \leq c \int_G \left| \left| \Gamma \left( y^{-1} \circ x \right) - \Gamma \left( y^{-1} \circ z \right) \right| \right| dy.
\]
\[
\left| \left|Lu(y)\right|\right| dy
\]
\[
= \int \left| \left| \Gamma \left( y^{-1} \circ x \right) - \Gamma \left( y^{-1} \circ z \right) \right| \right| dy + \int \left| \left|Lu(y)\right|\right| dy
\]
\[
\leq \int \left| \left| \Gamma \left( y^{-1} \circ x \right) - \Gamma \left( y^{-1} \circ z \right) \right| \right| dy + \int \left| \left|Lu(y)\right|\right| dy + \int \left| \left|Lu(y)\right|\right| dy.
\] (33)

It follows by Lemma 7 that
\[
|u(x) - u(z)|
\leq \int \left| \left| \Gamma \left( y^{-1} \circ x \right) - \Gamma \left( y^{-1} \circ z \right) \right| \right| dy + \int \left| \left|Lu(y)\right|\right| dy + \int \left| \left|Lu(y)\right|\right| dy.
\] (34)
Note that if \( \| y^{-1} \circ x \| \geq M \| x^{-1} \circ z \| \), then
\[
\| y^{-1} \circ x \| \geq M \| x^{-1} \circ z \| \geq \frac{M}{c_1} \| z^{-1} \circ x \| ; (35)
\]
if \( \| y^{-1} \circ x \| < M \| x^{-1} \circ z \| \), then
\[
\begin{align*}
\| y^{-1} \circ x \| &< M \| x^{-1} \circ z \| < M c_1 \| z^{-1} \circ x \| , \\
\| y^{-1} \circ z \| &< c_2 \left( \| y^{-1} \circ x \| + \| x^{-1} \circ z \| \right), \\
&< c_2 \left( M \| x^{-1} \circ z \| + \| x^{-1} \circ z \| \right) \\
&= c_2 (1 + M) \| x^{-1} \circ z \|. \quad (36)
\end{align*}
\]

It immediately derives by (34) that
\[
[u(x) - u(z)]
\]
\[
\begin{align*}
\leq & \int_{\{y^{-1} \circ x \| \geq (M/c_1) \| z^{-1} \circ x \| \}} c \| x^{-1} \circ z \| \| L(u)(y) \| dy \\
+ & \int_{\{y^{-1} \circ x \| < M \| x^{-1} \circ z \| \}} \frac{c}{c_1} \| y^{-1} \circ x \| ^{2-Q} \| L(u)(y) \| dy \\
+ & \int_{\{y^{-1} \circ z \| < (1+M) \| z^{-1} \circ x \| \}} \frac{c}{c_1} \| y^{-1} \circ z \| ^{2-Q} \| L(u)(y) \| dy \\
= & I_1 + I_2 + I_3.
\end{align*}
\]

If \( 1/Q < 1/p - 1/q \), it is shown by choosing \( \alpha = 1, \sigma = M/c_1 \) in Lemma 8 that there exists \( c = c(p, \lambda, \sigma) > 0 \) such that
\[
I_1 \leq c \| z^{-1} \circ x \| \| L u \| _{L^{p,q}(\omega^p,\omega^q,G)} \| z^{-1} \circ x \| ^{1-Q(1/p - 1/q)} 
= c \| L u \| _{L^{p,q}(\omega^p,\omega^q,G)} \| z^{-1} \circ x \| ^{2-Q(1/p - 1/q)} ; (38)
\]
if \( 2/Q > 1/p - 1/q \), we get from Lemma 8 (\( \beta = 2, \sigma = M/c_1 \) and \( \beta = 2, \sigma = c_2 (1 + M) \), resp.) that
\[
I_2 \leq c \| L u \| _{L^{p,q}(\omega^p,\omega^q,G)} \| z^{-1} \circ x \| ^{2-Q(1/p - 1/q)} ; \\
I_3 \leq c \| L u \| _{L^{p,q}(\omega^p,\omega^q,G)} \| z^{-1} \circ x \| ^{2-Q(1/p - 1/q)} . \quad (39)
\]

Putting (38) and (39) in (37), we have (9) and this finishes the proof. □

Proof of Theorem 2. For \( i = 1, \ldots, m \), we have from Lemma 6 that there exists \( M > 1 \) such that for any test function \( u \) and every \( x, z \in G, x \neq z \),
\[
|X_i u(x) - X_i u(z)| \leq \int_G \left| \Gamma_i \left( y^{-1} \circ x \right) - \Gamma_i \left( y^{-1} \circ z \right) \right| |L u(y)| dy 
\]
\[
= \int_{\{y^{-1} \circ x \| \geq M \| x^{-1} \circ z \| \}} \left| \Gamma_i \left( y^{-1} \circ x \right) - \Gamma_i \left( y^{-1} \circ z \right) \right| |L u(y)| dy 
+ \int_{\{y^{-1} \circ x \| < M \| x^{-1} \circ z \| \}} \left| \Gamma_i \left( y^{-1} \circ x \right) - \Gamma_i \left( y^{-1} \circ z \right) \right| |L u(y)| dy 
\]
\[
\leq \int_{\{y^{-1} \circ x \| \geq M \| x^{-1} \circ z \| \}} \left| \Gamma_i \left( y^{-1} \circ x \right) - \Gamma_i \left( y^{-1} \circ z \right) \right| |L u(y)| dy 
+ \int_{\{y^{-1} \circ x \| < M \| x^{-1} \circ z \| \}} \left| \Gamma_i \left( y^{-1} \circ x \right) - \Gamma_i \left( y^{-1} \circ z \right) \right| |L u(y)| dy .
\]

Summarizing (35) and (36) and using Lemma 7, it yields
\[
|X_i u(x) - X_i u(z)| \leq \int_{\{y^{-1} \circ x \| \geq M \| x^{-1} \circ z \| \}} c \| x^{-1} \circ z \| \| L u(y) \| dy 
+ \int_{\{y^{-1} \circ x \| < M \| x^{-1} \circ z \| \}} \frac{c}{c_1} \| y^{-1} \circ x \| ^{2-Q} \| L u(y) \| dy 
\]
\[
\leq \int_{\{y^{-1} \circ x \| \geq M \| x^{-1} \circ z \| \}} \left| \Gamma_i \left( y^{-1} \circ x \right) - \Gamma_i \left( y^{-1} \circ z \right) \right| |L u(y)| dy 
+ \int_{\{y^{-1} \circ x \| < M \| x^{-1} \circ z \| \}} \left| \Gamma_i \left( y^{-1} \circ x \right) - \Gamma_i \left( y^{-1} \circ z \right) \right| |L u(y)| dy .
\]

If \( 0 < 1/p - 1/q \), we have by using Lemma 8 with \( \alpha = 0 \) and \( \sigma = M/c_1 \) that there exists \( c = c(p, \lambda, \sigma) > 0 \) such that
\[
I_4 \leq c \| z^{-1} \circ x \| \| g \| _{L^{p,q}(\omega^p,\omega^q,G)} \| z^{-1} \circ x \| ^{1-Q(1/p - 1/q)} 
= c \| g \| _{L^{p,q}(\omega^p,\omega^q,G)} \| z^{-1} \circ x \| ^{2-Q(1/p - 1/q)} ; (42)
\]
if \( 1/Q > 1/p - 1/q \), it is shown by applying Lemma 8 (\( \beta = 1, \sigma = M/c_1 \) and \( \beta = 1, \sigma = c_2 (1 + M) \), resp.) that
\[
I_5 \leq c \| g \| _{L^{p,q}(\omega^p,\omega^q,G)} \| z^{-1} \circ x \| ^{1-Q(1/p - 1/q)} ; \\
I_6 \leq c \| g \| _{L^{p,q}(\omega^p,\omega^q,G)} \| z^{-1} \circ x \| ^{2-Q(1/p - 1/q)} . \quad (43)
\]

Substituting (42) and (43) into (41), we get (10). □
Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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