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Research Article

The Characteristic Properties of the Minimal L_p-Mean Width

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Giannopoulos proved that a smooth convex body K has minimal mean width position if and only if the measure $h_K(u)\sigma(du)$, supported on S^{n-1}, is isotropic. Further, Yuan and Leng extended the minimal mean width to the minimal L_p-mean width and characterized the minimal position of convex bodies in terms of isotropicity of a suitable measure. In this paper, we study the minimal L_p-mean width of convex bodies and prove the existence and uniqueness of the minimal L_p-mean width in its $\mathrm{SL}(n)$ images. In addition, we establish a characterization of the minimal L_p-mean width, conclude the average $M_p(K)$ with a variation of the minimal L_p-mean width position, and give the condition for the minimum position of $M_p(K)$.

1. Introduction

Let $\mathfrak{L}^n(\mathbb{R}^n)$ denote the space of linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $\mathrm{SL}(n) = \{T \in \mathfrak{L}^n(\mathbb{R}^n) : \det(T) = 1\}$. Suppose K is a convex body in $\mathbb{R}^n$; then $\{TK : T \in \mathrm{SL}(n)\}$ is the family of its positions. In [1] it was shown that for many natural functionals of the form $T \mapsto f(TK)$, $T \in \mathrm{SL}(n)$, the solution T_0 of the problem $\min\{f(TK) : T \in \mathrm{SL}(n)\}$ is isotropic with respect to an appropriate measure depending on f. The purpose of this note is to provide applications of this point of view in the case of the mean width functional $T \mapsto w(TK)$ under various constraints.

Recall that the width of K in the direction of $u \in S^{n-1}$ is defined by $w(K,u) = h_K(u) + h_K(-u)$, where $h_K(x)$ is the support function of convex K. The mean width of K is given by

$$w(K) = \int_{S^{n-1}} w(K,u)\,\sigma(du) = 2\int_{S^{n-1}} h_K(u)\,\sigma(du), \quad (1)$$

where σ is the rotationally invariant probability measure on the unit sphere S^{n-1}.

We say that K has minimal mean width if $w(K) \le w(TK)$ for every $T \in \mathrm{SL}(n)$. In [1], the authors show that a smooth enough convex body (that is, h_K is twice continuously differentiable) is in minimal mean width position if and only if the measure $h_K(u)\sigma(du)$, supported on S^{n-1}, is isotropic.

Theorem 1. *A smooth enough convex body K in $\mathbb{R}^n$ has minimal mean width if and only if*

$$\int_{S^{n-1}} \langle u,\theta\rangle^2 h_K(u)\,\sigma(du) = \frac{w(K)}{2n}, \quad \text{for every } \theta \in S^{n-1}. \quad (2)$$

Moreover, if $U \in \mathrm{SL}(n)$ and UK has minimal mean width, we must have $U \in \mathrm{O}(n)$, where $\mathrm{O}(n)$ denotes rotation transformation group.

Yuan et al. in [2] introduced the notion of the minimal L_p-mean width of convex body. Let K be a convex body in $\mathbb{R}^n$ and $p \ge 1$; the L_p-width of K in the direction of $u \in S^{n-1}$ is defined by $w_p(K,u) = h_K^p(u) + h_K^p(-u)$. The L_p-width function $w_p(K,\cdot)$ is translation invariant; therefore we may assume that $o \in \operatorname{int} K$. The L_p-mean width of K is given by

$$w_p(K) = \int_{S^{n-1}} w_p(K,u)\,\sigma(du) = 2\int_{S^{n-1}} h_K^p(u)\,\sigma(du). \quad (3)$$

If $w_p(K) \le w_p(TK)$ for every $T \in \mathrm{SL}(n)$, then we say that K has minimal L_p-mean width. The following isotropic characterization of the minimal L_p-mean width position was proved in [2].

Theorem 2. *A smooth enough convex body K in $\mathbb{R}^n$ has minimal L_p-mean width if and only if*

$$\int_{S^{n-1}} \langle u,\theta\rangle^2 h_K^p(u)\,\sigma(\mathrm{d}u) = \frac{w_p(K)}{2n}, \quad \text{for every } \theta \in S^{n-1}. \tag{4}$$

Moreover, if $U \in SL(n)$ and UK has minimal p-mean width, we must have $U \in O(n)$.

It is easily seen that the L_p-mean width belongs to the L_p-Brunn-Minkowski theory. The L_p-Brunn-Minkowski theory is far more general than the classical Brunn-Minkowski theory; we refer the reader to [3–25].

The aim of this article is to study the characterization of the minimal L_p-mean width, which enriches the theory of L_p-mean width of convex bodies. In Section 3, we discuss the continuity of the L_p-mean width of convex bodies. In Section 4, we mainly demonstrate that the body K has a unique SL(n) image with minimal L_p-mean width. In view of this fact, we define the minimal L_p-mean width of K by

$$\overline{w}_p(K) = \min\left\{w_p(TK) : T \in \mathrm{SL}(n)\right\}. \tag{5}$$

In Section 5, we prove the following equivalent conditions with the minimal L_p-mean width of convex bodies.

Theorem 3. *Suppose K is a smooth enough convex body in $\mathbb{R}^n$, $p \geq 1$, and $T_0 \in SL(n)$. Then the following assertions are equivalent:*

(1) $\overline{w}_p(K) = w_p(T_0K)$.

(2) *The measure $w_p(T_0K,\cdot)$ is isotropic on S^{n-1}.*

(3) *For all $x \in \mathbb{R}^n$, the transformation T_0 satisfies*

$$\begin{aligned}|x|^2 \int_{S^{n-1}} \left|T_0^{-t}u\right|^{-(n+p)} h_K^p(u)\,\sigma(\mathrm{d}u) \\ = n\int_{S^{n-1}} \left\langle x, T_0^{-t}u\right\rangle^2 \left|T_0^{-t}u\right|^{-(n+p+2)} h_K^p(u)\,\sigma(\mathrm{d}u).\end{aligned} \tag{6}$$

In Section 6, we consider the average

$$M_p(K) = \int_{S^{n-1}} \|x\|_K^p\,\sigma(\mathrm{d}x) \tag{7}$$

of the norm $\|\cdot\|_K$ on S^{n-1}. Thus we define $M_p^*(K) = M_p(K^\circ)$ and obtain the following condition for the minimum position.

Theorem 4. *Let K be a symmetric convex body in $\mathbb{R}^n$ and assume that $M_p(K)M_p^*(K) \leq M_p(TK)M_p^*(TK)$ for every $T \in SL(n)$. Then*

$$\begin{aligned}\frac{1}{M_p(K)}\int_{S^{n-1}} \langle u,\theta\rangle^2 \|u\|_K^p\,\sigma(\mathrm{d}u) \\ = \frac{1}{M_p^*(K)}\int_{S^{n-1}} \langle u,\theta\rangle^2 \|u\|_{K^\circ}^p\,\sigma(\mathrm{d}u), \\ \text{for every } \theta \in S^{n-1}.\end{aligned} \tag{8}$$

In addition, we also get the following condition for the minimum position.

Theorem 5. *Let K be a symmetric convex body in $\mathbb{R}^n$ satisfying $a(K) = 1$ and $M_p(K) \leq M_p(TK)$ for every $T \in GL(n)$ with $a(TK) = 1$. Then, for every $\theta \in S^{n-1}$ we can find contact points x_1, x_2 of K and B such that*

$$\begin{aligned}1 + p\left\langle x_1,\theta\right\rangle^2 \leq \frac{n+p}{M_p(K)}\int_{S^{n-1}} \|u\|_K^p \langle u,\theta\rangle^2\,\sigma(\mathrm{d}u) \\ \leq 1 + p\left\langle x_2,\theta\right\rangle^2,\end{aligned} \tag{9}$$

where the condition $a(TK) = 1$ means that $TK \subseteq B$ but there exist contact points of TK and B (see [1]).

Please see the next section for above interrelated notations, definitions, and their background materials.

2. Preliminaries

2.1. Notations. The setting will be the Euclidean n-space $\mathbb{R}^n$. As usual, $\langle x,y\rangle$ denotes the standard inner product of x and y in $\mathbb{R}^n$; $B = \{x \in \mathbb{R}^n : \langle x,x\rangle \leq 1\}$ and $S^{n-1} = \partial B$ denotes the unit ball and unit sphere in $\mathbb{R}^n$, respectively. The volume of B is $\omega_n = \pi^{n/2}/\Gamma(1+n/2)$. We often use $|\cdot|$ to denote the standard Euclidean norm, on occasion the total mass of a measure, and the absolute value of the determinant of an $n \times n$ matrix. For brevity, we write $\langle x\rangle = x/|x|$, for $x \in \mathbb{R}^n \setminus \{o\}$.

Given a compact convex set K in $\mathbb{R}^n$, its support function $h_K(\cdot) = h(K,\cdot) : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$h_K(x) = \max\left\{\langle x,y\rangle : y \in K\right\}. \tag{10}$$

The definition immediately gives that, for $T \in \mathrm{GL}(n)$,

$$h_{TK}(x) = h_K\left(T^t x\right), \quad x \in \mathbb{R}^n. \tag{11}$$

As usual, write $\mathscr{K}_o^n$ for the class of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. $\mathscr{K}_o^n$ is often equipped with the Hausdorff metric δ_H, which is defined by

$$\begin{aligned}\delta_H(K_1,K_2) &= \max\left\{\left|h_{K_1}(u) - h_{K_2}(u)\right| : u \in S^{n-1}\right\} \\ &= \left\|h_{K_1} - h_{K_2}\right\|_\infty, \quad \text{for } K_1, K_2 \in \mathscr{K}_o^n.\end{aligned} \tag{12}$$

For $K \in \mathscr{K}_o^n$, define $R_K, r_K \in (0,\infty)$ by

$$\begin{aligned}R_K &= \max_{u\in S^{n-1}} h_K(u), \\ r_K &= \min_{u\in S^{n-1}} h_K(u).\end{aligned} \tag{13}$$

A set $K \subset \mathbb{R}^n$ is said to be a star body about the origin, if the line segment from the origin to any point $x \in K$ is contained in K and K has continuous and positive radial function $\rho_K(\cdot)$. Here, the radial function of K, $\rho_K : \mathbb{R}^n \setminus \{o\} \to [0,\infty)$, is defined by

$$\rho_K(x) = \rho(K,x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{o\}. \tag{14}$$

We write $\mathcal{S}_o^n$ for the class of star bodies about the origin o in $\mathbb{R}^n$. $\mathcal{S}_o^n$ is often equipped with the dual Hausdorff metric $\widetilde{\delta}_H$, which is defined by

$$\begin{aligned}\widetilde{\delta}_H(K,L) &= \max\left\{\left|\rho_K(u)-\rho_L(u)\right| : u\in S^{n-1}\right\}\\ &\coloneqq \left|\rho_K(u)-\rho_L(u)\right|_\infty, \quad \text{for } K,L\in\mathcal{S}_o^n.\end{aligned} \tag{15}$$

If convex body K contains the origin o as its interior point, then the Minkowski functional $\|\cdot\|_K$ of K is defined by

$$\|x\|_K = \min\{\lambda>0 : x\in\lambda K\}. \tag{16}$$

In this case, $\|x\|_K = h_{K^\circ}(x)$, where K° denotes the polar set of K, which is defined by

$$K^\circ = \left\{x\in\mathbb{R}^n : \langle x,y\rangle \le 1 \ \forall y\in K\right\}. \tag{17}$$

If $K\in\mathcal{K}_o^n$ and any $T\in \mathrm{GL}(n)$, then

$$(TK)^\circ = T^{-t}K^\circ. \tag{18}$$

In addition, we easily see that if $K\in\mathcal{K}_o^n$, then the support and radial functions of K°, the polar body of K, are defined, respectively, by

$$\begin{aligned}h_{K^\circ} &= \rho_K^{-1},\\ \rho_{K^\circ} &= h_K^{-1}.\end{aligned} \tag{19}$$

2.2. Linear Operators. For $T\in\mathfrak{L}^n(\mathbb{R}^n)$, T^t and $\|T\|$ denote the transpose and norm of T, respectively. For $K\in\mathcal{K}_o^n$, let $d(K)$ denote its maximal principal radius.

Two facts are in order (see [26]). First, $T\in\mathfrak{L}^n(\mathbb{R}^n)$ is nondegenerated, if and only if the ellipsoid TB is nondegenerated. Second, for $T\in\mathfrak{L}^n(\mathbb{R}^n)$, since

$$\|T\| = \sup_{u\in S^{n-1}}|Tu| = \sup_{u\in S^{n-1}}\left|T^tu\right| = \left\|T^t\right\|, \tag{20}$$

it follows that

$$\begin{aligned}d(TB) &= \sup_{u\in S^{n-1}} h_{TB}(u) = \sup_{u\in S^{n-1}}\left|T^tu\right| = \sup_{u\in S^{n-1}}|Tu|\\ &= \sup_{u\in S^{n-1}} h_{T^tB}(u) = d\left(T^tB\right).\end{aligned} \tag{21}$$

Let $d_n(T_1,T_2) = \|T_1-T_2\|$, for $T_1,T_2\in\mathfrak{L}^n(\mathbb{R}^n)$. Then the metric space $(\mathfrak{L}^n(\mathbb{R}^n), d_n)$ is complete. Since $\mathfrak{L}^n(\mathbb{R}^n)$ is of finite dimension, a set in $(\mathfrak{L}^n(\mathbb{R}^n), d_n)$ is compact, if and only if it is bounded and closed.

Lemma 6 (see [26]). *Suppose $\{T_i\}_{i\in\mathbb{N}}\subset SL(n)$. Then $\|T_i\|\to\infty \Leftrightarrow \|T_i^{-1}\|\to\infty$.*

Thus, $\{T_i\}_{i\in\mathbb{N}}$ is bounded from above, if and only if $\{T_i^{-1}\}_{i\in\mathbb{N}}$ is bounded from above.

Lemma 7 (see [26]). *Suppose $\{T_i\}_{i\in\mathbb{N}}\subset SL(n)$ and $T_i\to T_0\subset SL(n)$ with respect to d_n. Then*

(1) *$T_i^tB\to T_0^tB$ with respect to δ_H.*

(2) *$T_i^{-1}\to T_0^{-1}$ with respect to d_n.*

(3) *$T_iB\to T_0B$ with respect to $\widetilde{\delta}_H$.*

2.3. Notion of Isotropy of Measures. For further discussion, we introduce the important notion of isotropy of measures. A nonnegative Borel measure μ on S^{n-1} is said to be isotropic if

$$\int_{S^{n-1}}\langle u,v\rangle^2\,\mathrm{d}\mu(u) = \frac{|\mu|}{n}, \quad \forall v\in S^{n-1}. \tag{22}$$

Here, $|\mu|$ denotes the total mass of μ. The definition immediately yields

$$\int_{S^{n-1}} u_i^2\,\mathrm{d}\mu(u) = \frac{|\mu|}{n}, \tag{23}$$

where u_i denotes the ith component of the coordinate of u. For nonzero $x\in\mathbb{R}^n$, the notation $x\otimes x$ represents the linear operator of the rank 1 on $\mathbb{R}^n$ that takes y to $\langle x,y\rangle x$. It immediately gives the fact that $\mathrm{tr}(x\otimes x) = |x|^2$. Equivalently, μ is isotropic if

$$\int_{S^{n-1}} u\otimes u\,\mathrm{d}\mu(u) = \frac{|\mu|}{n}I_n, \tag{24}$$

where I_n denotes the identity operator on $\mathbb{R}^n$. For more information about the isotropy, we refer to [1, 27–29].

The following fact will be needed.

Lemma 8 (see [30]). *Suppose that μ is a probability measure on a space X and $f: X\to I\subset\mathbb{R}$ is a μ-integrable function, where I is a possibly infinite interval. Jensen's inequality states that if $\varphi: I\to\mathbb{R}$ is a convex function, then*

$$\int_X \varphi(f(x))\,\mathrm{d}\mu(x) \ge \varphi\left(\int_X f(x)\,\mathrm{d}\mu(x)\right). \tag{25}$$

If φ is strictly convex, equality holds if and only if $f(x)$ is constant for μ, with almost all $x\in X$.

3. The Continuity of the L_p-Mean Width

Using $K_i\to K$ equivalent to $h_{K_i}\to h_K$ uniformly on S^{n-1}, together with the convexity of $\phi(t) = t^p$ and definition (3), we immediately obtain the following.

Lemma 9. *Suppose $K, K_i\in\mathcal{K}_o^n$ and $p, p_j\ge 1$, $i,j\in\mathbb{N}$. If $K_i\to K$ and $p_j\to p$, then*

$$\lim_{i,j\to\infty} w_{p_j}(K_i) = w_p(K). \tag{26}$$

Fixing a convex body K, the continuity of the L_p-mean width w_p with respect to $T\in \mathrm{SL}(n)$ is contained in the following.

Lemma 10. *Suppose $K \in \mathscr{K}_o^n$, $p \geq 1$, and $T_0, T_i \in SL(n)$. If $T_i \to T_0$, then*

$$\lim_{T_i \to T_0} w_p(T_i K) = w_p(T_0 K). \tag{27}$$

Proof. From (2) of Lemma 7, we have the following implications:

$$\begin{gathered} \|T_i - T_0\| \longrightarrow 0 \\ \Downarrow \\ \left\|T_i^{-1} - T_0^{-1}\right\| \longrightarrow 0 \\ \Downarrow \\ T_i^{-1}u \longrightarrow T_0^{-1}u, \\ \text{pointwise convergence for every } u \in S^{n-1} \\ \Updownarrow \\ T_i^{-1}B \longrightarrow T_0^{-1}B, \quad \text{with respect to } \delta_H \\ \Updownarrow \\ h_{T_i^{-1}B}(u) \longrightarrow h_{T_0^{-1}B}(u), \\ \text{uniformly for every } u \in S^{n-1}. \end{gathered} \tag{28}$$

From the above, (35), and Lemma 9, we get $\lim_{T_i \to T_0} w_p(T_i K) = w_p(T_0 K)$. □

Lemma 11. *Suppose $K \in \mathscr{K}_o^n$ and $p \geq 1$. Then*

$$\min_{\substack{T \in SL(n) \\ \|T\| \to \infty}} w_p(TK) = \infty. \tag{29}$$

Proof. Without loss of generality, we write T in the form $T = (O_1 A O_2)^t$, where A is an $n \times n$ diagonal matrix, with $\det(A) = 1$ and positive diagonal elements $a_1, \ldots, a_n$, and O_1, O_2 are $n \times n$ orthogonal matrices. For any $y \in \mathbb{R}^n$, let $\|y\|_1$ denote the l_1-norm of y, $u = (u_1, \ldots, u_n) \in S^{n-1}$ with $\epsilon_u = \min_{1 \leq i \leq n} |u_i| > \epsilon > 0$. Recall that there exists a positive C such that $|y| \geq C\|y\|_1$, and $\sum_{i=1}^n a_i \geq \max_{1 \leq i \leq n} a_i = \|A\| = \|T\|$. Then from the definition of L_p-mean width, Jensen's inequality (Lemma 8, see [30, 31]), and (13), it follows that

$$\begin{aligned} w_p(TK) &= 2 \int_{S^{n-1}} h_K^p(AO_2 u)\, \sigma(du) \\ &\geq 2 \left(\int_{S^{n-1}} h_K(Au)\, \sigma(du) \right)^p \\ &= 2 \left(\int_{S^{n-1}} |Au|^{n+1} h_K(v)\, \sigma(dv) \right)^p \quad \left(v = \frac{Au}{|Au|} \right) \\ &\geq 2 \left(\int_{S^{n-1}} C^{n+1} \|Au\|_1^{n+1} h_K(v)\, \sigma(dv) \right)^p \\ &= 2 \left(\int_{S^{n-1}} C^{n+1} \left(\sum_{i=1}^n |a_i u_i| \right)^{n+1} h_K(v)\, \sigma(dv) \right)^p \\ &> 2 \left(\int_{S^{n-1}} (\epsilon C)^{n+1} \left(\sum_{i=1}^n a_i \right)^{n+1} h_K(v)\, \sigma(dv) \right)^p \\ &\qquad\qquad (\epsilon \in (0, \epsilon_u] \subseteq (0, 1]) \\ &\geq 2 \left((\epsilon C)^{n+1} r_K \|T\|^{n+1} \right)^p. \end{aligned} \tag{30}$$

It immediately yields

$$\lim_{\substack{T \in \mathrm{SL}(n) \\ \|T\| \to \infty}} w_p(TK) = \infty. \tag{31}$$

□

4. The Minimal L_p-Mean Width

In order to demonstrate the existence and uniqueness of minimal L_p-mean width, we begin by proving some lemmas.

Lemma 12. *Suppose $K \in \mathscr{K}_o^n$, $p \geq 1$, and $T \in GL(n)$. Then*

$$w_p(TK) = 2\left|T^{-t}\right| \int_{S^{n-1}} h_{TB}^{n+p}(u)\, h_K^p(v)\, \sigma(dv), \quad v = \left\langle T^{-1}u \right\rangle \in S^{n-1}. \tag{32}$$

Proof. From (3) and (11), we have

$$\begin{aligned} w_p(TK) &= \int_{S^{n-1}} h_K^p\left(T^t u\right) \sigma(du) \\ &= 2\left|T^{-t}\right| \int_{S^{n-1}} \left|T^t u\right|^{n+p} h_K^p\left(\left\langle T^t u\right\rangle\right) \sigma\left(d\left(\left\langle T^t u\right\rangle\right)\right) \\ &= 2\left|T^{-t}\right| \int_{S^{n-1}} \left|T^t u\right|^{n+p} h_K^p(v)\, \sigma(dv) \\ &= 2\left|T^{-t}\right| \int_{S^{n-1}} h_{TB}^{n+p}(u)\, h_K^p(v)\, \sigma(dv). \end{aligned} \tag{33}$$

We conclude the proof. □

For $T \in \mathrm{GL}(n)$, from (18) and (19), we have

$$\begin{aligned} h_{TB}(u) &= \left|T^t u\right| = \left|T^{-t}\left\langle T^t u\right\rangle\right|^{-1} = \left|T^{-t} v\right|^{-1} \\ &= h_{T^{-1}B}^{-1}(v) = \rho_{T^t B}(v). \end{aligned} \tag{34}$$

Therefore, we can get

$$\begin{aligned} w_p(TK) &= 2\left|T^{-t}\right| \int_{S^{n-1}} h_{T^{-1}B}^{-(n+p)}(v)\, h_K^p(v)\, \sigma(dv) \\ &= 2\left|T^{-t}\right| \int_{S^{n-1}} \rho_{T^t B}^{n+p}(v)\, h_K^p(v)\, \sigma(dv). \end{aligned} \tag{35}$$

Given an origin-symmetric ellipsoid E in $\mathbb{R}^n$, let $d_E = \max\{h_E(u) : u \in S^{n-1}\}$. Then there exists a $v_E \in S^{n-1}$ such that, for all $u \in S^{n-1}$,

$$d_E |\langle u, v_E\rangle| \le h_E(u). \quad (36)$$

Lemma 13. *Suppose $K \in \mathscr{K}_o^n$, $p \ge 1$, and $T \in SL(n)$. Then*

$$d_{TB} \le \frac{\left(w_p(TK)/2r_K^p\right)^{1/(n+p)}}{\min_{v\in S^{n-1}} \int_{S^{n-1}} |\langle u, v\rangle| \sigma(dv)}. \quad (37)$$

Proof. From Lemma 12, (36), and Jensen's inequality, together with (13), we have

$$\begin{aligned} w_p(TK) &= 2\int_{S^{n-1}} h_{TB}^{n+p}(u) h_K^p(v)\sigma(dv) \\ &\ge 2\int_{S^{n-1}} \left(d_{TK}|\langle u, v_{TK}\rangle|\right)^{n+p} h_K^p(v)\sigma(dv) \\ &\ge 2r_K^p \int_{S^{n-1}} \left(d_{TK}|\langle u, v_{TK}\rangle|\right)^{n+p}\sigma(dv) \\ &\ge 2r_K^p \left(\int_{S^{n-1}} d_{TK}|\langle u, v_{TK}\rangle|\sigma(dv)\right)^{n+p} \\ &\ge 2r_K^p \left(d_{TK}\int_{S^{n-1}} |\langle u, v_{TK}\rangle|\sigma(dv)\right)^{n+p} \\ &\ge 2r_K^p d_{TK}^{n+p}\left(\min_{v\in S^{n-1}}\int_{S^{n-1}} |\langle u, v\rangle|\sigma(dv)\right)^{n+p}. \end{aligned} \quad (38)$$

We easily see that

$$\min_{v\in S^{n-1}}\int_{S^{n-1}} |\langle u, v\rangle|\sigma(dv) > 0. \quad (39)$$

Indeed, it is known that there exists a unit vector $v_0 \in S^{n-1}$ such that $\int_{S^{n-1}} |\langle u, v_0\rangle|\sigma(dv) = \min_{v\in S^{n-1}}\int_{S^{n-1}} |\langle u, v\rangle|\sigma(dv)$. Choose $u \in S^{n-1}$ which is not orthogonal to v_0. Then the statement follows by continuity. Accordingly, the desired inequality is derived. □

With the previous lemmas in hand, we can show that the minimal L_p-mean width is well-defined. Our purpose is to find necessary and sufficient conditions for a body K to have minimal L_p-mean width. We assume for simplicity that h_K is twice continuously differentiable (we then say that K is smooth enough).

Theorem 14. *Suppose $K \in \mathscr{K}_o^n$ and any $T \in SL(n)$. Then, modulo orthogonal transformations, there exists a unique solution to the minimization problem*

$$\min_{T\in SL(n)} w_p(TK). \quad (40)$$

Proof. Note that the proof is based on the idea of Zou and Xiong [26, 32]. Let $\{T_k\}_{k\in\mathbb{N}} \subset \mathrm{SL}(n)$ be a minimizing sequence for the problem; that is,

$$\lim_{k\to\infty} w_p(T_kK) = \inf\left\{w_p(TK) : T \in \mathrm{SL}(n)\right\}. \quad (41)$$

Note that

$$\inf\left\{w_p(TK) : T \in \mathrm{SL}(n)\right\} \le w_p(K) < \infty, \quad (42)$$

which implies $\{w_p(T_kK)\}_{k\in\mathbb{N}}$; therefore $\{(w_p(T_kK)/2r_K^p)^{1/(n+p)}\}_{k\in\mathbb{N}}$ is bounded from above. So, by Lemma 13, $\{T_kB\}_{k\in\mathbb{N}}$ is bounded with respect to the Hausdorff metric. From the Blaschke selection theorem, $\{T_kB\}_{k\in\mathbb{N}}$ has a convergent subsequence $\{T_{k_j}B\}_{j\in\mathbb{N}}$ that converges to a body E. Since volume functional is continuous with respect to the Hausdorff metric, and $V(T_{k_j}K) = \omega_n$ for each j, it yields that $V(E) = \omega_n$; since the convergence of $\{T_{k_j}B\}_{j\in\mathbb{N}}$ is equivalent to the uniform convergence of $\{h_{T_{k_j}B}\}_{j\in\mathbb{N}}$ on S^{n-1}, and $h_{T_{k_j}B}(u) = h_{T_{k_j}B}(-u)$ for all $u \in S^{n-1}$, it yields that $h_E(u) = h_E(-u)$ for all $u \in S^{n-1}$. Thus, E is a nondegenerated origin-symmetric ellipsoid.

Consequently, there exists a transformation $T_0 \in \mathrm{SL}(n)$ such that $E = T_0B$. This demonstrates the existence of solutions to the considered problem.

Now, we prove the uniqueness by contradiction. Assume that $T_1, T_2 \in \mathrm{SL}(n)$ both solve the considered minimization problem. Let $E_1 = T_1B$ and $E_2 = T_2B$. It is known that each $T \in \mathrm{SL}(n)$ can be represented in the form $T = PQ$, where P is symmetric, positive definite and Q is orthogonal. So, without loss of generality, we may assume that T_1 and T_2 are symmetric and positive definite.

The Minkowski inequality for symmetric positive definite matrices shows that

$$\begin{aligned} \det\left(\frac{T_1+T_2}{2}\right)^{1/n} &> \frac{1}{2}\det(T_1)^{1/n} + \frac{1}{2}\det(T_2)^{1/n} \\ &= 1. \end{aligned} \quad (43)$$

Let

$$T_3 = \det\left(\frac{T_1+T_2}{2}\right)^{-1/n}\left(\frac{T_1+T_2}{2}\right). \quad (44)$$

Then $T_3 \in \mathrm{SL}(n)$, and, for all $u \in S^{n-1}$,

$$\begin{aligned} h_{E_3}(u) &= h_{T_3B}(u) < h_{(1/2)(T_1+T_2)B}(u) = \left|\frac{T_1u+T_2u}{2}\right| \\ &\le \frac{|T_1u|+|T_2u|}{2} = \frac{1}{2}h_{T_1B}(u) + \frac{1}{2}h_{T_2B}(u). \end{aligned} \quad (45)$$

Using (32) and (45) and noticing that if $a, b \in \mathbb{R}$ and $p > 0$, then $(|a|+|b|)^p \le C_p(|a|^p+|b|^p)$ (where $C_p = 1$ if $0 < p \le 1$, and $C_p = 2^{p-1}$ if $p > 1$), we have

$$\begin{aligned} w_p(T_3K) &= 2\int_{S^{n-1}} h_{T_3B}^{n+p}(v) h_K^n(v)\sigma(dv) \\ &< 2\int_{S^{n-1}} \left(\frac{1}{2}h_{T_1B}(u) + \frac{1}{2}h_{T_2B}(u)\right)^{n+p} h_K^p(v)\sigma(dv) \end{aligned}$$

$$= \int_{S^{n-1}} h_{T_1 B}^{n+p}(u)\, h_K^p(v)\, \sigma(dv) + \int_{S^{n-1}} h_{T_2 B}^{n+p}(u)\, h_K^p(v)\, \sigma(dv) = \frac{1}{2} w_\phi(T_1 K) + \frac{1}{2} w_\phi(T_2 K) = w_\phi(T_1 K) = w_\phi(T_2 K). \quad (46)$$

Namely,

$$w_\phi(T_3 K) < w_\phi(T_1 K) = w_\phi(T_2 K). \quad (47)$$

However, by the previous assumption on T_1 and T_2, we have

$$w_\phi(T_3 K) \ge w_\phi(T_1 K) = w_\phi(T_2 K), \quad (48)$$

which is a contradiction. The proof is complete. □

In view of Theorem 14, naturally, we introduce the following.

Definition 15. Suppose $K \in \mathcal{K}_o^n$ and $p \ge 1$. The quantity

$$\overline{w}_p(K) = \min \{ w_p(TK) : T \in \mathrm{SL}(n) \} \quad (49)$$

is called the minimal L_p-mean width of the convex body K with respect to p.

5. The Characterization of the Minimal L_p-Mean Width

Theorem of this section characterizes the convex body with minimal L_p-mean width.

Theorem 16. *Suppose $K \in \mathcal{K}_o^n$, $p \ge 1$, and $T_0 \in SL(n)$. Then the following assertions are equivalent:*

(1) $\overline{w}_p(K) = w_p(T_0 K)$.

(2) *The measure $w_p(T_0 K, \cdot)$ is isotropic on S^{n-1}.*

(3) *For all $x \in \mathbb{R}^n$, the transformation T_0 satisfies*

$$|x|^2 \int_{S^{n-1}} \left| T_0^{-t} u \right|^{-(n+p)} h_K^p(u)\, \sigma(du) = n \int_{S^{n-1}} \left\langle x, T_0^{-t} u \right\rangle^2 \left| T_0^{-t} u \right|^{-(n+p+2)} h_K^p(u)\, \sigma(du). \quad (50)$$

Proof. First, we prove the equivalence of (1) and (2). Suppose that (1) holds. Since $\overline{w}_p(K)$ is $\mathrm{SL}(n)$ invariant, we may assume that T_0 is the $n \times n$ identity matrix I_n.

Let $T \in \mathfrak{L}(\mathbb{R}^n)$ be a linear transformation. Then there exists $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, the matrices $I_n + \varepsilon T$ are still positive definite. For $\varepsilon \in (0, \varepsilon_0)$, define

$$T_\varepsilon = \frac{(I_n + \varepsilon T)^t}{|I_n + \varepsilon T|^{1/n}}. \quad (51)$$

Then $T_\varepsilon \in \mathrm{SL}(n)$ is volume preserving. If ε is sufficiently small, from (33), together with the two equalities (see [1])

$$\begin{aligned} |(I_n + \varepsilon T) u| &= 1 + \varepsilon \langle u, Tu \rangle + O(\varepsilon^2), \\ |I_n + \varepsilon T|^{1/n} &= 1 + \frac{\varepsilon}{n} \operatorname{tr} T + O(\varepsilon^2), \end{aligned} \quad (52)$$

and the definition of w_p, we have

$$w_p(T_\varepsilon K) = 2 \int_{S^{n-1}} \left(\frac{1 + \varepsilon \langle u, Tu \rangle + O(\varepsilon^2)}{1 + (\varepsilon/n) \operatorname{tr} T + O(\varepsilon^2)} \right)^{n+p} h_K(v)\, \sigma(dv). \quad (53)$$

From the smoothness of $|T_\varepsilon^t u|$ in ε, the integrand depends smoothly on ε. Thus,

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} w_p(T_\varepsilon K) = 0. \quad (54)$$

Calculating it directly, we have

$$0 = 2 \int_{S^{n-1}} \left. \frac{\partial}{\partial \varepsilon} \right|_{\varepsilon=0} \left(\frac{1 + \varepsilon \langle u, Tu \rangle + O(\varepsilon^2)}{1 + (\varepsilon/n) \operatorname{tr} T + O(\varepsilon^2)} \right)^{n+p} h_K(v) \cdot \sigma(dv) = 2(n+p) \int_{S^{n-1}} \left(\langle u, Tu \rangle - \frac{\operatorname{tr} T}{n} \right) h_K(v) \cdot \sigma(dv). \quad (55)$$

Let $v \in S^{n-1}$ and $T = v \otimes v$. Using the facts $\operatorname{tr}(v \otimes v) = 1$ and $\langle u, (v \otimes v) u \rangle = \langle u, v \rangle^2$, it follows that

$$\int_{S^{n-1}} \langle u, v \rangle^2\, dw_p(K, v) = \frac{w_p(K)}{n}, \quad (56)$$

where $dw_p(K, v) = 2 h_K^p(v) \sigma(dv)$. Thus, $w_p(K, \cdot)$ is isotropic on S^{n-1}.

Next, we show the implication "(2) ⇒ (1)". The proof will be completed by two steps.

Firstly, for a point $a = (a_1, \dots, a_n) \in [0, \infty)^n$, let

$$\mathcal{F}(a) = 2 \int_{S^{n-1}} |\operatorname{diag}(a_1, \dots, a_n) u|^{n+p} h_K^p(v)\, \sigma(dv), \quad (57)$$

where $\operatorname{diag}(a_1, \dots, a_n)$ denotes the diagonal matrix with diagonal elements $a_1, \dots, a_n$.

We aim to show that

$$\mathcal{F}(a) \ge \mathcal{F}(e), \quad \text{whenever } a_1 \cdots a_n = 1. \quad (58)$$

Here, e denotes the point $(1, \dots, 1)$.

It can be checked that $\mathcal{F} : [0, \infty)^n \to [0, \infty)$ is continuous and convex and $\mathcal{F}(\lambda a)$ is strictly increasing in $\lambda \in [0, \infty)$, for any $a \in (0, \infty)^n$. Thus, $\mathcal{F}^{-1}([0, \mathcal{F}(e)])$ is compact, convex, and of nonempty interior. Namely, it is a convex body.

By using the fact that $|\mathrm{diag}(a_1,\dots,a_n)u|$ is smooth in $(a_1,\dots,a_n)$ uniformly for $u\in S^{n-1}$, we have

$$\begin{aligned}\left.\frac{\partial}{\partial a_j}\right|_{a=e} &\mathcal{F}(a)\\ &= 2\int_{S^{n-1}}\frac{\partial}{\partial a_j}\left|\mathrm{diag}(a_1,\dots,a_n)u\right|^{n+p}h_K^p(v)\,\sigma(\mathrm{d}v) \\ &= 2(n+p)\int_{S^{n-1}}u_j^2\mathrm{d}w_p(K,v).\end{aligned}\quad(59)$$

Meanwhile, since the boundary of $\mathcal{F}^{-1}([0,\mathcal{F}(e)])$ is given by the equation $\mathcal{F}(a)=\mathcal{F}(e)$ with $a\in\mathbb{R}_+^n$, so the vector $\int_{S^{n-1}}(u_1^2,\dots,u_n^2)\mathrm{d}w_p(K,v)$ is an outer normal of $\mathcal{F}^{-1}([0,\mathcal{F}(e)])$ at the boundary point e. Notice that $w_p(K,\cdot)$ is isotropic; it yields

$$\nabla\mathcal{F}(e)=\int_{S^{n-1}}\left(u_1^2,\dots,u_n^2\right)\mathrm{d}w_p(K,v)=\frac{\left|w_p(K,\cdot)\right|}{n}e. \quad(60)$$

Thus, e is an outer normal vector of $\mathcal{F}^{-1}([0,\mathcal{F}(e)])$ at the boundary point e. Consequently,

$$\mathcal{F}^{-1}([0,\mathcal{F}(e)])\subset\left\{a\in\mathbb{R}^n:\langle a,e\rangle\le n\right\}. \quad(61)$$

That is to say, for all $a\in[0,\infty)^n$, if $\mathcal{F}(a)\le\mathcal{F}(e)$, then $\langle a,e\rangle\le n$. In contrast, for all $b=(b_1,\dots,b_n)\in(0,\infty)^n$ with $b_1\cdots b_n=1$, the arithmetic-geometric mean inequality yields that $\langle a,e\rangle\ge n$, with equality if and only if $b=e$. Thus, (58) is derived.

Secondly, with (58) in hand, we aim to show that, for all $T\in\mathrm{SL}(n)$, $w_p(TK)\ge w_p(K)$, with equality if and only if T is orthogonal.

It is known that T^t can be represented in the form $T^t=PAQ$, where P and Q are $n\times n$ orthogonal matrices, and $A=\mathrm{diag}(a_1,\dots,a_n)$ is diagonal and positive definite. Therefore, by (33) and (58), we have

$$\begin{aligned} w_p(TK) &= 2\int_{S^{n-1}}|Au|^{n+p}h_{QK}^p(v)\,\sigma(\mathrm{d}v)\\ &= 2\int_{S^{n-1}}\left|\mathrm{diag}(a_1,\dots,a_n)u\right|^{n+p}h_{QK}^p(v)\,\sigma(\mathrm{d}v)\\ &\ge 2\int_{S^{n-1}}\left|\mathrm{diag}(1,\dots,1)u\right|^{n+p}h_{QK}^p(v)\,\sigma(\mathrm{d}v)\\ &= 2\int_{S^{n-1}}h_{QK}^p(v)\,\sigma(\mathrm{d}v)=w_p(QK)=w_p(K).\end{aligned}\quad(62)$$

Equality holds if and only if $(a_1,\dots,a_n)=(1,\dots,1)$, equivalently, if and only if T is orthogonal. Thus, the implication "(2) $\Rightarrow$ (1)" is shown.

In the remaining part, we prove the equivalence of (2) and (3). From the definitions of $w_p(T_0K,\cdot)$ and (11), we have

$$\begin{aligned}\mathrm{d}w_p(T_0K,u)&=2h_K^p\left(T_0^tu\right)\sigma(\mathrm{d}u)\\ &=2\left|T_0^{-t}\left\langle T_0^tu\right\rangle\right|^{-(n+p)}h_K^p\left(\left\langle T_0^tu\right\rangle\right)\sigma\left(\mathrm{d}\left(\left\langle T_0^tu\right\rangle\right)\right),\end{aligned}\quad(63)$$

which immediately yields that

$$2\int_{S^{n-1}}\left|T_0^{-t}v\right|^{-(n+p)}h_K^p(v)\,\sigma(\mathrm{d}v)=\left|w_p(T_0K,\cdot)\right|,\qquad v=\left\langle T_0^tu\right\rangle. \quad(64)$$

Meanwhile, for $x\in\mathbb{R}^n$, we have

$$\begin{aligned}\int_{S^{n-1}}\langle x,u\rangle^2\mathrm{d}w_p(T_0K,u)&=2\int_{S^{n-1}}\left\langle x,T_0^{-t}\left\langle T_0^tu\right\rangle\right\rangle^2\\ &\quad\cdot\left|T_0^{-t}\left\langle T_0^tu\right\rangle\right|^{-(n+p+2)}h_K^p\left(\left\langle T_0^tu\right\rangle\right)\sigma\left(\mathrm{d}\left(\left\langle T_0^tu\right\rangle\right)\right)\\ &=2\int_{S^{n-1}}\left\langle x,T_0^{-t}v\right\rangle^2\left|T_0^{-t}v\right|^{-(n+p+2)}h_K^p(v)\,\sigma(\mathrm{d}v).\end{aligned}\quad(65)$$

Therefore,

$$\begin{aligned}|x|^2&\int_{S^{n-1}}\left|T_0^{-t}u\right|^{-(n+p)}h_K^p(u)\,\sigma(\mathrm{d}u)\\ &=n\int_{S^{n-1}}\left\langle x,T_0^{-t}u\right\rangle^2\left|T_0^{-t}u\right|^{-(n+p+2)}h_K^p(u)\,\sigma(\mathrm{d}u).\end{aligned}\quad(66)$$

With these, the equivalence of (2) and (3) is shown. The proof is complete. □

6. The Characterization of the Average $M_p(K)$

In this section, we consider the average

$$M_p(K)=\int_{S^{n-1}}\|x\|_K^p\,\sigma(\mathrm{d}x) \quad(67)$$

of the norm $\|\cdot\|_K$ on S^{n-1} and define $M_p^*(K)=M_p(K^\circ)$.

We conclude this section with a variation of the minimal L_p-mean width position: consider a symmetric convex body K in $\mathbb{R}^n$ and the problem of minimizing $M_p(TK)M_p^*(TK)$ over all $\mathrm{SL}(n)$.

The following fact will be needed.

Lemma 17 (see [2]). *Let K be a smooth enough convex body in $\mathbb{R}^n$. We define*

$$I_p(K,\theta)=\int_{S^{n-1}}h_K^{p-1}(u)\left\langle\nabla h_K(u),\theta\right\rangle\langle u,\theta\rangle\,\sigma(\mathrm{d}u),\qquad \theta\in S^{n-1}. \quad(68)$$

Then

$$\begin{aligned}\frac{w_p(K)}{2}&+pI_p(K,\theta)\\ &=(n+p)\int_{S^{n-1}}h_K^p(u)\langle u,\theta\rangle^2\,\sigma(\mathrm{d}u)\end{aligned}\quad(69)$$

for every $\theta\in S^{n-1}$.

Now, we obtain the following condition for the minimum position.

Theorem 18. *Let K be a symmetric convex body in $\mathbb{R}^n$ and assume that $M_p(K)M_p^*(K) \le M_p(TK)M_p^*(TK)$ for every $T \in SL(n)$. Then*

$$\frac{1}{M_p(K)} \int_{S^{n-1}} \langle u,\theta\rangle^2 \|u\|_K^p \,\sigma(du) = \frac{1}{M_p^*(K)} \int_{S^{n-1}} \langle u,\theta\rangle^2 \|u\|_{K^\circ}^p \,\sigma(du), \tag{70}$$

for every $\theta \in S^{n-1}$.

Proof. Without loss of generality, we may assume that K is smooth enough. Let $R \in \mathrm{SL}(n)$ and $\varepsilon > 0$ be small enough, and write $T^{-1} = I_n + \varepsilon R$. Then (see [1])

$$T^t = \left(I_n + \varepsilon R^t\right)^{-1} = I_n + \sum_{k=1}^{\infty} (-1)^k \varepsilon^k \left(R^k\right)^t. \tag{71}$$

Our assumption about K takes the form

$$\begin{aligned} M_p(K) M_p^*(K) &\le \int_{S^{n-1}} \|u + \varepsilon R u\|_K^p \,\sigma(du) \int_{S^{n-1}} \left\|u - \varepsilon R^t u\right\|_{K^\circ}^p \sigma(du) \\ &\quad + O\left(\varepsilon^2\right). \end{aligned} \tag{72}$$

Note that there are two equalities (see [33])

$$\begin{aligned} \|u + \varepsilon R u\|_K^p &= h_{K^\circ}(u + \varepsilon R u)^p \\ &= h_{K^\circ}(u)^p \\ &\quad + \varepsilon p h_{K^\circ}(u)^{p-1} \left\langle \nabla h_{K^\circ}(u), Ru\right\rangle \\ &\quad + O\left(\varepsilon^2\right), \\ \left\|u - \varepsilon R^t u\right\|_{K^\circ}^p &= h_K\left(u - \varepsilon R^t u\right)^p \\ &= h_K(u)^p \\ &\quad - \varepsilon p h_K(u)^{p-1} \left\langle \nabla h_K(u), R^t u\right\rangle \\ &\quad + O\left(\varepsilon^2\right), \end{aligned} \tag{73}$$

which implies

$$\begin{aligned} &M_p(K) M_p^*(K) \\ &\quad \le M_p(K) M_p^*(K) \\ &\qquad - p M_p(K) \int_{S^{n-1}} \varepsilon h_K^{p-1}(u) \left\langle \nabla h_K(u), R^t u\right\rangle \sigma(du) \\ &\qquad + p M_p^*(K) \int_{S^{n-1}} \varepsilon h_{K^\circ}^{p-1}(u) \left\langle \nabla h_{K^\circ}(u), Ru\right\rangle \sigma(du) \\ &\qquad + O\left(\varepsilon^2\right). \end{aligned} \tag{74}$$

Letting $\varepsilon \to 0^+$ and replacing R by $-R$, we have

$$\begin{aligned} &\frac{1}{M_p(K)} \int_{S^{n-1}} p h_{K^\circ}^{p-1}(u) \left\langle \nabla h_{K^\circ}(u), Ru\right\rangle \sigma(du) \\ &\quad = \frac{1}{M_p^*(K)} \int_{S^{n-1}} p h_K^{p-1}(u) \left\langle \nabla h_K(u), R^t u\right\rangle \sigma(du) \end{aligned} \tag{75}$$

for every $R \in \mathrm{SL}(n)$. Using (75) with $R_\theta(x) = \langle x,\theta\rangle\theta$, $\theta \in S^{n-1}$, we get

$$\begin{aligned} &\frac{1}{M_p(K)} \int_{S^{n-1}} h_{K^\circ}^{p-1}(u) \left\langle \nabla h_{K^\circ}(u), \theta\right\rangle \langle u,\theta\rangle \sigma(du) \\ &\quad = \frac{1}{M_p^*(K)} \int_{S^{n-1}} h_K^{p-1}(u) \left\langle \nabla h_K(u), \theta\right\rangle \langle u,\theta\rangle \sigma(du) \end{aligned} \tag{76}$$

for every $\theta \in S^{n-1}$. Taking into account Lemma 17, we conclude the proof. □

When $p = 1$, it is easy to show that Theorem 18 reduces to Giannopoulos and Milman's result (see [1]).

Recall that the condition $a(TK) = 1$ means that $TK \subseteq B$ but there exist contact points of TK and B (see [1]). We also obtain the following result.

Theorem 19. *Let K be a symmetric convex body in $\mathbb{R}^n$ satisfying $a(K) = 1$ and $M_p(K) \le M_p(TK)$ for every $T \in GL(n)$ with $a(TK) = 1$. Then, for every $\theta \in S^{n-1}$ we can find contact points x_1, x_2 of K and B such that*

$$\begin{aligned} 1 + p\left\langle x_1,\theta\right\rangle^2 &\le \frac{n+p}{M_p(K)} \int_{S^{n-1}} \|u\|_K^p \langle u,\theta\rangle^2 \sigma(du) \\ &\le 1 + p\left\langle x_2,\theta\right\rangle^2. \end{aligned} \tag{77}$$

Proof. Let $T \in \mathfrak{L}^n(\mathbb{R}^n)$ and $\varepsilon > 0$ be small enough. Then

$$T_1 := \left(\min_{u\in S^{n-1}} \|u + \varepsilon T u\|_K\right)\left(I_n + \varepsilon T\right)^{-1} \tag{78}$$

satisfies $a(T_1 K) = 1$. Therefore

$$\begin{aligned} &\int_{S^{n-1}} \|u + \varepsilon T u\|^p \sigma(du) \\ &\quad \ge M_p(K) \left(\min_{x\in S^{n-1}} \|x + \varepsilon T x\|\right)^p. \end{aligned} \tag{79}$$

Notice that (see [33])

$$\begin{aligned} \|u + \varepsilon T u\|_K^p &= \|u\|_K^p + \varepsilon p \|u\|_K^{p-1} \left\langle \nabla h_{K^\circ}(u), Tu\right\rangle \\ &\quad + O\left(\varepsilon^2\right). \end{aligned} \tag{80}$$

It follows from (79) that

$$\begin{aligned} &\int_{S^{n-1}} p \|u\|_K^{p-1} \left\langle \nabla h_{K^\circ}(u), Tu\right\rangle \sigma(du) + O(\varepsilon) \\ &\quad \ge M_p(K) \frac{\left(\min_{x\in S^{n-1}} \|x + \varepsilon T x\|\right)^p - 1}{\varepsilon}. \end{aligned} \tag{81}$$

Let x_ε be a point on S^{n-1} at which the minimum is attained. Since $x_\varepsilon \in S^{n-1}$ and $\|\cdot\|_K \geq |\cdot|$, and noticing that

$$\left|x_\varepsilon + \varepsilon T x_\varepsilon\right|^p = 1 + \varepsilon p \left\langle x_\varepsilon, T x_\varepsilon \right\rangle + O\left(\varepsilon^2\right), \tag{82}$$

(81) takes the form

$$\begin{aligned} &\int_{S^{n-1}} p \|u\|_K^{p-1} \left\langle \nabla h_{K^\circ}(u), Tu \right\rangle \sigma(\mathrm{d}u) + O(\varepsilon) \\ &\geq M_p(K) \frac{\left(\min_{x \in S^{n-1}} \|x + \varepsilon T x\|\right)^p - 1}{\varepsilon} \\ &\geq M_p(K) \frac{\left|x_\varepsilon + \varepsilon T x_\varepsilon\right|^p - 1}{\varepsilon} \\ &= M_p(K) \left(p \left\langle x_\varepsilon, T x_\varepsilon \right\rangle\right) + O(\varepsilon). \end{aligned} \tag{83}$$

If x is a contact point of K and B, we must have

$$\begin{aligned} 1 + \varepsilon \|T\| \geq \|x + \varepsilon T x\|_K &\geq \left\|x_\varepsilon + \varepsilon T x_\varepsilon\right\|_K \\ &\geq \left\|x_\varepsilon\right\|_K - \varepsilon \|T\|. \end{aligned} \tag{84}$$

It follows that

$$1 \leq \left\|x_\varepsilon\right\|_K \leq 1 + 2\varepsilon \|T\|. \tag{85}$$

Also, $x \in S^{n-1}$ and using (85) we get $\|x\| = \lim_{k\to\infty} \|x_{\varepsilon_k}\| = 1$. That is, x is a contact point of K and B. Now, we can find a sequence $\varepsilon_k \to 0$ and a point $x \in S^{n-1}$ such that $x_{\varepsilon_k} \to x$. Letting $k \to \infty$ in (83), we obtain

$$\begin{aligned} &\int_{S^{n-1}} \|u\|_K^{p-1} \left\langle \nabla h_{K^\circ}(u), Tu \right\rangle \sigma(\mathrm{d}u) \\ &\quad \geq M_p(K) \langle x, Tx \rangle. \end{aligned} \tag{86}$$

Replacing T by $-T$ we find another contact point x' of K and B such that

$$\begin{aligned} &\int_{S^{n-1}} \|u\|_K^{p-1} \left\langle \nabla h_{K^\circ}(u), Tu \right\rangle \sigma(\mathrm{d}u) \\ &\quad \leq M_p(K) \left\langle x', Tx' \right\rangle. \end{aligned} \tag{87}$$

Choosing $T_\theta(x) = \langle x, \theta \rangle \theta$, $\theta \in S^{n-1}$, and applying Lemma 17, we obtain (77). □

When $p = 1$, it is easy to show that Theorem 19 reduces to Giannopoulos and Milman's result (see [1]).

Conflicts of Interest

The author declare that they have no conflicts of interest.

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