Research Article

Positive Solutions of Fractional Differential Equations with p -Laplacian

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The multiplicity of positive solution for a new class of four-point boundary value problem of fractional differential equations with p -Laplacian operator is investigated. By the use of the Leggett-Williams fixed-point theorem, the multiplicity results of positive solution are obtained. An example is given to illustrate the main results.

1. Introduction

In recent years, boundary value problems of nonlinear fractional differential equations have been studied extensively (see [1–23] and the references therein). By the use of some fixed-point theorem, the existence results of positive solutions are obtained for singular fractional problem [2, 4, 24, 25], impulsive fractional problem [3, 20], nonlocal problem [1, 12, 14, 16, 22, 23], numerical solution problem [17, 18], initial value problem [19, 20], Dirichlet value problem [6, 9, 15], iterative solution problem [5], and so on. Cui [7] considered the following boundary values problems:

$$\begin{aligned} D_{0+}^{\alpha} x(t) + p(t) f(t, x(t)) + q(x(t)) &= 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u(1) &= 0, \end{aligned} \quad (1)$$

where $2 < \alpha \leq 3$ is a real number and D_{0+}^{α} is the standard Riemann-Liouville differentiation. Under the assumption that $f(t, x)$ is a Lipschitz continuous function, by the use of u_0 -positive operator, they studied the uniqueness results for the fractional differential equation.

On the other hand, because of the wide mathematical and physical background, the existence of positive solutions for nonlinear integer-order boundary values problems with p -Laplacian operator has received wide attention (see [8, 12, 13, 21, 24–29]). For example, Su et al. [28] considered

the following four-point boundary values problems with p -Laplacian operator:

$$\begin{aligned} (\phi_p(u'(t)))' + a(t) f(u(t)) &= 0, \quad 0 < t < 1, \\ \alpha \phi_p(u(0)) - \beta \phi_p(u'(\xi)) &= 0, \\ \gamma \phi_p(u(1)) - \delta \phi_p(u'(\eta)) &= 0, \end{aligned} \quad (2)$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. Liu et al. [13], Dong et al. [8], and Zhang et al. [25] studied p -Laplacian boundary value problems with fractional derivative. By using the fixed-point index theory, they obtained the existence of positive solutions.

In this paper, we investigate the multiplicity of positive solution for a new class of four-point boundary value problem of fractional differential equations with p -Laplacian operator:

$$\begin{aligned} D_{0+}^{\gamma} (\phi_p(D_{0+}^{\alpha} u(t))) &= f(t, u(t)), \quad 0 < t < 1, \\ u(0) = D_{0+}^{\alpha} u(0) &= 0, \\ D_{0+}^{\beta} u(1) &= \lambda u(\xi), \\ D_{0+}^{\alpha} u(1) &= \mu D_{0+}^{\alpha} u(\eta), \end{aligned} \quad (3)$$

where $\alpha, \beta, \gamma \in \mathbf{R}$, $1 < \alpha$, $\gamma \leq 2$, $\beta > 0$, $1 + \beta \leq \alpha$ and $\xi, \eta \in (0, 1)$, $\lambda, \mu \in [0, +\infty)$, $(1 - \beta)\Gamma(\alpha) \geq \lambda\Gamma(\alpha - \beta)\xi^{\alpha-2}$, $1 - \mu^{p-1}\eta^{\gamma-2} \geq 0$ and $\phi_p(s) = |s|^{p-2}s$, $p > 1$, D_{0+}^α is the standard Riemann-Liouville differentiation and $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$. By using the Leggett-Williams fixed-point theorem on a cone, the multiplicity results of positive solution are obtained.

2. Preliminaries

Lemma 1 (see [10]). Assume that $x \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, 1) \cap L(0, 1)$. Then

$$I_{0+}^\alpha D_{0+}^\alpha x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_N t^{\alpha-N}, \quad (4)$$

$$c_i \in \mathbf{R}, \quad i = 1, 2, \dots, N,$$

where N is the smallest integer greater than or equal to α .

Lemma 2. If $h \in C[0, 1]$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_q = (\phi_p)^{-1}$, $1/p + 1/q = 1$, $\alpha, \beta, \gamma \in \mathbf{R}$, $1 < \alpha$, $\gamma \leq 2$, $\beta > 0$, $1 + \beta \leq \alpha$ and $\xi, \eta \in (0, 1)$, $\lambda, \mu \in [0, +\infty)$, $\mathcal{M} := \lambda\Gamma(\alpha - \beta)\xi^{\alpha-1}$, $\mathcal{N} := \mu^{p-1}\eta^{\gamma-1}$, then the problem

$$\begin{aligned} D_{0+}^\gamma (\phi_p(D_{0+}^\alpha u(t))) &= h(t), \quad 0 < t < 1, \\ u(0) &= D_{0+}^\alpha u(0) = 0, \\ D_{0+}^\beta u(1) &= \lambda u(\xi), \\ D_{0+}^\alpha u(1) &= \mu D_{0+}^\alpha u(\eta) \end{aligned} \quad (5)$$

has a unique solution

$$u(t) = \int_0^1 G(t, s) \phi_q \left(\int_0^1 K(s, \tau) h(\tau) d\tau \right) ds, \quad (6)$$

where

$$G(t, s) = \begin{cases} \frac{\Gamma(\alpha) t^{\alpha-1} (1-s)^{\alpha-\beta-1} - \lambda \Gamma(\alpha - \beta) t^{\alpha-1} (\xi - s)^{\alpha-1}}{\Gamma(\alpha) (\Gamma(\alpha) - \mathcal{M})} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \quad s \leq \xi, \\ \frac{\Gamma(\alpha) t^{\alpha-1} (1-s)^{\alpha-\beta-1} - (\Gamma(\alpha) - \mathcal{M}) (t-s)^{\alpha-1}}{\Gamma(\alpha) (\Gamma(\alpha) - \mathcal{M})}, & 0 < \xi \leq s \leq t \leq 1, \\ \frac{\Gamma(\alpha) t^{\alpha-1} (1-s)^{\alpha-\beta-1} - \lambda \Gamma(\alpha - \beta) t^{\alpha-1} (\xi - s)^{\alpha-1}}{\Gamma(\alpha) (\Gamma(\alpha) - \mathcal{M})}, & 0 \leq t \leq s \leq \xi < 1, \\ \frac{t^{\alpha-1} (1-s)^{\alpha-\beta-1}}{\Gamma(\alpha) - \mathcal{M}}, & 0 \leq t \leq s \leq 1, \quad \xi \leq s. \end{cases} \quad (7)$$

$$K(s, \tau) = \begin{cases} \frac{[s(1-\tau)]^{\gamma-1} - \mu^{p-1} [s(\eta - \tau)]^{\gamma-1} - (1 - \mathcal{N}) (s - \tau)^{\gamma-1}}{(1 - \mathcal{N}) \Gamma(\gamma)}, & 0 \leq \tau \leq s \leq 1, \quad \tau \leq \eta, \\ \frac{[s(1-\tau)]^{\gamma-1} - (1 - \mathcal{N}) (s - \tau)^{\gamma-1}}{(1 - \mathcal{N}) \Gamma(\gamma)}, & 0 < \eta \leq \tau \leq s \leq 1, \\ \frac{[s(1-\tau)]^{\gamma-1} - \mu^{p-1} [s(\eta - \tau)]^{\gamma-1}}{(1 - \mathcal{N}) \Gamma(\gamma)}, & 0 \leq s \leq \tau \leq \eta < 1, \\ \frac{[s(1-\tau)]^{\gamma-1}}{(1 - \mathcal{N}) \Gamma(\gamma)}, & 0 \leq s \leq \tau \leq 1, \quad \eta \leq \tau. \end{cases}$$

Proof. Suppose u is a solution of the problem (5). By Lemma 1, there is

$$\phi_p(D_{0+}^\alpha u(t)) = I^\gamma h(t) + d_1 t^{\gamma-1} + d_2 t^{\gamma-2}, \quad (8)$$

for some $d_1, d_2 \in \mathbf{R}$. Taking into account the fact that $D_{0+}^\alpha u(0) = 0$, we have $d_2 = 0$ and

$$\phi_p(D_{0+}^\alpha u(t)) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - \tau)^{\gamma-1} h(\tau) d\tau + d_1 t^{\gamma-1}. \quad (9)$$

Thus,

$$\begin{aligned} \phi_p(D_{0+}^\alpha u(1)) &= \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \tau)^{\gamma-1} h(\tau) d\tau + d_1, \\ \phi_p(D_{0+}^\alpha u(\eta)) &= \frac{1}{\Gamma(\gamma)} \int_0^\eta (\eta - \tau)^{\gamma-1} h(\tau) d\tau \\ &\quad + d_1 \eta^{\gamma-1}. \end{aligned} \quad (10)$$

By $D_{0+}^\alpha u(1) = \mu D_{0+}^\alpha u(\eta)$, (10), it holds that

$$\begin{aligned} d_1 &= - \int_0^1 \frac{(1 - \tau)^{\gamma-1}}{\Gamma(\gamma) (1 - \mathcal{N})} h(\tau) d\tau \\ &\quad + \int_0^\eta \frac{\mu^{p-1} (\eta - \tau)^{\gamma-1}}{\Gamma(\gamma) (1 - \mathcal{N})} h(\tau) d\tau. \end{aligned} \quad (11)$$

So,

$$\begin{aligned} \phi_p(D_{0+}^\alpha u(t)) &= \int_0^t \frac{(t-\tau)^{\gamma-1}}{\Gamma(\gamma)} h(\tau) d\tau \\ &\quad - \int_0^1 \frac{t^{\gamma-1}(1-\tau)^{\gamma-1}}{\Gamma(\gamma)(1-\mathcal{N})} h(\tau) d\tau \\ &\quad + \int_0^\eta \frac{\mu^{p-1} t^{\gamma-1} (\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)(1-\mathcal{N})} h(\tau) d\tau \\ &= - \int_0^1 K(t, \tau) h(\tau) d\tau, \end{aligned} \quad (12)$$

$$D_{0+}^\alpha u(t) + \phi_q \left(\int_0^1 K(t, \tau) h(\tau) d\tau \right) = 0. \quad (13)$$

Applying Lemma 1, we can reduce (13) to an equivalent integral equation

$$\begin{aligned} u(t) &= -I_{0+}^\alpha \phi_q \left(\int_0^1 K(t, \tau) h(\tau) d\tau \right) + c_1 t^{\alpha-1} \\ &\quad + c_2 t^{\alpha-2}, \end{aligned} \quad (14)$$

for some $c_1, c_2 \in \mathbb{R}$. With the condition $u(0) = 0$, there is $c_2 = 0$. Consequently,

$$u(t) = -I_{0+}^\alpha \phi_q \left(\int_0^1 K(t, \tau) h(\tau) d\tau \right) + c_1 t^{\alpha-1}. \quad (15)$$

By (15), one has

$$\begin{aligned} D_{0+}^\beta u(t) &= -D_{0+}^\beta I_{0+}^\alpha \phi_q \left(\int_0^1 K(t, \tau) h(\tau) d\tau \right) \\ &\quad + c_1 D_{0+}^\beta t^{\alpha-1} \\ &= -I_{0+}^{\alpha-\beta} \phi_q \left(\int_0^1 K(t, \tau) h(\tau) d\tau \right) \\ &\quad + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}. \end{aligned} \quad (16)$$

So,

$$\begin{aligned} D^\beta u(1) &= - \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \phi_q \left(\int_0^1 K(s, \tau) h(\tau) d\tau \right) ds \\ &\quad + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}, \\ u(\xi) &= - \int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 K(s, \tau) h(\tau) d\tau \right) ds \\ &\quad + c_1 \xi^{\alpha-1}. \end{aligned} \quad (17)$$

By $D^\beta u(1) = \lambda u(\xi)$, (17), we have

$$\begin{aligned} c_1 &= \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)-\mathcal{M}} \left\{ \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \phi_q \left(\int_0^1 K(s, \tau) h(\tau) d\tau \right) ds \right. \\ &\quad \left. - \lambda \int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 K(s, \tau) h(\tau) d\tau \right) ds \right\}. \end{aligned} \quad (18)$$

So, the unique solution of problem (5) is

$$\begin{aligned} u(t) &= - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 K(s, \tau) h(\tau) d\tau \right) ds \\ &\quad + \frac{\Gamma(\alpha-\beta)t^{\alpha-1}}{\Gamma(\alpha)-\mathcal{M}} \left\{ \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \phi_q \left(\int_0^1 K(s, \tau) h(\tau) d\tau \right) ds \right. \\ &\quad \left. - \lambda \int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\int_0^1 K(s, \tau) h(\tau) d\tau \right) ds \right\} \\ &= \int_0^1 G(t, s) \phi_q \left(\int_0^1 K(s, \tau) h(\tau) d\tau \right) ds. \end{aligned} \quad (19)$$

The proof is completed. $\square$

Lemma 3. Suppose that $(1-\beta)\Gamma(\alpha) \geq \lambda\Gamma(\alpha-\beta)\xi^{\alpha-2}$, $1-\mu^{p-1}\eta^{\gamma-2} \geq 0$. The functions $G(t, s)$ and $K(t, s)$ satisfy the following:

- (1) $G(t, s), K(t, s) \in C([0, 1] \times [0, 1])$; $G(t, s) > 0$, $K(t, s) > 0$ for $t, s \in (0, 1)$,
- (2) $G(t, s) \leq G(s, s)$, $K(t, s) \leq K(s, s)$ for $t, s \in [0, 1]$,
- (3) There exists a positive function $\psi(s) \in C(0, 1)$ such that

$$\min_{\xi \leq t \leq 1} G(t, s) \geq \psi(s) \max_{0 \leq t \leq 1} G(t, s) = \psi(s) G(s, s), \quad s \in (0, 1). \quad (20)$$

Proof.

The Proof of the Statement (1). From definitions, it is clear that $G(t, s), K(t, s) \in C([0, 1] \times [0, 1])$. By $(1-\beta)\Gamma(\alpha) \geq \lambda\Gamma(\alpha-\beta)\xi^{\alpha-2}$, $1-\mu^{p-1}\eta^{\gamma-2} \geq 0$, we have

$$\begin{aligned} \Gamma(\alpha) &\geq (1-\beta)\Gamma(\alpha) \geq \lambda\Gamma(\alpha-\beta)\xi^{\alpha-2} \\ &> \lambda\Gamma(\alpha-\beta)\xi^{\alpha-1} = \mathcal{M}, \end{aligned} \quad (21)$$

$$1-\mathcal{N} = 1-\mu^{p-1}\eta^{\gamma-1} > 1-\mu^{p-1}\eta^{\gamma-2} \geq 0.$$

If $0 < s \leq t < 1$, $s \leq \xi$, let

$$g(t, s) = \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}. \quad (22)$$

It is obvious that $g(t, s) > 0$ for $0 < s \leq t < 1$. Hence, we have

$$\begin{aligned}
G(t, s) &= \frac{\Gamma(\alpha) t^{\alpha-1} (1-s)^{\alpha-\beta-1} - (\Gamma(\alpha) - \mathcal{M})(t-s)^{\alpha-1} - \lambda \Gamma(\alpha - \beta) t^{\alpha-1} (\xi - s)^{\alpha-1}}{\Gamma(\alpha) (\Gamma(\alpha) - \mathcal{M})} \\
&= \frac{1}{\Gamma(\alpha)} \left(1 + \frac{\mathcal{M}}{\Gamma(\alpha) - \mathcal{M}} \right) t^{\alpha-1} (1-s)^{\alpha-\beta-1} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\lambda \Gamma(\alpha - \beta) t^{\alpha-1} (\xi - s)^{\alpha-1}}{\Gamma(\alpha) (\Gamma(\alpha) - \mathcal{M})} \\
&= \frac{t^{\alpha-1} (1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\lambda \Gamma(\alpha - \beta) t^{\alpha-1} [\xi^{\alpha-1} (1-s)^{\alpha-\beta-1} - (\xi - s)^{\alpha-1}]}{\Gamma(\alpha) (\Gamma(\alpha) - \mathcal{M})} \\
&= g(t, s) + \frac{\lambda \Gamma(\alpha - \beta) t^{\alpha-1}}{\Gamma(\alpha) - \mathcal{M}} g(\xi, s) > 0.
\end{aligned} \tag{23}$$

By using the analogous argument, it holds that $G(t, s) > 0$ for other situations. Hence, $G(t, s) > 0$ for $t, s \in (0, 1)$.

If $0 < s \leq t < 1$, $s \leq \eta$, let

$$k(t, s) = \frac{[t(1-s)]^{\gamma-1} - (t-s)^{\gamma-1}}{\Gamma(\gamma)}. \tag{24}$$

It is obvious that $k(t, s) > 0$ for $0 < s \leq t < 1$. Hence, we have

$$\begin{aligned}
K(t, s) &= \frac{[t(1-s)]^{\gamma-1} - \mu^{p-1} [t(\eta-s)]^{\gamma-1} - (1-\mathcal{N})(t-s)^{\gamma-1}}{(1-\mathcal{N}) \Gamma(\gamma)} \\
&= \left(1 + \frac{\mathcal{N}}{1-\mathcal{N}} \right) \frac{[t(1-s)]^{\gamma-1}}{\Gamma(\gamma)} - \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)} \\
&\quad - \frac{\mu^{p-1} [t(\eta-s)]^{\gamma-1}}{(1-\mathcal{N}) \Gamma(\gamma)} \\
&= \frac{[t(1-s)]^{\gamma-1} - (t-s)^{\gamma-1}}{\Gamma(\gamma)} \\
&\quad + \frac{\mu^{p-1} t^{\gamma-1} [\eta^{\gamma-1} (1-s)^{\gamma-1} - (\eta-s)^{\gamma-1}]}{(1-\mathcal{N}) \Gamma(\gamma)} \\
&= k(t, s) + \frac{\mu^{p-1} t^{\gamma-1}}{1-\mathcal{N}} k(\eta, s) > 0.
\end{aligned} \tag{25}$$

Similarly, it holds that

$$K(t, s) > 0 \text{ for } 0 < \eta \leq s \leq t < 1 \text{ or } 0 < t \leq s \leq \eta < 1 \text{ or } 0 < t \leq s < 1, \eta \leq s.$$

Hence, $K(t, s) > 0$ for $t, s \in (0, 1)$.

The Proof of the Statement (2). Let

$$\begin{aligned}
g_1(t, s) &= \frac{\Gamma(\alpha) t^{\alpha-1} (1-s)^{\alpha-\beta-1} - \lambda \Gamma(\alpha - \beta) t^{\alpha-1} (\xi - s)^{\alpha-1}}{\Gamma(\alpha) (\Gamma(\alpha) - \mathcal{M})} \\
&\quad - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 \leq s \leq t \leq 1, \quad s \leq \xi, \\
g_2(t, s) &= \frac{\Gamma(\alpha) t^{\alpha-1} (1-s)^{\alpha-\beta-1} - (\Gamma(\alpha) - \mathcal{M})(t-s)^{\alpha-1}}{\Gamma(\alpha) (\Gamma(\alpha) - \mathcal{M})}, \\
&\quad 0 < \xi \leq s \leq t \leq 1, \\
g_3(t, s) &= \frac{\Gamma(\alpha) t^{\alpha-1} (1-s)^{\alpha-\beta-1} - \lambda \Gamma(\alpha - \beta) t^{\alpha-1} (\xi - s)^{\alpha-1}}{\Gamma(\alpha) (\Gamma(\alpha) - \mathcal{M})}, \\
&\quad 0 \leq t \leq s \leq \xi < 1, \\
g_4(t, s) &= \frac{t^{\alpha-1} (1-s)^{\alpha-\beta-1}}{\Gamma(\alpha) - \mathcal{M}}, \quad 0 \leq t \leq s \leq 1, \quad \xi \leq s.
\end{aligned} \tag{26}$$

It is easy to check that $g_3(t, s)$ and $g_4(t, s)$ are increasing with respect to t on $[0, s]$. We will show that $g_1(t, s)$ and $g_2(t, s)$ are decreasing with respect to t on $[s, 1]$. For $0 \leq s \leq t \leq 1$, $s \leq \xi$, let

$$h_1(t, s) = \Gamma(\alpha) (\Gamma(\alpha) - \mathcal{M}) g_1(t, s); \tag{27}$$

then we have

$$\begin{aligned}
h_1(t, s) &= \Gamma(\alpha) t^{\alpha-1} (1-s)^{\alpha-\beta-1} - (\Gamma(\alpha) - \mathcal{M})(t-s)^{\alpha-1} \\
&\quad - \lambda \Gamma(\alpha - \beta) t^{\alpha-1} (\xi - s)^{\alpha-1}, \\
\frac{\partial h_1(t, s)}{\partial t} &= (\alpha - 1) t^{\alpha-2} \left[\Gamma(\alpha) (1-s)^{\alpha-\beta-1} \right. \\
&\quad \left. - (\Gamma(\alpha) - \mathcal{M}) \left(1 - \frac{s}{t} \right)^{\alpha-2} \right]
\end{aligned}$$

$$\begin{aligned}
& -\lambda\Gamma(\alpha-\beta)(\xi-s)^{\alpha-1} \Big] \leq (\alpha-1) \\
& \cdot t^{\alpha-2} \left[\Gamma(\alpha)(1-s)^{\alpha-\beta-1} - (\Gamma(\alpha) - \mathcal{M})(1-s)^{\alpha-2} \right. \\
& - \lambda\Gamma(\alpha-\beta)(\xi-s)^{\alpha-1} \Big] = (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2} \\
& \cdot \left[\Gamma(\alpha)(1-s)^{1-\beta} - (\Gamma(\alpha) - \mathcal{M}) \right. \\
& - \lambda\Gamma(\alpha-\beta)(\xi-s)^{\alpha-1}(1-s)^{2-\alpha} \Big] \leq (\alpha-1) \\
& \cdot t^{\alpha-2}(1-s)^{\alpha-2} \left[\Gamma(\alpha)(1-(1-\beta)s) \right. \\
& - (\Gamma(\alpha) - \mathcal{M}) - \lambda\Gamma(\alpha-\beta)(\xi-s)^{\alpha-1}(1-s)^{2-\alpha} \Big] \\
& = (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2} \left[\mathcal{M} \right. \\
& \left. - \Gamma(\alpha)(1-\beta)s\lambda\Gamma(\alpha-\beta)(\xi-s)^{\alpha-1}(1-s)^{2-\alpha} \right]. \quad (28)
\end{aligned}$$

For $0 \leq s \leq \xi$, let

$$\begin{aligned}
\omega(s) &= \mathcal{M} - \Gamma(\alpha)(1-\beta)s \\
& - \lambda\Gamma(\alpha-\beta)(\xi-s)^{\alpha-1}(1-s)^{2-\alpha} \\
& = \lambda\Gamma(\alpha-\beta)\xi^{\alpha-1} - \Gamma(\alpha)(1-\beta)s \\
& - \lambda\Gamma(\alpha-\beta)(\xi-s)^{\alpha-1}(1-s)^{2-\alpha}. \quad (29)
\end{aligned}$$

We have

$$\begin{aligned}
\omega(0) &= \lambda\Gamma(\alpha-\beta)\xi^{\alpha-1} - \lambda\Gamma(\alpha-\beta)\xi^{\alpha-1} = 0, \\
\omega(\xi) &= \lambda\Gamma(\alpha-\beta)\xi^{\alpha-1} - (1-\beta)\xi\Gamma(\alpha) \\
&= \xi \left[\lambda\Gamma(\alpha-\beta)\xi^{\alpha-2} - (1-\beta)\Gamma(\alpha) \right] \leq 0, \quad (30) \\
\omega'(s) &= (\beta-1)\Gamma(\alpha) + \lambda\Gamma(\alpha-\beta)(\xi-s)^{\alpha-1} \\
& \cdot (1-s)^{2-\alpha} \frac{(\alpha-1)(1-s) + (\xi-s)(2-\alpha)}{(1-s)(\xi-s)}.
\end{aligned}$$

If there exists $s^* \in (0, \xi)$ such that $\omega'(s^*) = 0$, then

$$\begin{aligned}
& \lambda\Gamma(\alpha-\beta)(\xi-s^*)^{\alpha-1}(1-s^*)^{2-\alpha} \\
& = (1-\beta)\Gamma(\alpha) \frac{(1-s^*)(\xi-s^*)}{(\alpha-1)(1-s^*) + (\xi-s^*)(2-\alpha)}. \quad (31)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\omega(s^*) &= \lambda\Gamma(\alpha-\beta)\xi^{\alpha-1} - (1-\beta)\Gamma(\alpha) \\
& \cdot \left(s^* + \frac{(1-s^*)(\xi-s^*)}{(\alpha-1)(1-s^*) + (\xi-s^*)(2-\alpha)} \right) = \lambda\Gamma(\alpha-\beta) \\
& \cdot \xi^{\alpha-1} - (1-\beta)\Gamma(\alpha) \\
& \times \frac{s^*[(\alpha-1)(1-s^*) + (\xi-s^*)(2-\alpha)] + (1-s^*)(\xi-s^*)}{(\alpha-1)(1-s^*) + (\xi-s^*)(2-\alpha)}
\end{aligned}$$

$$\begin{aligned}
& = \lambda\Gamma(\alpha-\beta)\xi^{\alpha-1} - (1-\beta)\Gamma(\alpha) \\
& \cdot \left(\frac{\xi(\alpha-1)(1-s^*) - (\xi-s^*)(\alpha-1)(1-s^*)}{(\alpha-1)(1-s^*) + (\xi-s^*)(2-\alpha)} \right. \\
& \left. + \frac{\xi(\xi-s^*)(2-\alpha) - (\xi-s^*)^2(2-\alpha) + (1-s^*)(\xi-s^*)}{(\alpha-1)(1-s^*) + (\xi-s^*)(2-\alpha)} \right) \\
& = \lambda\Gamma(\alpha-\beta)\xi^{\alpha-1} - (1-\beta)\Gamma(\alpha) \\
& \cdot \left(\xi + \frac{(\xi-s^*)(2-\alpha)(1-s^*) - (\xi-s^*)^2(2-\alpha)}{(\alpha-1)(1-s^*) + (\xi-s^*)(2-\alpha)} \right) \\
& = \lambda\Gamma(\alpha-\beta)\xi^{\alpha-1} - (1-\beta)\Gamma(\alpha) \\
& \cdot \left(\xi + \frac{(\xi-s^*)(2-\alpha)(1-\xi)}{(\alpha-1)(1-s^*) + (\xi-s^*)(2-\alpha)} \right) = \lambda\Gamma(\alpha-\beta) \\
& \cdot \xi^{\alpha-1} - (1-\beta)\Gamma(\alpha)\xi - (1-\beta)\Gamma(\alpha) \\
& \cdot \frac{(\xi-s^*)(2-\alpha)(1-\xi)}{(\alpha-1)(1-s^*) + (\xi-s^*)(2-\alpha)} \\
& \leq \left[\lambda\Gamma(\alpha-\beta)\xi^{\alpha-2} - (1-\beta)\Gamma(\alpha) \right] \xi \leq 0. \quad (32)
\end{aligned}$$

In fact, taking into account the fact that $\lambda\Gamma(\alpha-\beta)\xi^{\alpha-2} - (1-\beta)\Gamma(\alpha) \leq 0$ and $2-\alpha \geq 0$, one has

$$-(1-\beta)\Gamma(\alpha) \frac{(\xi-s^*)(2-\alpha)(1-\xi)}{(\alpha-1)(1-s^*) + (\xi-s^*)(2-\alpha)} \leq 0. \quad (33)$$

So, we have $\max_{0 \leq s \leq \xi} \omega(s) \leq 0$, which implies that $\partial h_1(t, s)/\partial t \leq 0$. Hence $g_1(t, s)$ is decreasing with respect to t on $[s, 1]$.

For $0 < \xi \leq s \leq t \leq 1$, let $h_2(t, s) = \Gamma(\alpha)(\Gamma(\alpha) - \mathcal{M})g_2(t, s)$; then we have

$$\begin{aligned}
h_2(t, s) &= \Gamma(\alpha)t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (\Gamma(\alpha) - \mathcal{M})(t \\
& - s)^{\alpha-1}, \quad (34)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial h_2(t, s)}{\partial t} &= (\alpha-1)t^{\alpha-2} \left[\Gamma(\alpha)(1-s)^{\alpha-\beta-1} \right. \\
& - (\Gamma(\alpha) - \mathcal{M}) \left(1 - \frac{s}{t} \right)^{\alpha-2} \Big] \leq (\alpha-1) \\
& \cdot t^{\alpha-2} \left[\Gamma(\alpha)(1-s)^{\alpha-\beta-1} - (\Gamma(\alpha) - \mathcal{M})(1-s)^{\alpha-2} \right] \\
& = (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2} \left[\Gamma(\alpha)(1-s)^{1-\beta} \right. \\
& - (\Gamma(\alpha) - \mathcal{M}) \Big] \leq (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2} \\
& \cdot \left[\Gamma(\alpha)(1-(1-\beta)s) - (\Gamma(\alpha) - \mathcal{M}) \right] \leq (\alpha-1) \\
& \cdot t^{\alpha-2}(1-s)^{\alpha-2} \left[\lambda\Gamma(\alpha-\beta)\xi^{\alpha-1} - (1-\beta)\xi\Gamma(\alpha) \right] \\
& = (\alpha-1)t^{\alpha-2}(1-s)^{\alpha-2}\xi \left[\lambda\Gamma(\alpha-\beta)\xi^{\alpha-2} \right. \\
& \left. - (1-\beta)\Gamma(\alpha) \right] \leq 0, \quad (35)
\end{aligned}$$

which implies that $g_2(t, s)$ is decreasing with respect to t on $[s, 1]$.

We can conclude that $G(t, s)$ is increasing with respect to t for $t \leq s$ and $G(t, s)$ is decreasing with respect to t for $t \geq s$. Hence, $G(t, s) \leq G(s, s)$ for $t, s \in [0, 1]$.

By using the analogous method, we can conclude that $K(t, s)$ is increasing with respect to t for $t \leq s$ and $K(t, s)$ is decreasing with respect to t for $t \geq s$. Hence, $K(t, s) \leq K(s, s)$ for $t, s \in [0, 1]$.

The Proof of the Statement (3). Taking into account the definition of $G(t, s)$, there is

$$\begin{aligned} \min_{\xi \leq t \leq 1} G(t, s) \\ = \begin{cases} \min_{\xi \leq t \leq 1} g_1(t, s), & 0 \leq s \leq \xi, \\ \min \left\{ \min_{\xi \leq t \leq 1} g_2(t, s), \min_{\xi \leq t \leq 1} g_4(t, s) \right\}, & \xi \leq s \leq 1, \end{cases} \end{aligned}$$

$$= \begin{cases} g_1(1, s), & 0 \leq s \leq \xi, \\ \varphi(s), & \xi \leq s \leq 1, \end{cases} \quad (36)$$

where $\varphi(s) = \min\{g_2(1, s), g_4(\xi, s)\}$. It is easy to see that $\varphi(s)$ is continuous on $[\xi, 1]$ and $\varphi(s) > 0$ for all $s \in [\xi, 1]$. Let

$$\psi(s) = \begin{cases} \frac{g_1(1, s)}{G(s, s)}, & 0 < s \leq \xi, \\ \frac{\varphi(s)}{G(s, s)}, & \xi \leq s < 1, \end{cases} \quad (37)$$

where

$$G(s, s) = \begin{cases} \frac{\Gamma(\alpha) s^{\alpha-1} (1-s)^{\alpha-\beta-1} - \lambda \Gamma(\alpha-\beta) s^{\alpha-1} (\xi-s)^{\alpha-1}}{\Gamma(\alpha) (\Gamma(\alpha) - \mathcal{M})}, & 0 \leq s \leq \xi, \\ \frac{s^{\alpha-1} (1-s)^{\alpha-\beta-1}}{\Gamma(\alpha) - \mathcal{M}}, & \xi \leq s \leq 1. \end{cases} \quad (38)$$

Then

$$\min_{\xi \leq t \leq 1} G(t, s) \geq \psi(s) \max_{0 \leq t \leq 1} G(t, s) = \psi(s) G(s, s), \quad (39)$$

$s \in (0, 1).$

The proof is completed. $\square$

Lemma 4 (see [11]). Let P be a cone in a real Banach space E , $P_c = \{u \in P \mid \|u\| \leq c\}$, θ be a nonnegative continuous concave functional on a cone P such that $\theta(u) \leq \|u\|$, for all $u \in \overline{P}_c$, and $P(\theta, b, d) = \{u \in P \mid b \leq \theta(u), \|u\| \leq d\}$. Suppose $T : \overline{P}_c \rightarrow \overline{P}_c$ is completely continuous and there exist constants $0 < a < b < d \leq c$ such that

- (A₁) $\{u \in P(\theta, b, d) \mid \theta(u) > b\} \neq \emptyset$ and $\theta(Tu) > b$ for $u \in P(\theta, b, d)$;
- (A₂) $\|Tu\| < a$ for $\|u\| \leq a$;
- (A₃) $\theta(Tu) > b$ for $u \in P(\theta, b, c)$ with $\|Tu\| > d$.

Then T has at least three fixed points u_1, u_2, u_3 satisfying

$$\|u_1\| < a, \quad b < \theta(u_2) \text{ and } \|u_3\| > a \text{ with } \theta(u_3) < b.$$

3. Main Result

Let $E = C[0, 1]$ be a Banach space with $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$. Define the cone $P \subset E$ by $P = \{u \in E \mid u(t) \geq 0, 0 \leq t \leq 1\}$. Let the nonnegative continuous concave functional θ on the P be defined by

$$\theta(u) = \min_{\xi \leq t \leq 1} |u(t)|. \quad (40)$$

Lemma 5. Let $T : P \rightarrow E$ be an operator defined by

$$\begin{aligned} (Tu)(t) \\ = \int_0^1 G(t, s) \phi_q \left(\int_0^1 K(s, \tau) f(\tau, u(\tau)) d\tau \right) ds. \end{aligned} \quad (41)$$

Then $T : P \rightarrow P$ is completely continuous.

Proof. $T : P \rightarrow P$ is continuous in view of nonnegativity and continuity of $G(t, s)$, $K(t, s)$ and $f(t, u)$. Furthermore it is easy to see that by the Arzela-Ascoli theorem and Lebesgue dominated convergence theorem $T : P \rightarrow P$ is completely continuous. $\square$

For convenience, we introduce the following notations:

$$\begin{aligned} N &= \left[\int_0^1 G(s, s) \phi_q \left(\int_0^1 K(s, \tau) d\tau \right) ds \right]^{-1}, \\ L &= \left[\int_\xi^1 \psi(s) G(s, s) \phi_q \left(\int_0^1 K(s, \tau) d\tau \right) ds \right]^{-1}. \end{aligned} \quad (42)$$

Theorem 6. Suppose $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$ and there exist constants $0 < a < b < c$ such that

- (B₁) $f(t, u) < \phi_p(Na)$, for $(t, u) \in [0, 1] \times [0, a]$;
- (B₂) $f(t, u) \geq \phi_p(Lb)$, for $(t, u) \in [\xi, 1] \times [b, c]$;
- (B₃) $f(t, u) \leq \phi_p(Nc)$, for $(t, u) \in [0, 1] \times [0, c]$.

Then problem (3) possesses at least three positive solutions u_1, u_2 , and u_3 with

$$\begin{aligned}
\max_{0 \leq t \leq 1} |u_1(t)| &< a, \\
b &< \min_{\xi \leq t \leq 1} |u_2(t)| < \max_{0 \leq t \leq 1} |u_2(t)| \leq c, \\
a &< \max_{0 \leq t \leq 1} |u_3(t)| \leq c, \\
\min_{\xi \leq t \leq 1} |u_3(t)| &< b.
\end{aligned} \tag{43}$$

Proof. In order to apply Lemma 4, we divide the proof into four steps.

Step 1. If $u \in \bar{P}_c$, then $\|u\| \leq c$. Assumption (B_3) implies $f(t, u(t)) \leq \phi_p(Nc)$ for $0 \leq t \leq 1$. Consequently,

$$\begin{aligned}
\|Tu\| &= \max_{0 \leq t \leq 1} \left| \int_0^1 G(t, s) \phi_q \left(\int_0^1 K(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \right| \\
&\leq \int_0^1 G(s, s) \phi_q \left(\int_0^1 K(s, \tau) \phi_p(Nc) d\tau \right) ds \\
&\leq Nc \int_0^1 G(s, s) \phi_q \left(\int_0^1 K(s, \tau) d\tau \right) ds = c.
\end{aligned} \tag{44}$$

Hence $T : \bar{P}_c \rightarrow \bar{P}_c$.

Step 2. We claim that the condition (A_1) of Lemma 4 is satisfied. Let $u(t) = (b+c)/2$, $0 \leq t \leq 1$. we can easily see that $u(t) = (b+c)/2 \in P(\theta, b, c)$, $\theta(u) = \theta((b+c)/2) > b$; consequently, $\{u \in P(\theta, b, c) \mid \theta(u) > b\} \neq \emptyset$. Hence, if $u \in P(\theta, b, c)$, then $b \leq u(t) \leq c$ for $\xi \leq t \leq 1$. From assumption (B_2) , we have

$$f(t, u(t)) \geq \phi_p(Lb), \quad \xi \leq t \leq 1. \tag{45}$$

So

$$\begin{aligned}
\theta(Tu) &= \min_{\xi \leq t \leq 1} |(Tu)(t)| \\
&= \min_{\xi \leq t \leq 1} \left| \int_0^1 G(t, s) \phi_q \left(\int_0^1 K(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \right| \\
&\geq \min_{\xi \leq t \leq 1} \left| \int_0^1 G(t, s) \phi_q \left(\int_0^1 K(s, \tau) \phi_p(Lb) d\tau \right) ds \right| \\
&> \int_\xi^1 \psi(s) G(s, s) \phi_q \left(\int_0^1 K(s, \tau) \phi_p(Lb) d\tau \right) ds \\
&= Lb \int_\xi^1 \psi(s) G(s, s) \phi_q \left(\int_0^1 K(s, \tau) d\tau \right) ds = b.
\end{aligned} \tag{46}$$

Then, we have

$$\theta(Tu) > b, \quad \forall u \in P(\theta, b, c). \tag{47}$$

This implies that condition (A_1) of Lemma 4 is satisfied.

Step 3. We now prove the condition (A_2) of Lemma 4 is satisfied. If $u \in \bar{P}_a$, then $\|u\| \leq a$. Assumption (B_1) implies $f(t, u(t)) < \phi_p(Na)$ for $0 \leq t \leq 1$. Thus

$$\begin{aligned}
\|Tu\| &= \max_{0 \leq t \leq 1} \left| \int_0^1 G(t, s) \phi_q \left(\int_0^1 K(s, \tau) f(\tau, u(\tau)) d\tau \right) ds \right| \\
&< \int_0^1 G(s, s) \phi_q \left(\int_0^1 K(s, \tau) \phi_p(Na) d\tau \right) ds \\
&= Na \int_0^1 G(s, s) \phi_q \left(\int_0^1 K(s, \tau) d\tau \right) ds = a.
\end{aligned} \tag{48}$$

Hence, the condition (A_2) of Lemma 4 is also satisfied.

Step 4. Finally, we prove that the condition (A_3) of Lemma 4 is satisfied. If $u \in P(\theta, b, c)$, by *Step 2*, we have $\theta(Tu) > b$. Hence, the condition (A_3) of Lemma 4 is also satisfied.

By Lemma 4, problem (3) has at least three positive solutions u_1 , u_2 , and u_3 with

$$\begin{aligned}
\max_{0 \leq t \leq 1} |u_1(t)| &< a, \\
b &< \min_{\xi \leq t \leq 1} |u_2(t)| < \max_{0 \leq t \leq 1} |u_2(t)| \leq c, \\
a &< \max_{0 \leq t \leq 1} |u_3(t)| \leq c,
\end{aligned} \tag{49}$$

$$\min_{\xi \leq t \leq 1} |u_3(t)| < b.$$

The proof is complete. $\square$

Example 7. Consider the following boundary value problem:

$$D^{3/2}(\phi_{3/2}(D^{3/2}u(t))) = f(t, u(t)), \quad 0 < t < 1,$$

$$u(0) = D^{3/2}u(0) = 0,$$

$$D^{1/2}u(1) = \frac{1}{4}u\left(\frac{1}{2}\right), \tag{50}$$

$$D^{3/2}u(1) = \frac{1}{8}D^{3/2}u\left(\frac{1}{4}\right),$$

where

$$f(t, u) = \begin{cases} \frac{1}{8}t^2 + 2u^2, & u < 1, \\ \frac{13}{7} + \frac{1}{8}t^2 + \frac{1}{7}u, & u \geq 1. \end{cases} \tag{51}$$

Choose $p = \alpha = \gamma = 3/2$, $\beta = \xi = 1/2$, $\eta = \lambda = 1/4$, $\mu = 1/8$. It is easy to check that $(1 - \beta)\Gamma(\alpha) - \lambda\Gamma(\alpha - \beta)\xi^{\alpha-2} > 0$ and $1 - \mu^{p-1}\eta^{\gamma-2} > 0$; by simple computation, we have

$$\begin{aligned}
N &\approx 1.3788, \\
L &\approx 3.9213.
\end{aligned} \tag{52}$$

Let $a = 1/4$, $b = 1$, $c = 7$; one can check that the function $f(t, u)$ satisfies

- (S₁) $f(t, u) = (1/8)t^2 + 2u^2 \leq 0.25 < \phi_{3/2}(Na) \approx 0.5871$, for $(t, u) \in [0, 1] \times [0, 1/4]$;
- (S₂) $f(t, u) = 13/7 + (1/8)t^2 + (1/7)u > 2.0312 > \phi_{3/2}(Lb) \approx 1.9802$, for $(t, u) \in [1/2, 1] \times [1, 7]$;
- (S₃) $f(t, u) = 13/7 + (1/8)t^2 + (1/7)u < 2.9822 < \phi_{3/2}(Nc) \approx 3.1067$, for $(t, u) \in [0, 1] \times [0, 7]$.

That is to say that all the conditions of Theorem 6 hold. Thus Theorem 6 implies the problem (50) has at least three positive solutions u_1 , u_2 , and u_3 such that

$$\begin{aligned} \max_{0 \leq t \leq 1} |u_1(t)| &< \frac{1}{4}, \\ 1 &< \min_{1/2 \leq t \leq 1} |u_2(t)| < \max_{0 \leq t \leq 1} |u_2(t)| \leq 7, \\ \frac{1}{4} &< \max_{0 \leq t \leq 1} |u_3(t)| \leq 7, \\ \min_{1/2 \leq t \leq 1} |u_3(t)| &< 1. \end{aligned} \quad (53)$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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