Research Article

Busemann-Petty Problems for Quasi $L_p$ Intersection Bodies

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We investigate the $L_p$ dual geominimal surface area and volume forms of Busemann-Petty problems for the quasi $L_p$ intersection bodies and establish some new geometric inequalities. Our results provide a significant complement to the researches on Busemann-Petty problems for intersection bodies.

1. Introduction and Main Results

Let $S^{n-1}$ denote the unit sphere in Euclidean space $\mathbb{R}^n$. If $K$ is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, then its radial function, $\rho_K: \mathbb{R}^n \setminus \{0\} \rightarrow [0, \infty)$, is defined by (see [1, 2])

$$\rho_K(\mathbf{u}) = \max \{ \lambda \geq 0 : \lambda \mathbf{u} \in K \}, \quad \mathbf{u} \in S^{n-1}. \quad (1)$$

If $\rho_K$ is positive and continuous, then $K$ will be called a star body (about the origin), and $\mathcal{S}^n$ denotes the set of star bodies in $\mathbb{R}^n$. We use $\mathcal{S}_o^n$ and $\mathcal{S}_e^n$ to denote the subset of star bodies in $\mathcal{S}^n$ containing the origin in their interiors and origin-symmetric star bodies, respectively. Two star bodies $K$ and $L$ are said to be the dilation of one another if $\rho_K(\mathbf{u})/\rho_L(\mathbf{u})$ is independent of $\mathbf{u} \in S^{n-1}$.

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean $\mathbb{R}^n$. For $u \in S^{n-1}$, $u^\perp$ denotes the $(n-1)$-dimensional subspace orthogonal to $u$. We use $V_k(M)$ to denote the $k$-dimensional volume of a $k$-dimensional compact convex set $M$. Instead of $V_n$ we usually write $V$. For the standard unit ball $B$ in $\mathbb{R}^n$, we write $\omega_n = V(B)$ for its volume.

Busemann and Petty posed a problem [3]: let $K$ and $L$ be origin-symmetric convex bodies in $\mathbb{R}^n$. Is it true that, for any $u \in S^{n-1}$,

$$V_{n-1}(K \cap u^\perp) \leq V_{n-1}(L \cap u^\perp) \quad \Rightarrow \quad V(K) \leq V(L)? \quad (2)$$

A long list of authors contributed to the solution of this famous problem over a period of 40 years; see [4–18]. The question has a negative answer for $n \geq 5$ and an affirmative answer for $n = 3, 4$. For a detailed account of the interesting history of the Busemann-Petty problem, see the books by Gardner [1, Chapter 8] and Koldobsky [19, Chapter 5].

The crucial idea in the solution of the problem was to define new convex body which was called intersection body. This was done by Lutwak [15] whose work is considered the starting point of the solution of the Busemann-Petty problem in all dimensions. For $K \in \mathcal{S}_o^n$, the intersection body, $IK$, of $K$ is a star body whose radial function in the direction $u \in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of $K$ by $u^\perp$; that is,

$$\rho(IK, u) = V_{n-1}(K \cap u^\perp). \quad (3)$$

The intersection bodies have been intensively studied in recent years (see [20–28] and the books [19, 29]). From (3) and the fact that star bodies $K$ and $L$ satisfy $K \subseteq L$ if and only if $\rho(K, \cdot) \leq \rho(L, \cdot)$, we see that the Busemann-Petty problem can be rephrased in the following way: for $K, L \in \mathcal{S}_o^n$, is it true that

$$IK \subseteq IL \quad \Rightarrow \quad V(K) \leq V(L)? \quad (4)$$

Lutwak [15] showed that the problem has an affirmative answer if the body $K$ is restricted to the class of intersection
bodies. In addition, Lutwak proved that if \( L \) is a sufficiently smooth origin-symmetric star body with positive radial function which is not an intersection body, then there exists an origin-symmetric star body \( K \) such that \( IK \subseteq IL \) but \( V(K) > V(L) \). Further, Busemann-Petty problems have been considered in the context of \( L_p \) Brunn-Minkowski Theory (see [30–41]). In particular, Haberl and Ludwig [42] generalized the intersection body to \( L_p \) form and introduced the notion of \( L_p \) intersection body: let \( L \) be a star body and nonzero \( p < 1 \). The \( L_p \) intersection body of \( L \), \( I_p L \), is the symmetric star body whose radial function is defined by

\[
\rho(I_p L, u)^p = \int_L |u \cdot x|^{-p} \, dx. \tag{5}
\]

After that, associated with \( L_p \) intersection bodies, Yuan and Cheung [41] gave an affirmative form of Busemann-Petty problem for the \( L_p \) intersection bodies.

**Theorem 1.** Let \( K \) be \( L_p \) intersection body and \( L \) be a star body in \( \mathbb{R}^n \). If \( I_p K \subseteq I_p L \), then

\[
V(K) \leq V(L), \quad \text{for } 0 < p < 1, \tag{6}
\]

\[
V(K) \geq V(L), \quad \text{for } p < 0.
\]

In both cases equality holds if and only if \( K = L \).

In 2007, Yu et al. [40] defined the quasi \( L_p \) intersection body as follows: let \( K \) be a star body and \( p \geq 1 \). The quasi \( L_p \) intersection body, \( I_p K \), of \( K \) is defined by

\[
\rho(I_p K, u) = \left( \frac{V_p(K, B \cap u^\perp)}{V_p(B, B \cap u^\perp)} \right)^{1/p}, \tag{7}
\]

for \( u \in S^{n-1} \), where \( V_p \) denotes the \( L_p \) dual mixed volume (see (22)).

Suppose that \( f \) is a Borel function on \( S^{n-1} \). The spherical Radon transform \( Rf \) [43] of \( f \) is defined by

\[
(Rf)(u) = \int_{S^{n-1} \cap u^\perp} f(v) \, dS_{n-2}(v), \tag{8}
\]

for \( u \in S^{n-1} \). Using the spherical Radon transform, the definition of \( I_p K \) is rewritten by

\[
\rho(I_p K, u) = \left( \frac{\tilde{V}_p(K, B \cap u^\perp)}{\tilde{V}_p(B, B \cap u^\perp)} \right)^{1/p},
\]

\[
= \left( \frac{1}{(n-1) \omega_{n-1}} R(\rho_K)^{n-p}(u) \right)^{1/p}, \tag{9}
\]

\[
= \left( \frac{1}{(n-1) \omega_{n-1}} \int_{S^{n-1} \cap u^\perp} \rho(K, v)^{n-p} \, dS_{n-2}(v) \right)^{1/p},
\]

for \( u \in S^{n-1} \).

One aim of this paper is to establish the volume forms of the Busemann-Petty problems for the quasi \( L_p \) intersection bodies. For convenience, let \( I_p \) denote the set of quasi \( L_p \) intersection bodies.

**Theorem 2.** Let \( K, L \in \delta^n_o \) and \( p \geq 1 \). If \( 1 \leq p < n \) and \( K \in I_p \), then

\[
I_p K \subseteq I_p L \implies V(K) \leq V(L). \tag{10}
\]

If \( p > n \) and \( L \in I_p \), then

\[
I_p K \subseteq I_p L \implies V(K) \geq V(L). \tag{11}
\]

And \( V(K) = V(L) \) if and only if \( K = L \).

**Remark 3.** Theorem 2 can be found in [40]. However, what should be noted is that we give a new method of proof in this paper.

**Theorem 4.** For \( 1 \leq p < n \), if \( K \notin \delta^n_o \), then there exists \( L \in \delta^n_e \) such that

\[
I_p K \subset I_p L, \tag{12}
\]

but

\[
V(K) > V(L). \tag{13}
\]

Recall the definition of \( L_p \) dual geominimal surface area, \( \overline{G}_p(K) \), of \( K \in \delta^n_o \) for \( p > 0 \) in [44]:

\[
\omega_n^{p/n} \overline{G}_p(K) = \sup \left\{ n\tilde{V}_p(K, Q) V(Q^*)^{p/n} : Q \in \delta^n_e \right\}. \tag{14}
\]

Another aim of this paper is to give the \( L_p \) dual geominimal surface area forms of Busemann-Petty problems for the quasi \( L_p \) intersection bodies. If \( M \in I_p \), then we rewrite the definition of \( L_p \) dual geominimal surface area by

\[
\omega_n^{p/n} \overline{G}_p(K) = \sup \left\{ n\tilde{V}_p(K, M) V(M^*)^{p/n} : M \in I_p \right\}. \tag{15}
\]

**Theorem 5.** If \( p \geq 1 \) and \( K, L \in \delta^n_o \), then

\[
I_p K \subseteq I_p L, \tag{16}
\]

implying

\[
\overline{G}_p(K) \leq \overline{G}_p(L). \tag{17}
\]

And \( \overline{G}_p(K) = \overline{G}_p(L) \) if and only if \( K = L \).

**Theorem 6.** For \( 1 \leq p < n \), if \( K \notin \delta^n_o \), then there exists \( L \in \delta^n_e \) such that

\[
I_p K \subset I_p L, \tag{18}
\]

but

\[
\overline{G}_p(K) > \overline{G}_p(L). \tag{19}
\]
2. Preliminaries

2.1. $L_p$ Dual Mixed Volume. For $K, L \in \mathcal{D}^n_\omega$, $p > 0$, and $\lambda, \mu \geq 0$ ($\lambda + \mu \neq 0$), the $L_p$ radial combination, $\lambda \ast K + \mu \ast L \in \mathcal{D}^n_\omega$, of $K$ and $L$ is defined by (see [40])

$$\rho \left( \lambda \ast K + \mu \ast L, \cdot \right)^p = \lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p. \quad (20)$$

The polar coordinate formula for the volume of a body $K \in \mathcal{D}^n_\omega$ is

$$V(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^p dS(u). \quad (21)$$

For $p \geq 1$, the $L_p$ dual mixed volume, $\overline{V}_p(K, L)$, of $K, L \in \mathcal{D}^n_\omega$ was defined by (see [40])

$$\overline{V}_p(K, L) = \frac{n}{p} \lim_{\varepsilon \to 0^+} \frac{V((K + \varepsilon L, \cdot)^p) - V(K)}{\varepsilon}. \quad (22)$$

From definition (22), the following integral representation of $L_p$ dual mixed volume was given (see [40]): if $K, L \in \mathcal{D}^n_\omega$ and $p \geq 1$, then

$$\overline{V}_p(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-p} \rho(L, u)^p dS(u). \quad (23)$$

Obviously,

$$\overline{V}_p(K, K) = V(K). \quad (24)$$

The Minkowski inequality for the $L_p$ dual mixed volume was established in [40]: if $K, L \in \mathcal{D}^n_\omega$, then, for $1 \leq p < n$,

$$\overline{V}_p(K, L) \leq V(K)^{(n-p)/n} V(L)^{p/n}; \quad (25)$$

for $p > n$,

$$\overline{V}_p(K, L) \geq V(K)^{(n-p)/n} V(L)^{p/n}. \quad (26)$$

In every case, equality holds if and only if $K$ is a dilation of $L$.

2.2. $L_p$ Dual Blaschke Body. For $K, L \in \mathcal{D}^n_\omega$, $0 < p < n$, and $\lambda, \mu \geq 0$ ($\lambda + \mu \neq 0$), the $L_p$ dual Blaschke combination, $\lambda \ast K + \mu \ast L \in \mathcal{D}^n_\omega$, of $K$ and $L$ is defined by (see [28])

$$\rho \left( \lambda \ast K + \mu \ast L, \cdot \right)^p = \lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p. \quad (27)$$

Taking $\lambda = \mu = 1/2$, $L = -K$ in (27), the $L_p$ dual Blaschke body, $\overline{V}_pK$, of $K$ is given by (see [28])

$$\overline{V}_pK = \frac{1}{2} \ast \frac{1}{2} \ast (-K). \quad (28)$$

Obviously, the $L_p$ dual Blaschke body is origin-symmetric.

3. Proofs of Theorems 2–6

The proof of Theorem 2 needs the following Lemma.

Lemma 7 (see [40]). If $K, L \in \mathcal{D}^n_\omega$, then, for $p \geq 1$,

$$\overline{V}_p(K, I_pL) = \overline{V}_p(L, I_pK). \quad (29)$$

Proof of Theorem 2. For a star body $\overline{K}$ with $I_p\overline{K} = K$, it follows from Lemma 7 that

$$V(K) = \overline{V}_p(K, K) = \overline{V}_p(K, I_pK) = \overline{V}_p(\overline{K}, I_p\overline{K}); \quad (30)$$

$$\overline{V}_p(L, K) = \overline{V}_p(L, I_p\overline{K}) = \overline{V}_p(\overline{K}, I_pL).$$

Since

$$\overline{V}_p(K, I_pK) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-p} \rho(I_pK, u)^p \rho(I_pL, u)^p \rho(I_pL, u)^p dS(u) \leq \max_{u \in S^{n-1}} \left( \frac{\rho(I_pK, u)}{\rho(I_pL, u)} \right)^p \cdot \overline{V}_p(\overline{K}, I_pL), \quad (31)$$

we have

$$V(K) \leq \overline{V}_p(K, I_pL) \leq \overline{V}_p(\overline{K}, I_pL) \leq V(L)^{(n-p)/n} V(K)^{p/n}; \quad (32)$$

that is,

$$V(K) \leq V(L). \quad (34)$$

From the inequality condition of (25), we know that $V(K) = V(L)$ if and only if $K = L$.

From a star body $\overline{K}$ with $I_p\overline{K} = K$, By Lemma 7, we have

$$V(K) = \overline{V}_p(I_pK); \quad (35)$$

Thus

$$\overline{V}_p(I_pK) = \frac{1}{n} \int_{S^{n-1}} \rho(I_pK, u)^{n-p} \rho(I_pK, u)^p \rho(I_pL, u)^p \rho(I_pL, u)^p dS(u) \leq \max_{u \in S^{n-1}} \left( \frac{\rho(I_pK, u)}{\rho(I_pL, u)} \right)^p \cdot \overline{V}_p(I_pK), \quad (36)$$

$$\overline{V}_p(I_pL) = \overline{V}_p(I_pK).$$
that is,
\[
\frac{V_p(K, L)}{V(L)} \leq \max_{u \in \mathbb{S}^{n-1}} \left( \frac{\rho(I_p K, u)}{\rho(I_p L, u)} \right)^p. \tag{37}
\]
For \(p > n\), it follows from \(I_p K \subseteq I_p L\) that
\[
V(K) \geq V_p(K, L) \geq V(K)^{(n-p)/n} V(L)^{p/n}, \tag{38}
\]
namely,
\[
V(K) \geq V(L). \tag{39}
\]

**Lemma 8** (see [28]). If \(K, L \in \mathcal{S}_o^n\) and \(\lambda, \mu \geq 0, (\lambda + \mu) \neq 0\), then, for \(0 < p < n\),
\[
V\left(\lambda K + \mu L\right)^{(n-p)/n} \leq \lambda V(K)^{(n-p)/n} + \mu V(L)^{(n-p)/n},
\]
with equality if and only if \(K\) is a dilation of \(L\).

Let \(\lambda = \mu = 1/2, L = -K\) in (40), the following is an immediate result of Lemma 8.

**Corollary 9.** If \(K \in \mathcal{S}_o^n\), then, for \(0 < p < n\),
\[
V\left(\tilde{V}_p K\right) \leq V(K),
\]
with equality if and only if \(K\) is origin-symmetric.

**Lemma 10.** If \(K \in \mathcal{S}_o^n\), then, for \(p \geq 1\),
\[
I_p\left(\tilde{V}_p K\right) = I_p K.
\]

**Proof.** From (9), (27), and (28), we have
\[
\rho\left(I_p\left(\tilde{V}_p K\right), u\right)^p = \frac{1}{(n-1) \omega_{n-1}} \int_{S^{n-1}} \rho\left(\frac{1}{2}, K + \mu \frac{1}{2} (-K), v\right)^{n-p} dS_{n-2}(v)
\]
\[
+ \frac{1}{2} \rho\left(-K, v\right)^{n-p} dS_{n-2}(v) = \frac{1}{2} \rho\left(I_p K, u\right)^p + \frac{1}{2} \rho\left(I_p(-K), u\right)^p.
\]
Since \(I_p(-K) = I_p K\), we have
\[
\rho\left(I_p\left(\tilde{V}_p K\right), u\right)^p = \rho\left(I_p K, u\right)^p; \tag{44}
\]
that is,
\[
I_p\left(\tilde{V}_p K\right) = I_p K. \tag{45}
\]

**Proof of Theorem 4.** Since \(K \notin \mathcal{S}_r^n\), Corollary 9 implies
\[
V\left(\tilde{V}_p K\right) < V(K). \tag{46}
\]
Let \(\varepsilon > 0\) such that \(V((1 + \varepsilon)\tilde{V}_p K) < V(K)\). Taking \(L = (1 + \varepsilon)\tilde{V}_p K\) we have
\[
V(K) > V(L). \tag{47}
\]
Combining with Lemma 10 we get
\[
I_p L = I_p\left(1 + \varepsilon\right) \tilde{V}_p K = (1 + \varepsilon)^{(n-p)/p} I_p\left(\tilde{V}_p K\right)
\]
\[
= (1 + \varepsilon)^{(n-p)/p} I_p K \geq I_p K.
\]

**Proof of Theorem 5.** From the condition \(I_p K \subseteq I_p L\), we have, for arbitrary \(N \in \mathcal{S}_o^n\),
\[
\tilde{V}_p\left(N, I_p K\right) \leq \tilde{V}_p\left(N, I_p L\right). \tag{49}
\]
By Lemma 7, we obtain
\[
\tilde{V}_p\left(K, I_p N\right) \leq \tilde{V}_p\left(L, I_p N\right). \tag{50}
\]
Let \(M = I_p N\) in (50). Together with (15), it follows that
\[
\overline{G}_p(K) \leq \overline{G}_p(L),
\]
with equality if and only if \(K = L\).

**Lemma 11.** If \(K \in \mathcal{S}_r^n\) and \(1 \leq p < n\), then
\[
\overline{G}_p\left(\tilde{V}_p K\right) \leq \overline{G}_p(K),
\]
with equality if and only if \(K\) is origin-symmetric.

**Proof.** From (14), (27), and (28), we get
\[
\omega_n^{p/n} \overline{G}_p\left(\tilde{V}_p K\right) = \sup \left\{ \int_{S^{n-1}} \rho\left(\frac{1}{2}, K + \mu \frac{1}{2} (-K), Q\right)^{n-p} dS(u) \cdot V(Q)^{p/n} : Q \in \mathcal{S}_r^n\right\}
\]
\[
= \sup \left\{ \int_{S^{n-1}} \rho\left(\frac{1}{2}, K + \mu \frac{1}{2} (-K), u\right)^{n-p} \cdot \rho(L, u)^p dS(u) V(Q)^{p/n} : Q \in \mathcal{S}_r^n\right\}
\]
\[
= \sup \left\{ \int_{S^{n-1}} \left[ \frac{1}{2} \rho(K, u)^{n-p} + \frac{1}{2} \rho(-K, u)^{n-p} \right] \cdot \rho(L, u)^p dS(u) V(Q)^{p/n} : Q \in \mathcal{S}_r^n\right\}
\]
\[
\leq \frac{1}{2} \left[ \omega_n^{p/n} \overline{G}_p(K), Q\right) V(Q)^{p/n} + \frac{n}{2} \right.
\]
\[
\cdot \overline{V}_p(-K, Q) V(Q)^{p/n} : Q \in \mathcal{S}_r^n \leq \frac{1}{2}.
\]
\[
\cdot \sup \left\{ nV_p(K, Q) V(Q^*)^{p/n} : Q \in S^n \right\} + \frac{1}{2} \\
\cdot \sup \left\{ nV_p(-K, Q) V(Q^*)^{p/n} : Q \in S^n \right\} 
\]

(53)

Note that \( Q \in S^n \); thus we have \( V_p(-K, Q) = V_p(K, Q) \). Together with (53), this yields

\[
\overline{G}_p \left( \overline{V}_p K \right) \leq \overline{G}_p \left( K \right). 
\]

Equality holds in (53) if and only if \( K \) is a dilation of \(-K\). This gives \( K = -K \). Namely, \( K \) is origin-symmetric. Hence, equality holds in (54) if and only if \( K \) is origin-symmetric. \( \square \)

**Proof of Theorem 6.** Since \( K \notin S^n \), from Lemma 11 we have

\[
\overline{G}_p \left( \overline{V}_p K \right) < \overline{G}_p \left( K \right). 
\]

Choose \( \varepsilon > 0 \) such that \( \overline{G}_p \left( (1 + \varepsilon)\overline{V}_p K \right) < \overline{G}_p \left( K \right) \). Taking \( L = (1 + \varepsilon)\overline{V}_p K \) we obtain

\[
\overline{G}_p \left( K \right) > \overline{G}_p \left( L \right). 
\]

It follows from Lemma 10 that

\[
I_p L = I_p \left( (1 + \varepsilon)\overline{V}_p K \right) = (1 + \varepsilon)^{(n-p)/p} I_p \left( \overline{V}_p K \right) \\
= (1 + \varepsilon)^{(n-p)/p} I_p K \supset I_p K. 
\]

(57)

\( \square \)

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

All authors read and approved the final manuscript.

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**References**


