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*Research Article*

# Busemann-Petty Problems for Quasi $L_p$ Intersection Bodies

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We investigate the $L_p$ dual geominimal surface area and volume forms of Busemann-Petty problems for the quasi $L_p$ intersection bodies and establish some new geometric inequalities. Our results provide a significant complement to the researches on Busemann-Petty problems for intersection bodies.

## 1. Introduction and Main Results

Let $S^{n-1}$ denote the unit sphere in Euclidean space $\mathbb{R}^n$. If $K$ is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \to [0,\infty)$, is defined by (see [1, 2])

$$\rho(K,u) = \max\{\lambda \geq 0 : \lambda u \in K\}, \quad u \in S^{n-1}. \tag{1}$$

If $\rho_K$ is positive and continuous, then $K$ will be called a star body (about the origin), and $\mathcal{S}^n$ denotes the set of star bodies in $\mathbb{R}^n$. We will use $\mathcal{S}_o^n$ and $\mathcal{S}_e^n$ to denote the subset of star bodies in $\mathcal{S}^n$ containing the origin in their interiors and origin-symmetric star bodies, respectively. Two star bodies $K$ and $L$ are said to be the dilation of one another if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean $\mathbb{R}^n$. For $u \in S^{n-1}$, $u^\perp$ denotes the $(n-1)$-dimensional subspace orthogonal to $u$. We use $V_k(M)$ to denote the $k$-dimensional volume of a $k$-dimensional compact convex set $M$. Instead of $V_n$ we usually write $V$. For the standard unit ball $B$ in $\mathbb{R}^n$, we write $\omega_n = V(B)$ for its volume.

Busemann and Petty posed a problem [3]: let $K$ and $L$ be origin-symmetric convex bodies in $\mathbb{R}^n$. Is it true that, for any $u \in S^{n-1}$,

$$\begin{gathered} V_{n-1}(K \cap u^\perp) \leq V_{n-1}(L \cap u^\perp) \Longrightarrow \\ V(K) \leq V(L)? \end{gathered} \tag{2}$$

A long list of authors contributed to the solution of this famous problem over a period of 40 years; see [4–18]. The question has a negative answer for $n \geq 5$ and an affirmative answer for $n = 3, 4$. For a detailed account of the interesting history of the Busemann-Petty problem, see the books by Gardner [1, Chapter 8] and Koldobsky [19, Chapter 5].

The crucial idea in the solution of the problem was to define new convex body which was called intersection body. This was done by Lutwak [15] whose work is considered the starting point of the solution of the Busemann-Petty problem in all dimensions. For $K \in \mathcal{S}_o^n$, the intersection body, $IK$, of $K$ is a star body whose radial function in the direction $u \in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of $K$ by $u^\perp$; that is,

$$\rho(IK,u) = V_{n-1}(K \cap u^\perp). \tag{3}$$

The intersection bodies have been intensively studied in recent years (see [20–28] and the books [19, 29]). From (3) and the fact that star bodies $K$ and $L$ satisfy $K \subseteq L$ if and only if $\rho(K,\cdot) \leq \rho(L,\cdot)$, we see that the Busemann-Petty problem can be rephrased in the following way: for $K, L \in \mathcal{S}_o^n$, is it true that

$$\begin{gathered} IK \subseteq IL \Longrightarrow \\ V(K) \leq V(L)? \end{gathered} \tag{4}$$

Lutwak [15] showed that the problem has an affirmative answer if the body $K$ is restricted to the class of intersection

bodies. In addition, Lutwak proved that if $L$ is a sufficiently smooth origin-symmetric star body with positive radial function which is not an intersection body, then there exists an origin-symmetric star body $K$ such that $IK \subseteq IL$ but $V(K) > V(L)$. Further, Busemann-Petty problems have been considered in the context of $L_p$ Brunn-Minkowski Theory (see [30–41]). In particular, Haberl and Ludwig [42] generalized the intersection body to $L_p$ form and introduced the notion of $L_p$ intersection body: let $L$ be a star body and nonzero $p < 1$. The $L_p$ intersection body of $L$, $I_pL$, is the symmetric star body whose radial function is defined by

$$\rho\left(I_pL,u\right)^p = \int_L |u\cdot x|^{-p}\,dx. \tag{5}$$

After that, associated with $L_p$ intersection bodies, Yuan and Cheung [41] gave an affirmative form of Busemann-Petty problem for the $L_p$ intersection bodies.

**Theorem 1.** *Let $K$ be $L_p$ intersection body and $L$ be a star body in $\mathbb{R}^n$. If $I_pK \subseteq I_pL$, then*

$$\begin{aligned} V(K) &\le V(L), \quad \text{for } 0 < p < 1,\\ V(K) &\ge V(L), \quad \text{for } p < 0. \end{aligned} \tag{6}$$

*In both cases equality holds if and only if $K = L$.*

In 2007, Yu et al. [40] defined the quasi $L_p$ intersection body as follows: let $K$ be a star body and $p \ge 1$. The quasi $L_p$ intersection body, $I_pK$, of $K$ is defined by

$$\rho\left(I_pK,u\right) = \left(\frac{\widetilde{V}_p\left(K,B\cap u^{\perp}\right)}{\widetilde{V}_p\left(B,B\cap u^{\perp}\right)}\right)^{1/p}, \tag{7}$$

for $u \in S^{n-1}$, where $\widetilde{V}_p$ denotes the $L_p$ dual mixed volume (see (22)).

Suppose that $f$ is a Borel function on $S^{n-1}$. The spherical Radon transform $Rf$ [43] of $f$ is defined by

$$\left(Rf\right)(u) = \int_{S^{n-1}\cap u^{\perp}} f(v)\,dS_{n-2}(v), \tag{8}$$

for $u \in S^{n-1}$. Using the spherical Radon transform, the definition of $I_pK$ is rewritten by

$$\begin{aligned} \rho\left(I_pK,u\right) &= \left(\frac{\widetilde{V}_p\left(K,B\cap u^{\perp}\right)}{\widetilde{V}_p\left(B,B\cap u^{\perp}\right)}\right)^{1/p}\\ &= \left(\frac{1}{(n-1)\,\omega_{n-1}} R\left(\rho_K^{n-p}\right)(u)\right)^{1/p}\\ &= \left(\frac{1}{(n-1)\,\omega_{n-1}} \int_{S^{n-1}\cap u^{\perp}} \rho(K,v)^{n-p}\,dS_{n-2}(v)\right)^{1/p}, \end{aligned} \tag{9}$$

for $u \in S^{n-1}$.

One aim of this paper is to establish the volume forms of the Busemann-Petty problems for the quasi $L_p$ intersection bodies. For convenience, let $\mathbb{I}_p$ denote the set of quasi $L_p$ intersection bodies.

**Theorem 2.** *Let $K, L \in \mathcal{S}_o^n$ and $p \ge 1$. If $1 \le p < n$ and $K \in \mathbb{I}_p$, then*

$$\begin{aligned} I_pK \subseteq I_pL &\Longrightarrow\\ V(K) \le V(L)&. \end{aligned} \tag{10}$$

*If $p > n$ and $L \in \mathbb{I}_p$, then*

$$\begin{aligned} I_pK \subseteq I_pL &\Longrightarrow\\ V(K) \ge V(L)&. \end{aligned} \tag{11}$$

*And $V(K) = V(L)$ if and only if $K = L$.*

*Remark 3.* Theorem 2 can be found in [40]. However, what should be noted is that we give a new method of proof in this paper.

**Theorem 4.** *For $1 \le p < n$, if $K \notin \mathcal{S}_e^n$, then there exists $L \in \mathcal{S}_e^n$ such that*

$$I_pK \subset I_pL, \tag{12}$$

*but*

$$V(K) > V(L). \tag{13}$$

Recall the definition of $L_p$ dual geominimal surface area, $\widetilde{G}_p(K)$, of $K \in \mathcal{S}_o^n$ for $p > 0$ in [44]:

$$\omega_n^{p/n}\widetilde{G}_p(K) = \sup\left\{n\widetilde{V}_p(K,Q)\,V\left(Q^*\right)^{p/n} : Q \in \mathcal{S}_e^n\right\}. \tag{14}$$

Another aim of this paper is to give the $L_p$ dual geominimal surface area forms of Busemann-Petty problems for the quasi $L_p$ intersection bodies. If $M \in \mathbb{I}_p$, then we rewrite the definition of $L_p$ dual geominimal surface area by

$$\begin{aligned} &\omega_n^{p/n}\overline{G}_p(K)\\ &\quad = \sup\left\{n\widetilde{V}_p(K,M)\,V\left(M^*\right)^{p/n} : M \in \mathbb{I}_p\right\}. \end{aligned} \tag{15}$$

**Theorem 5.** *If $p \ge 1$ and $K, L \in \mathcal{S}_o^n$, then*

$$I_pK \subseteq I_pL, \tag{16}$$

*implying*

$$\overline{G}_p(K) \le \overline{G}_p(L). \tag{17}$$

*And $\overline{G}_p(K) = \overline{G}_p(L)$ if and only if $K = L$.*

**Theorem 6.** *For $1 \le p < n$, if $K \notin \mathcal{S}_e^n$, then there exists $L \in \mathcal{S}_e^n$ such that*

$$I_pK \subset I_pL, \tag{18}$$

*but*

$$\widetilde{G}_p(K) > \widetilde{G}_p(L). \tag{19}$$

## 2. Preliminaries

*2.1. $L_p$ Dual Mixed Volume.* For $K, L \in \mathcal{S}_o^n$, $p > 0$, and $\lambda, \mu \geq 0$ $(\lambda + \mu \neq 0)$, the $L_p$ radial combination, $\lambda \circ K \tilde{+}_p \mu \circ L \in \mathcal{S}_o^n$, of $K$ and $L$ is defined by (see [40])

$$\rho\left(\lambda \circ K \tilde{+}_p \mu \circ L, \cdot\right)^p = \lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p. \tag{20}$$

The polar coordinate formula for the volume of a body $K \in \mathcal{S}_o^n$ is

$$V(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^n \, dS(u). \tag{21}$$

For $p \geq 1$, the $L_p$ dual mixed volume, $\widetilde{V}_p(K, L)$, of $K, L \in \mathcal{S}_o^n$ was defined by (see [40])

$$\frac{n}{p} \widetilde{V}_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V\left(K \tilde{+}_p \varepsilon \circ L\right) - V(K)}{\varepsilon}. \tag{22}$$

From definition (22), the following integral representation of $L_p$ dual mixed volume was given (see [40]): if $K, L \in \mathcal{S}_o^n$ and $p \geq 1$, then

$$\widetilde{V}_p(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-p} \rho(L, u)^p \, dS(u). \tag{23}$$

Obviously,

$$\widetilde{V}_p(K, K) = V(K). \tag{24}$$

The Minkowski inequality for the $L_p$ dual mixed volume was established in [40]: if $K, L \in \mathcal{S}_o^n$, then, for $1 \leq p < n$,

$$\widetilde{V}_p(K, L) \leq V(K)^{(n-p)/n} V(L)^{p/n}; \tag{25}$$

for $p > n$,

$$\widetilde{V}_p(K, L) \geq V(K)^{(n-p)/n} V(L)^{p/n}. \tag{26}$$

In every case, equality holds if and only if $K$ is a dilation of $L$.

*2.2. $L_p$ Dual Blaschke Body.* For $K, L \in \mathcal{S}_o^n$, $0 < p < n$, and $\lambda, \mu \geq 0$ $(\lambda + \mu \neq 0)$, the $L_p$ dual Blaschke combination, $\lambda \star K \check{+}_p \mu \star L \in \mathcal{S}_o^n$, of $K$ and $L$ is defined by (see [28])

$$\rho\left(\lambda \star K \check{+}_p \mu \star L, \cdot\right)^{n-p} = \lambda \rho(K, \cdot)^{n-p} + \mu \rho(L, \cdot)^{n-p}. \tag{27}$$

Taking $\lambda = \mu = 1/2$, $L = -K$ in (27), the $L_p$ dual Blaschke body, $\check{\nabla}_p K$, of $K$ is given by (see [28])

$$\check{\nabla}_p K = \frac{1}{2} \star K \check{+}_p \frac{1}{2} \star (-K). \tag{28}$$

Obviously, the $L_p$ dual Blaschke body is origin-symmetric.

## 3. Proofs of Theorems 2–6

The proof of Theorem 2 needs the following Lemma.

**Lemma 7** (see [40]). *If $K, L \in \mathcal{S}_o^n$, then, for $p \geq 1$,*

$$\widetilde{V}_p\left(K, I_p L\right) = \widetilde{V}_p\left(L, I_p K\right). \tag{29}$$

*Proof of Theorem 2.* For a star body $\overline{K}$ with $I_p \overline{K} = K$, it follows from Lemma 7 that

$$\begin{aligned} V(K) &= \widetilde{V}_p(K, K) = \widetilde{V}_p\left(K, I_p \overline{K}\right) = \widetilde{V}_p\left(\overline{K}, I_p K\right); \\ \widetilde{V}_p(L, K) &= \widetilde{V}_p\left(L, I_p \overline{K}\right) = \widetilde{V}_p\left(\overline{K}, I_p L\right). \end{aligned} \tag{30}$$

Since

$$\begin{aligned} \widetilde{V}_p\left(\overline{K}, I_p K\right) &= \frac{1}{n} \int_{S^{n-1}} \rho\left(\overline{K}, u\right)^{n-p} \left(\frac{\rho\left(I_p K, u\right)}{\rho\left(I_p L, u\right)}\right)^p \\ &\quad \cdot \rho\left(I_p L, u\right)^p dS(u) \leq \max_{u \in S^{n-1}} \left(\frac{\rho\left(I_p K, u\right)}{\rho\left(I_p L, u\right)}\right)^p \\ &\quad \cdot \widetilde{V}_p\left(\overline{K}, I_p L\right), \end{aligned} \tag{31}$$

we have

$$\frac{V(K)}{\widetilde{V}_p\left(\overline{K}, I_p L\right)} \leq \max_{u \in S^{n-1}} \left(\frac{\rho\left(I_p K, u\right)}{\rho\left(I_p L, u\right)}\right)^p. \tag{32}$$

From $I_p K \subseteq I_p L$, we obtain that, for $1 \leq p < n$,

$$\begin{aligned} V(K) &\leq \widetilde{V}_p\left(\overline{K}, I_p L\right) = \widetilde{V}_p(L, K) \\ &\leq V(L)^{(n-p)/n} V(K)^{p/n}; \end{aligned} \tag{33}$$

that is,

$$V(K) \leq V(L). \tag{34}$$

From the equality condition of (25), we know that $V(K) = V(\mathrm{L})$ if and only if $K = L$.

From a star body $\overline{L}$ with $I_p \overline{L} = L$, By Lemma 7, we have

$$\begin{aligned} V(L) &= \widetilde{V}_p(L, L) = \widetilde{V}_p\left(L, I_p \overline{L}\right) = \widetilde{V}_p\left(\overline{L}, I_p L\right); \\ \widetilde{V}_p(K, L) &= \widetilde{V}_p\left(K, I_p \overline{L}\right) = \widetilde{V}_p\left(\overline{L}, I_p K\right). \end{aligned} \tag{35}$$

Thus

$$\begin{aligned} \widetilde{V}_p\left(\overline{L}, I_p K\right) &= \frac{1}{n} \int_{S^{n-1}} \rho\left(\overline{L}, u\right)^{n-p} \left(\frac{\rho\left(I_p K, u\right)}{\rho\left(I_p L, u\right)}\right)^p \\ &\quad \cdot \rho\left(I_p L, u\right)^p dS(u) \leq \max_{u \in S^{n-1}} \left(\frac{\rho\left(I_p K, u\right)}{\rho\left(I_p L, u\right)}\right)^p \\ &\quad \cdot \widetilde{V}_p\left(\overline{L}, I_p L\right); \end{aligned} \tag{36}$$

that is,

$$\frac{\widetilde{V}_p(K,L)}{V(L)} \le \max_{u\in S^{n-1}} \left(\frac{\rho\left(I_pK,u\right)}{\rho\left(I_pL,u\right)}\right)^p. \tag{37}$$

For $p > n$, it follows from $I_pK \subseteq I_pL$ that

$$V(L) \ge \widetilde{V}_p(K,L) \ge V(K)^{(n-p)/n} V(L)^{p/n}; \tag{38}$$

namely,

$$V(K) \ge V(L). \tag{39}$$

□

**Lemma 8** (see [28]). *If $K, L \in \mathcal{S}_o^n$ and $\lambda, \mu \ge 0$ $(\lambda + \mu \ne 0)$, then, for $0 < p < n$,*

$$\begin{aligned} &V\left(\lambda \star K \tilde{+}_p \mu \star L\right)^{(n-p)/n} \\ &\quad \le \lambda V(K)^{(n-p)/n} + \mu V(L)^{(n-p)/n}, \end{aligned} \tag{40}$$

*with equality if and only if $K$ is a dilation of $L$.*

Let $\lambda = \mu = 1/2$, $L = -K$ in (40), the following is an immediate result of Lemma 8.

**Corollary 9.** *If $K \in \mathcal{S}_o^n$, then, for $0 < p < n$,*

$$V\left(\check{\nabla}_p K\right) \le V(K), \tag{41}$$

*with equality if and only if $K$ is origin-symmetric.*

**Lemma 10.** *If $K \in \mathcal{S}_o^n$, then, for $p \ge 1$,*

$$I_p\left(\check{\nabla}_p K\right) = I_p K. \tag{42}$$

*Proof.* From (9), (27), and (28), we have

$$\begin{aligned} \rho\left(I_p\left(\check{\nabla}_p K\right),u\right)^p &= \frac{1}{(n-1)\omega_{n-1}} \int_{S^{n-1}\cap u^\perp} \rho\left(\frac{1}{2}\right. \\ &\quad \left.\star K \tilde{+}_p \frac{1}{2} \star (-K), v\right)^{n-p} dS_{n-2}(v) \\ &= \frac{1}{(n-1)\omega_{n-1}} \int_{S^{n-1}\cap u^\perp} \left[\frac{1}{2}\rho(K,v)^{n-p}\right. \\ &\quad \left.+ \frac{1}{2}\rho(-K,v)^{n-p}\right] dS_{n-2}(v) = \frac{1}{2} \\ &\quad \cdot \rho\left(I_pK,u\right)^p + \frac{1}{2}\rho\left(I_p(-K),u\right)^p. \end{aligned} \tag{43}$$

Since $I_p(-K) = I_p(K)$, we have

$$\rho\left(I_p\left(\check{\nabla}_p K\right),u\right)^p = \rho\left(I_pK,u\right)^p; \tag{44}$$

that is,

$$I_p\left(\check{\nabla}_p K\right) = I_p K. \tag{45}$$

□

*Proof of Theorem 4.* Since $K \notin \mathcal{S}_e^n$, Corollary 9 implies

$$V\left(\check{\nabla}_p K\right) < V(K). \tag{46}$$

Let $\varepsilon > 0$ such that $V((1+\varepsilon)\check{\nabla}_p K) < V(K)$. Taking $L = (1+\varepsilon)\check{\nabla}_p K$ we have

$$V(K) > V(L). \tag{47}$$

Combining with Lemma 10 we get

$$\begin{aligned} I_pL &= I_p\left((1+\varepsilon)\check{\nabla}_p K\right) = (1+\varepsilon)^{(n-p)/p} I_p\left(\check{\nabla}_p K\right) \\ &= (1+\varepsilon)^{(n-p)/p} I_pK \supset I_pK. \end{aligned} \tag{48}$$

□

*Proof of Theorem 5.* From the condition $I_pK \subseteq I_pL$, we have, for arbitrary $N \in \mathcal{S}_o^n$,

$$\widetilde{V}_p\left(N, I_pK\right) \le \widetilde{V}_p\left(N, I_pL\right). \tag{49}$$

By Lemma 7, we obtain

$$\widetilde{V}_p\left(K, I_pN\right) \le \widetilde{V}_p\left(L, I_pN\right). \tag{50}$$

Let $M = I_pN$ in (50). Together with (15), it follows that

$$\overline{G}_p(K) \le \overline{G}_p(L), \tag{51}$$

with equality if and only if $K = L$. □

**Lemma 11.** *If $K \in \mathcal{S}_o^n$ and $1 \le p < n$, then*

$$\widetilde{G}_p\left(\check{\nabla}_p K\right) \le \widetilde{G}_p(K), \tag{52}$$

*with equality if and only if $K$ is origin-symmetric.*

*Proof.* From (14), (27), and (28), we get

$$\begin{aligned} \omega_n^{p/n}\widetilde{G}_p\left(\check{\nabla}_p K\right) &= \sup\left\{n\widetilde{V}_p\left(\check{\nabla}_p K, Q\right) V(Q^*)^{p/n} : Q\right. \\ &\quad \left.\in \mathcal{S}_e^n\right\} = \sup\left\{n\widetilde{V}_p\left(\frac{1}{2}\star K \tilde{+}_p \frac{1}{2}\star(-K), Q\right)\right. \\ &\quad \left.\cdot V(Q^*)^{p/n} : Q \in \mathcal{S}_e^n\right\} \\ &= \sup\left\{\int_{S^{n-1}} \rho\left(\frac{1}{2}\star K \tilde{+}_p \frac{1}{2}\star(-K), u\right)^{n-p}\right. \\ &\quad \cdot \rho(L,u)^p\, dS(u)\, V(Q^*)^{p/n} : Q \\ &\quad \left.\in \mathcal{S}_e^n\right\} \\ &= \sup\left\{\int_{S^{n-1}} \left[\frac{1}{2}\rho(K,u)^{n-p} + \frac{1}{2}\rho(-K,u)^{n-p}\right]\right. \\ &\quad \cdot \rho(L,u)^p\, dS(u)\, V(Q^*)^{p/n} : Q \\ &\quad \left.\in \mathcal{S}_e^n\right\} = \sup\left\{\frac{n}{2}\widetilde{V}_p(K,Q)\, V(Q^*)^{p/n} + \frac{n}{2}\right. \\ &\quad \left.\cdot \widetilde{V}_p(-K,Q)\, V(Q^*)^{p/n} : Q \in \mathcal{S}_e^n\right\} \le \frac{1}{2} \end{aligned}$$

$$\cdot \sup\left\{n\widetilde{V}_p(K,Q)\,V(Q^*)^{p/n} : Q \in \mathcal{S}_e^n\right\} + \frac{1}{2}$$
$$\cdot \sup\left\{n\widetilde{V}_p(-K,Q)\,V(Q^*)^{p/n} : Q \in \mathcal{S}_e^n\right\}. \tag{53}$$

Note that $Q \in \mathcal{S}_e^n$; thus we have $\widetilde{V}_p(-K,Q) = \widetilde{V}_p(K,Q)$. Together with (53), this yields

$$\widetilde{G}_p\left(\check{\nabla}_p K\right) \le \widetilde{G}_p(K). \tag{54}$$

Equality holds in (53) if and only if $K$ is a dilation of $-K$. This gives $K = -K$. Namely, $K$ is origin-symmetric. Hence, equality holds in (54) if and only if $K$ is origin-symmetric. □

*Proof of Theorem 6.* Since $K \notin \mathcal{S}_e^n$, from Lemma 11 we have

$$\widetilde{G}_p\left(\check{\nabla}_p K\right) < \widetilde{G}_p(K). \tag{55}$$

Choose $\varepsilon > 0$ such that $\widetilde{G}_p((1+\varepsilon)\check{\nabla}_p K) < \widetilde{G}_p(K)$. Taking $L = (1+\varepsilon)\check{\nabla}_p K$ we obtain

$$\widetilde{G}_p(K) > \widetilde{G}_p(L). \tag{56}$$

It follows from Lemma 10 that

$$\begin{aligned} I_p L &= I_p\left((1+\varepsilon)\check{\nabla}_p K\right) = (1+\varepsilon)^{(n-p)/p} I_p\left(\check{\nabla}_p K\right) \\ &= (1+\varepsilon)^{(n-p)/p} I_p K \supset I_p K. \end{aligned} \tag{57}$$

□

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors read and approved the final manuscript.

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