Research Article

Identification of Fully Measurable Grand Lebesgue Spaces

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We consider the Banach function spaces, called *fully measurable grand Lebesgue spaces*, associated with the function norm $\rho(f) = \operatorname{ess\,sup}_{x \in X} \delta(x) \rho_{p(x)}(f)$, where $\rho_{p(x)}$ denotes the norm of the Lebesgue space of exponent $p(x)$, and $p(\cdot)$ and $\delta(\cdot)$ are measurable functions over a measure space (X, ν) , $p(x) \in [1, \infty]$, and $\delta(x) \in (0, 1]$ almost everywhere. We prove that every such space can be expressed equivalently replacing $p(\cdot)$ and $\delta(\cdot)$ with functions defined everywhere on the interval $(0, 1)$, decreasing and increasing, respectively (hence the full measurability assumption in the definition does not give an effective generalization with respect to the pointwise monotone assumption and the essential supremum can be replaced with the simple supremum). In particular, we show that, in the case of bounded $p(\cdot)$, the class of fully measurable Lebesgue spaces coincides with the class of generalized grand Lebesgue spaces introduced by Capone, Formica, and Giova.

Let (Ω, μ) be a finite measure space, and let $\mathcal{M}(\Omega)$ be the set of all measurable functions on Ω with values in $[-\infty, +\infty]$, $\mathcal{M}^+(\Omega)$ be the subset of the nonnegative functions, and $\mathcal{M}_0(\Omega)$ be the subset of the real valued functions.

Let (X, ν) be a measure space, and define $\mathcal{M}(X)$ similarly as above. Let $p(\cdot), \delta(\cdot) \in \mathcal{M}(X)$ such that $p(x) \in [1, \infty]$ and $\delta(x) \in (0, 1]$ for ν -almost every $x \in X$. Following [1], for $f \in \mathcal{M}^+(X)$, we set

$$\rho_{p[\cdot], \delta(\cdot)}(f) = \operatorname{ess\,sup}_{x \in X} \delta(x) \rho_{p(x)}(f), \quad (1)$$

where

$$\rho_p(f) = \begin{cases} \left(\frac{1}{\mu(\Omega)} \int_{\Omega} f^p d\mu \right)^{1/p} & \text{if } 1 \leq p < \infty; \\ \operatorname{ess\,sup}_{\Omega} f & \text{if } p = \infty. \end{cases} \quad (2)$$

Then $\rho_{p[\cdot], \delta(\cdot)}$ is a Banach function norm, and the associated Banach function space

$$L^{p[\cdot], \delta(\cdot)}(\Omega) = \{f \in \mathcal{M}_0(\Omega) : \|f\|_{p[\cdot], \delta(\cdot)} = \rho_{p[\cdot], \delta(\cdot)}(|f|) < \infty\} \quad (3)$$

is called *fully measurable grand Lebesgue space*. This definition is one of the most abstract generalizations of the original grand Lebesgue spaces originated in the paper by Iwaniec and Sbordone [2].

Given $p(\cdot) \in \mathcal{M}(X)$ as above, let us set

$$\begin{aligned} p_+ &= \operatorname{ess\,sup}_X p(\cdot), \\ p_- &= \operatorname{ess\,inf}_X p(\cdot), \end{aligned} \quad (4)$$

so that $1 \leq p_- \leq p_+ \leq \infty$. In the following, we exclude the noninteresting case of constant $p(\cdot)$, that is, the case $p_- = p_+ =: p_0$, because in such case we know (see [1]) that $L^{p[\cdot], \delta(\cdot)}(\Omega) = L^{p_0}(\Omega)$.

Given a function $\varphi = \varphi(\cdot)$ on $[p_-, p_+)$ that is positive and decreasing (i.e., $p_- \leq p_1 < p_2 < p_+ \Rightarrow \varphi(p_1) \geq \varphi(p_2) > 0$), we consider the Banach function norm:

$$\rho_\varphi(f) = \sup_{p \in [p_-, p_+)} \varphi(p) \left(\frac{1}{\mu(\Omega)} \int_\Omega |f|^p d\mu \right)^{1/p}. \quad (5)$$

Theorem 1. Let $p(\cdot), \delta(\cdot) \in \mathcal{M}(X)$ be such that $p(\cdot)$ is nonconstant, $p(\cdot) \in [1, \infty]$, and $\delta(\cdot) \in (0, 1]$ ν -almost everywhere. Then there exists a positive, decreasing, and left-continuous function $\varphi = \varphi(\cdot)$ on $[p_-, p_+)$, such that

$$\rho_\varphi(f) = \rho_{p[\cdot], \delta(\cdot)}(f), \quad f \in \mathcal{M}^+(\Omega). \quad (6)$$

Some comments on this result are in order. If $(X, \nu), (\Omega, \mu)$ coincide with the interval $I = (0, 1) \subseteq \mathbb{R}$ endowed with the Lebesgue measure, and if $p_+ < \infty$, then the class of spaces associated with the Banach function norm ρ_φ coincides with the class of spaces associated with the Banach function norm $\rho_{p[\cdot], \delta(\cdot)}$, where $\delta(\cdot)$ is increasing. In fact, substituting $p = p_+ - \varepsilon$, we have

$$\begin{aligned} \rho_\varphi(f) &= \sup_{p \in [p_-, p_+)} \varphi(p) \left(\frac{1}{\mu(\Omega)} \int_\Omega |f|^p d\mu \right)^{1/p} \\ &= \sup_{\varepsilon \in (0, p_+ - p_-]} \varphi(p_+ - \varepsilon) \left(\frac{1}{\mu(\Omega)} \int_\Omega |f|^{p_+ - \varepsilon} d\mu \right)^{1/(p_+ - \varepsilon)}. \end{aligned} \quad (7)$$

In other words, Theorem 1 shows that the class of fully measurable Lebesgue spaces introduced in [1], where both X and Ω coincide with the interval $I = (0, 1) \subseteq \mathbb{R}$, coincides in the case of bounded exponents with the class of spaces introduced in [3]. As a consequence, it does not make a difference whether one defines the fully measurable Lebesgue spaces $L^{p[\cdot], \delta(\cdot)}(I)$ with measurable δ as above or with increasing δ defined on the interval $(0, 1)$ and such that $\delta(0+) = 0$. This fact has a couple of consequences to be stressed.

First, to consider “essential suprema” for generalized grand Lebesgue spaces, and also for their associate spaces (which are called generalized small Lebesgue spaces), does not lead to more generality. Second, the attempts to look for the associate spaces of the fully measurable Lebesgue spaces, made in [4, 5], both solve completely the characterization of the duality. Our result may be relevant also for the weighted variant of the fully measurable Lebesgue spaces which recently attracted much interest ([6–9]).

Proof of Theorem 1. Let us set

$$\varphi(p) := \operatorname{ess\,sup}_{\{x \in X: p(x) \geq p\}} \delta(x) \quad \forall p \in [p_-, p_+). \quad (8)$$

Then φ is positive, decreasing, and left-continuous. Let $f \in \mathcal{M}^+(\Omega)$. Let $k < \rho_\varphi(f)$, and let $p \in [p_-, p_+)$ be such that

$$k < \varphi(p) \left(\frac{1}{\mu(\Omega)} \int_\Omega |f|^p d\mu \right)^{1/p}. \quad (9)$$

For any $\sigma > 1$, let $E_{p, \sigma} \subseteq X$ be defined by

$$E_{p, \sigma} := \{x \in X : p(x) \geq p, \sigma \delta(x) > \varphi(p)\}. \quad (10)$$

Then $E_{p, \sigma}$ is measurable and has strictly positive ν -measure. For any $x \in E_{p, \sigma}$ we have, by Hölder’s inequality,

$$\varphi(p) \left(\frac{1}{\mu(\Omega)} \int_\Omega |f|^p d\mu \right)^{1/p} \leq \sigma \delta(x) \rho_{p(x)}(f), \quad (11)$$

from which

$$\begin{aligned} \varphi(p) \left(\frac{1}{\mu(\Omega)} \int_\Omega |f|^p d\mu \right)^{1/p} &\leq \operatorname{ess\,sup}_{E_{p, \sigma}} \sigma \delta(x) \rho_{p(x)}(f) \\ &\leq \sigma \operatorname{ess\,sup}_X \delta(x) \rho_{p(x)}(f) = \sigma \rho_{p[\cdot], \delta(\cdot)}(f). \end{aligned} \quad (12)$$

Therefore, for any $k < \rho_\varphi(f)$,

$$k \leq \rho_{p[\cdot], \delta(\cdot)}(f), \quad (13)$$

and hence

$$\rho_\varphi(f) \leq \rho_{p[\cdot], \delta(\cdot)}(f). \quad (14)$$

On the other hand, since $\delta(x) \leq \varphi(p(x))$ for ν -almost every $x \in X$,

$$\begin{aligned} \delta(x) \rho_{p(x)}(f) &\leq \varphi(p(x)) \rho_{p(x)}(f) \\ &\leq \sup_{p \in [p_-, p_+)} \varphi(p) \left(\frac{1}{\mu(\Omega)} \int_\Omega |f|^p d\mu \right)^{1/p} \end{aligned} \quad (15)$$

for ν -almost every $x \in X$,

and hence

$$\rho_\varphi(f) \geq \rho_{p[\cdot], \delta(\cdot)}(f). \quad (16)$$

□

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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