Research Article

Boundedness of Fractional Oscillatory Integral Operators and Their Commutators in Vanishing Generalized Weighted Morrey Spaces

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In this article, we give the boundedness conditions in terms of Zygmund-type integral inequalities for oscillatory integral operators and fractional oscillatory integral operators on the vanishing generalized weighted Morrey spaces. Moreover, we investigate corresponding commutators.

1. Introduction

The classical Morrey spaces $M^{p,\lambda}$ that play important role in the theory of partial differential equations were introduced by Morrey [1] in 1938. Since then Morrey spaces have been studied by various authors. We refer readers to the survey [2] and to the elegant book [3] for further information about these spaces and references on recent developments in this field.

So far various generalizations of Morrey spaces have been defined. Mizuhara [4] introduced the generalized Morrey space $M^{p,\varphi}$ and Komori and Shirai [5] defined the weighted Morrey spaces $L^{p,\varphi}(w)$. Guliyev [6] gave the notion of generalized weighted Morrey space $M^{p,\varphi}(w)$ which can be accepted as an extension of $M^{p,\varphi}$ and $L^{p,\varphi}(w)$. Eroglu [7] proved the boundedness of oscillatory integral operators, fractional oscillatory integral operators, and the corresponding commutators on $M^{p,\varphi}$ and Shi et al. [8] proved the boundedness of these operators and commutators on $L^{p,\varphi}(w)$. In [9], Lu et al. obtained the boundedness of sublinear operators with rough kernels on $M^{p,\varphi}$ and in [10], Shi and Fu showed the boundedness of these operators on $L^{p,\varphi}(w)$. The boundedness of some sublinear operators and their commutators on $L^{p,\varphi}(w)$ was obtained by Shi et al. [11] and the boundedness of sublinear operators on $M^{p,\varphi}(w)$ was proved by Mustafayev [12].

Vanishing Morrey spaces $VM^{p,\lambda}(\mathbb{R}^n)$ are subspaces of functions in Morrey spaces which were introduced by Vitanza [13] satisfying the condition

$$\limsup_{r \to 0} r^{-\lambda/p} \left\| \int_{B(x,t)} f \right\|_{L^p(B(x,r))} = 0.$$ (1)

The properties and applications of vanishing Morrey Spaces were given in [14]. On vanishing Morrey spaces, the boundedness of commutators of the multidimensional Hardy type operators was proved in [15]. The vanishing generalized Morrey spaces $VM^{p,\varphi}(\mathbb{R}^n)$ were introduced and studied by Samko in [16].

In this study, as distinct from [8], we focus on vanishing generalized weighted Morrey spaces and give Zygmund-type conditions to prove the boundedness of oscillatory and the fractional oscillatory integral operators and their commutators in these spaces. In Section 2 we recall some definitions and necessary preliminaries and in Section 3 we give our main results; namely, we prove our theorems on vanishing generalized weighted Morrey spaces.
Throughout this paper, $C, c, c_1$ and so on are used as positive constant that can change from one line to another. $A \leq B$ means that $A \leq cB$ with some positive constant $c$. If $A \leq B$ and $B \leq A$, then we say $A = B$ which means $A$ and $B$ are equivalent.

## 2. Preliminaries

Let $w$ be a weight function on $\mathbb{R}^n$, such that $w(x) > 0$ for almost every $x \in \mathbb{R}^n$. $E$ is a measurable set with Lebesgue measure notated by $|E|$ and we define $w(E) = \int_E w(x)\,dx$.

For $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ we denote the weighted Lebesgue space by $L^p(w)(\Omega)$ with the norm

$$\|f\|_{L^p(w)(\Omega)} = \left(\int_\Omega |f(x)|^p w(x)\,dx\right)^{1/p} < \infty.$$  \hfill (2)

Let $1 \leq p < \infty$, $\varphi(x, r)$ be a positive continuous function on $\mathbb{R}^n \times (0, \infty)$ and let $w$ be a weight function on $\mathbb{R}^n$. We show the generalized weighted Morrey space by $M^{pq}(\mathbb{R}^n; w) = M^{pq}(w)$, which is space of all functions $f \in L^{pq}(\mathbb{R}^n)$ with finite quasi norm

$$\|f\|_{M^{pq}(w)} = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{\varphi^{1/p}(x, r)} \|f\|_{L^{pq}(\Omega)} ,$$  \hfill (3)

where $\Omega$ is an open set in $\mathbb{R}^n$.

The fractional integral operator (Riesz potential) $I_\alpha$ and fractional Maximal operator $M_\alpha$, which play important roles in real and harmonic analysis, are defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n - \alpha}}\,dy, \quad 0 < \alpha < n,$$

$$M_\alpha f(x) = \sup_{r > 0} \frac{1}{r^{\alpha - n}} \int_{B(x, r)} f(y)\,dy, \quad 0 \leq \alpha < n,$$  \hfill (4)

where $f \in L^{\infty}(\mathbb{R}^n)$. If $\alpha = 0$, then $M \equiv M_0$ is the Hardy-Littlewood maximal operator.

The class of $A_p$ weights was introduced by Muckenhoupt in [17] to show that Hardy-Littlewood maximal function $M$ is bounded on weighted Lebesgue spaces $L^p(w)(\mathbb{R}^n)$ if and only if $w \in A_p$, $1 < p < \infty$.

Now we define Muckenhoupt class $A_p$. Let $1 \leq p < \infty$. A weight $w$ is said to be an $A_p$ weight, if there exists a positive constant $c_p$, such that, for every ball $B \subset \mathbb{R}^n$,

$$\left(\int_B w(x)\,dx\right) \left(\int_B w(x)^{-p'}\,dx\right)^{p-1} \leq c_p |B|^p ,$$  \hfill (5)

when $1 < p < \infty$, and for $p = 1$

$$\int_B w(y)\,dy \leq c_1 w(x)|B|$$  \hfill (6)

for almost everywhere $x \in B$. The smallest $c_p$ is shown by $[w]_{A_p}$. We define $A\infty = \bigcup_{p \geq 1} A_p$.

A weight $w$ belongs to $A_{p,q}$ for $1 < p < q < \infty$ if there exists $C > 1$ such that

$$\left(\int_B w(x)^q\,dx\right)^{1/q} \left(\int_B w(x)^{-p'}\,dx\right)^{1/p'} \leq C |B|^{1/q - 1/p},$$  \hfill (7)

where $1/p + 1/p' = 1$. $A_{p,q}$ class was introduced by Muckenhoupt and Wheeden [18] to study weighted norm inequalities for fractional integral operators.

### 2.1. Vanishing Generalized Weighted Morrey Spaces.

Let $\Omega$ be an open set in $\mathbb{R}^n$ and let $\Pi$ be an arbitrary subset of $\Omega$. Let also $\varphi(x, r)$ be a measurable nonnegative function on $\Pi \times [0, l]$ ($l = \text{diam } \Omega$) and positive for all $(x, t) \in \Pi \times (0, l)$. Let $w$ be a weight function on $\Omega$; then we denote by $VM_{\Pi}^{pq}(\Omega; w) = VM_{\Pi}^{pq}(w)$ the vanishing weighted Morrey spaces which are defined as the spaces of functions $f \in L^{\infty}(\mathbb{R}^n)$ with finite quasi norm:

$$\|f\|_{VM_{\Pi}^{pq}(w)} = \sup_{x \in \Pi, r > 0} \frac{1}{\varphi^{1/p}(x, r)} \|f\|_{L^{pq}(\tilde{B}(x, r))}$$  \hfill (8)

such that

$$\lim_{r \to 0} \sup_{x \in \Pi} \frac{1}{r^{1/p}} \|f\|_{L^{pq}(\tilde{B}(x, r))} = 0,$$  \hfill (9)

where $\tilde{B}(x, r) = B(x, r) \cap \Omega$ and $1 \leq p \leq \infty$.

Naturally, it is suitable to impose the function $\varphi(x, r)$ on the following conditions:

$$\lim_{r \to 0} \sup_{x \in \Pi} \frac{\|\varphi\|_{L^{\infty}(\tilde{B}(x, r))}}{\varphi(x, r)} = 0, \quad \inf_{r > 0} \varphi(x, r) > 0,$$  \hfill (10)

which makes $VM_{\Pi}^{pq}(w)$ nontrivial, since bounded functions which have compact support belong to this space.

Henceforth we denote by $\varphi \in \mathcal{B}(w)$ if $\varphi(x, r)$ is a nonnegative measurable function on $\Pi \times [0, l]$ and positive for all $(x, t) \in \Pi \times (0, l)$ and satisfies conditions (10).

### 2.2. Oscillatory Integral Operators and Fractional Oscillatory Integral Operators.

Oscillatory integral operators appear in many fields of mathematics and physics. Furthermore oscillatory integrals have been an essential part of harmonic analysis. Many important operators in harmonic analysis are some versions of oscillatory integrals, such as the Fourier transform, Bochner-Riesz means, and Radon transform. Properties of oscillatory integral operators have been studied by Stein in [19].

A distribution kernel $K$ is called a standard Calderón-Zygmund (in short C-Z) kernel when it satisfies the following conditions:

$$|K(x, y)| \leq \frac{C}{|x - y|^{n'\gamma}}, \quad x \neq y$$  \hfill (11)

$$|\nabla_2 K(x, y)| + |\nabla_2 K(x, y)| \leq \frac{C}{|x - y|^{n\gamma}}, \quad x \neq y.$$
C-Z integral operator $T$ and the oscillatory integral operator $S$ are defined by

$$
Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x,y)f(y)\,dy, 
$$

$$
Sf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)}K(x,y)f(y)\,dy,
$$

where $P(x,y)$ is a real valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$. Lu and Zhang [20] used $L^2$-boundedness of $T$ to get $L^p$-boundedness of $S$ with $1 < p < \infty$.

Ricci and Stein [21] introduced the standard fractional C-Z kernel, $K_\alpha$ with $0 < \alpha < n$, where

$$
|K_\alpha(x,y)| \leq \frac{C}{|x-y|^{n-\alpha}},
$$

$$
|\nabla_x K_\alpha(x,y)| + |\nabla_y K_\alpha(x,y)| \leq \frac{C}{|x-y|^{n+1-\alpha}}, \quad x \neq y.
$$

The fractional oscillatory integral operator is defined in [22] as

$$
S_\alpha f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)}K_\alpha(x,y)f(y)\,dy,
$$

where $P(x,y)$ is a real valued polynomial and defined on $\mathbb{R}^n \times \mathbb{R}^n$ and $K_\alpha$ is a standard fractional C-Z kernel; note that when $\alpha = 0$, $S_\alpha = S$ and $K_\alpha = K$.

**Lemma 1** (see [23]). If $K$ is a standard C-Z kernel and the C-Z singular integral operator $T$ is of type $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$, then for any real polynomial $P(x,y)$ and $w \in A_p$ ($1 < p < \infty$), there exists constants $C > 0$ independent of the coefficients of $P$ such that $\|SF\|_{L^{p,w}(\mathbb{R}^n)} \leq C\|f\|_{L^{p,w}(\mathbb{R}^n)}$.

**Lemma 2** (see [24]). Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $w \in A_{pq}$, $\varphi \in \mathfrak{B}(w^p)$, and $\psi \in \mathfrak{B}(w^q)$. Then the operators $M_\alpha$ and $I_\alpha$ are bounded from $V_{p,q}^\infty(\mathbb{R}^n; w^p)$ to $V_{1,q}^\infty(\mathbb{R}^n; w^q)$, if

$$
\int_0^\infty \sup_{x \in \mathbb{R}^n} \frac{\varphi^{1/p}(x,t)}{w_{L^q(B(x,t))}} \frac{dt}{t} < \infty
$$

for every $\delta > 0$, and

$$
\int_r^\infty \frac{\psi^{1/p}(x,r)}{\|w\|_{L^q(B(x,r))}} \frac{dt}{t} \leq C \frac{\psi^{1/p}(x,r)}{\|w\|_{L^q(B(x,r))}},
$$

where $C$ does not depend on $x \in \mathbb{R}^n$ and $r$.

**2.3. The Commutator Operators $S_b$ and $S_{ab}$.** Let $f \in L^1_{loc}(\mathbb{R}^n)$; we define

$$
\|S_b f\|_p = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_B(x,r)| \, dy
$$

< \infty,

where

$$
f_B(x,r) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \, dy.
$$

In harmonic analysis a function of bounded mean oscillation, also known as a BMO function, is a real valued function whose mean oscillation is bounded. BMO$(\mathbb{R}^n)$ is the set of all locally integrable functions $f$ on $\mathbb{R}^n$ with $\|f\|_{\text{osc}} < \infty$.

Next we shall introduce the commutators of oscillatory integral operators and fractional oscillatory integral operators.

Let $b$ be a locally integrable function; the commutator operators $S_b$ and $S_{ab}$ which are formed by $b$ and $f$ are defined by

$$
S_b f(x) = Sf(x)b(x) - S(bf)(x),
$$

$$
S_{ab} f(x) = S_a f(x)b(x) - S_a(bf)(x).
$$

**Lemma 3** (see [25]). Suppose that $K$ is a standard C-Z kernel, $w \in A_p$ ($1 < p < \infty$), and the operator $T$ is of type $(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$. Then for any $b \in \text{BMO}(\mathbb{R}^n)$, there exists constants $C > 0$ independent on the coefficients of $P$ such that

$$
\|S_b f\|_{L^{p,w}(\mathbb{R}^n)} \leq C\|b\|_{\text{BMO}} \|f\|_{L^{p,w}(\mathbb{R}^n)}.
$$

**Lemma 4** (see [18]). Let $w \in A_{\infty}$. Then the norm of BMO$(\mathbb{R}^n)$ is equivalent to the norm of BMO$(w)$, where

$$
\text{BMO}(w) = \left\{ b : \|b\|_{\text{BMO}(w)} = \sup_{x \in \mathbb{R}^n, r > 0} \left( \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r),w}| w(y) \, dy \right) < \infty \right\},
$$

$$
b_{B(x,r),w}(y) = \frac{1}{w(B(x,r))} \int_{B(x,r)} b(y) w(y) \, dy.
$$

**Remark 5** (the John-Nirenberg inequality). There are possible constants $C_1, C_2 > 0$ such that for all $b \in \text{BMO}(\mathbb{R}^n)$ and $\beta > 0$

$$
|\{ x \in B : |b(x) - b_B| > \beta \}| \leq C_1 |B| e^{-C_2 \beta |\text{BMO}(w)|},
$$

for all $B \subset \mathbb{R}^n$. The John-Nirenberg inequality implies that

$$
\|b\|_{\text{BMO}(w)} = \sup_{x \in \mathbb{R}^n, r > 0} \left( \frac{1}{w(B(x,r))} \int_{B(x,r)} |b(y) - b_{B(x,r),w}|^p w(y) \, dy \right)^{1/p}
$$

for $1 < p < \infty$ and $w \in A_{\infty}$. 

Lemma 6 (see [26]). (i) Let \( w \in A_p \) and let \( b \) be a function in \( \text{BMO}(\mathbb{R}^n) \). Let also \( 1 \leq p < \infty \), \( x \in \mathbb{R}^n \), and \( r_1, r_2 > 0 \). Then
\[
\left( \frac{1}{w(B(x, r_1))} \right) \cdot \int_{B(x, r_1)} \left| b(y) - b_{B(x, r_2), w} \right|^p w(y) \, dy \right)^{1/p} + \ln \frac{r_1}{r_2} \| b \|_w \leq C \left( 1 + \ln \frac{r_1}{r_2} \right) \| b \|_w
\]
where \( C > 0 \) is independent of \( f, x, r_1, \) and \( r_2 \).

(ii) Let \( w \in A_p \) and let \( b \) be a function in \( \text{BMO}(\mathbb{R}^n) \). Let also \( 1 < p < \infty \), \( x \in \mathbb{R}^n \), and \( r_1, r_2 > 0 \). Then
\[
\left( \frac{1}{w^{1/p'}(B(x, r_1))} \right) \cdot \int_{B(x, r_1)} \left| b(y) - b_{B(x, r_2), w} \right|^{p'} w(y)^{1-p'} \, dy \right)^{1/p'} 
\]
where \( C > 0 \) is independent of \( f, x, r_1, \) and \( r_2 \).

Lemma 7 (see [24]). Let \( 1 < p < \infty \), \( b \in \text{BMO}(\mathbb{R}^n) \), \( 0 < \alpha < n/p \), \( 1/q = 1/p - \alpha/n \), \( w \in A_{q/p}, \) \( \varphi \in \mathcal{R}(w^p) \), and \( y \in \mathcal{R}(w^p) \). Then \( M_{ab} \) and \( L_{ab} \) are bounded from \( \text{VM}_p^{p,q}(\mathbb{R}^n; w^p) \) to \( \text{VM}_p^{q,w}(\mathbb{R}^n; w^p) \) if
\[
\sup_{0 < \delta < \delta} \left\{ \int_0^\infty \left[ \ln\left( \frac{e + t}{r} \right) \sup_{x \in B(x, t)} \frac{\varphi^{1/p}(x, t)}{\| w \|_{L^q(B(x, t))}} \right] \, dt \right\} < \infty
\]
for every \( \delta > 0 \), and
\[
\int_r^\infty \frac{\varphi^{1/p}(x, t)}{\| w \|_{L^q(B(x, t))}} \, dt \leq C \frac{\varphi^{1/p}(x, r)}{\| w \|_{L^q(B(x, r))}}.
\]
where \( C \) does not depend on \( x \) and \( r \).

3. Main Results

3.1. Boundedness of \( S \) and \( S_a \) in \( \text{VM}_p^{p,q}(\mathbb{R}^n; w) \)

Lemma 8. Let \( 1 < p < \infty \), \( w \in A_p \), and \( K \) is a standard C-Z kernel and C-Z singular operator \( T \) is of type \( (L^2(\mathbb{R}^n), L^2(\mathbb{R}^n)) \).

Then for any ball \( B = B(x_0, r) \) in \( \mathbb{R}^n \) and any real polynomial \( P(x, y) \) the following inequality holds:
\[
\| Sf \|_{L^p(B)} \leq c \| b \|_w \left( \int_0^\infty \frac{\| f \|_{L^p(B(x, t))}^{1/p} \| w \|_{L^q(B(x, t))}^{1/p}}{t} \, dt \right)
\]
where the constant \( c > 0 \) does not depend on \( B \) and \( f \).
Thus,
\[ \|Sf\|_{L^{p,w}(B)} \leq \|w\|^{1/p}_{L^{w}(B)} \int_{2r}^{\infty} \frac{\|f\|_{L^{p,w}(B(x,t))} \|w\|^{-1/p}_{L^{w}(B(x,t))}}{t} dt. \] (37)

On the other hand,
\[ \|f\|_{L^{p,w}(2B)} \approx \tau^n \|f\|_{L^{p,w}(2B(0,\tau^n \cdot 2r))} \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \] (38)
\[ \leq \|w\|^{1/p}_{L^{w}(B)} \int_{2r}^{\infty} \|f\|_{L^{p,w}(B(x,t))} \|w\|^{-1/p}_{L^{w}(B(x,t))} \frac{dt}{t^{n+1}}. \]

By (31) and (38) we write
\[ \|Sf\|_{L^{p,w}(B)} \leq \|w\|^{1/p}_{L^{w}(B)} \int_{2r}^{\infty} \|f\|_{L^{p,w}(B(x,t))} \|w\|^{-1/p}_{L^{w}(B(x,t))} \frac{dt}{t^{n+1}}. \] (39)

Then by (37) and (39) we get the inequality (29).

Remark 9. Note that Lemma 8 was proved in [7] in the case \( w \equiv 1 \).

**Theorem 10.** Let \( 1 < p < \infty, w \in A_p, \) and \( \varphi, \psi \in B(w) \). The oscillatory integral operator \( T \) is bounded from \( VM_{\Pi}^{p,w}(\mathbb{R}^n; w) \) to \( VM_{\Pi}^{p,w}(\mathbb{R}^n; w) \) for any real polynomial \( P(x,y) \) if \( K \) is a standard C-Z kernel, the C-Z singular operator \( T \) is of type \( (L^p, L^2) \):
\[ c_0 := \frac{1}{\delta} \sup_{x \in \Pi} \|w\|^{1/p}_{L^w(B(x,\delta))} < \infty \] (40)

for every \( \delta > 0 \), and
\[ \int_{r}^{\infty} \frac{\varphi^{1/p}(x,t)}{\|w\|^{1/p}_{L^w(B(x,t))}} \frac{dt}{t} \leq c_0 \frac{\psi^{1/p}(x,r)}{\|w\|^{1/p}_{L^w(B(x,r))}}, \] (41)

where \( c_0 \) does not depend on \( x \in \Pi \) and \( r > 0 \).

Proof. To estimate the norm of the operator we use (28) and get
\[ \|Sf\|_{VM_{\Pi}^{p,w}(w)} \leq c \sup_{x \in \Pi, r > 0} \frac{\psi^{1/p}(x,r)}{\|w\|^{1/p}_{L^w(B(x,r))}} \|Sf\|_{L^{p,w}(B(x,r))} \]
\[ \leq c \sup_{x \in \Pi, r > 0} \frac{1}{\psi^{1/p}(x,r)} \|w\|^{1/p}_{L^w(B(x,r))} \int_{r}^{\infty} t^{-1} \|f\|_{L^{p,w}(B(x,t))} \|w\|^{-1/p}_{L^w(B(x,t))} \frac{dt}{t}. \] (42)

By using (41) we can write
\[ \|Sf\|_{VM_{\Pi}^{p,w}(w)} \leq c \frac{1}{\psi^{1/p}(x,r)} \|w\|^{1/p}_{L^w(B(x,r))} \|Sf\|_{L^{p,w}(B(x,r))}. \] (43)

Thus,
\[ \|Sf\|_{VM_{\Pi}^{p,w}(w)} \leq c \frac{1}{\psi^{1/p}(x,r)} \|w\|^{1/p}_{L^w(B(x,r))}. \] (44)

Now we prove that it belongs to \( VM_{\Pi}^{p,w}(\mathbb{R}^n; w) \); that is,
\[ \lim_{r \to 0} \sup_{x \in \Pi} \frac{1}{\psi^{1/p}(x,r)} \|f\|_{L^{p,w}(B(x,r))} = 0 \implies \lim_{r \to 0} \sup_{x \in \Pi} \frac{1}{\psi^{1/p}(x,r)} \|Sf\|_{L^{p,w}(B(x,r))} = 0. \] (45)

To show that \( \sup_{x \in \Pi} (1/\psi^{1/p}(x,r)) \|Sf\|_{L^{p,w}(B(x,r))} < \epsilon \) for small \( r \), we split the right-hand side of (28):
\[ \frac{1}{\varphi^{1/p}(x,r)} \|Sf\|_{L^{p,w}(B(x,r))} \leq C \left[ I_{\delta_0}(x,r) + J_{\delta_0}(x,r) \right], \] (46)

where \( \delta_0 > 0 \) (we may take \( \delta_0 < 1 \) and
\[ I_{\delta_0}(x,r) = \frac{\|w\|^{1/p}_{L^w(B(x,r))}}{\psi^{1/p}(x,r)} \left( \int_{0<|\tau|<\delta_0} \sup_{x \in \Pi} \frac{1}{\psi^{1/p}(x,r)} \|f\|_{L^{p,w}(B(x,\tau))} \frac{dt}{t} \right), \]
\[ J_{\delta_0}(x,r) = \frac{\|w\|^{1/p}_{L^w(B(x,r))}}{\psi^{1/p}(x,r)} \left( \int_{r}^{\infty} \frac{\varphi^{1/p}(x,t)}{\|w\|^{1/p}_{L^w(B(x,t))}} \frac{dt}{t} \right), \] (47)

and it is supposed that \( r < \delta_0 \).

Now we fix \( \delta_0 > 0 \) such that \( \sup_{x \in \Pi} \sup_{0 < r < \delta_0} (1/\varphi^{1/p}(x,r)) \|f\|_{L^{p,w}(B(x,r))} < \epsilon/2C_0 \), where \( C_0 \) and \( C \) are constants from (41) and (46), respectively. Then we can write
\[ \sup_{x \in \Pi} C_0 I_{\delta_0}(x,r) < \frac{\epsilon}{2} \quad 0 < r < \delta_0. \] (48)

By choosing \( r \) sufficiently small and considering (40) we have
\[ J_{\delta_0}(x,r) \leq c_0 \|w\|^{1/p}_{L^w(B(x,r))} \|Sf\|_{VM_{\Pi}^{p,w}(w)} \leq \epsilon/2C_0, \]

where \( c_0 \) is the constant from (40). Then by (10) we choose \( r \) small enough such that
\[ \sup_{x \in \Pi} \frac{\|w\|^{1/p}_{L^w(B(x,r))}}{\psi^{1/p}(x,r)} < \left( \frac{\epsilon}{2C_0 \|Sf\|_{VM_{\Pi}^{p,w}(w)}} \right)^p, \] (50)

which completes the proof. \( \square \)
Theorem 11. Let $1 < p < \infty$, $0 < \alpha < n/p$, $1/q = 1/p - \alpha/n$, $P(x, y)$ be real polynomial; $w \in A_p, \psi \in \mathcal{B}(w^p)$, and $\psi \in \mathcal{B}(w^p)$. Then $S_\alpha$ is bounded from $VM^{p,q}_w(w^p)$ to $VM^{p,q}_w(w^p)$ if

$$
\int_0^\infty \sup_{x \in \Pi} \frac{\psi^{1/p}(x, t)}{t} dt < \infty
$$

for every $\delta > 0$, and

$$
\int_0^\infty \frac{\psi^{1/p}(x, t)}{t} \frac{dt}{\|u\|_{L^p(B(x, r))}} \leq C \psi^{1/p}(x, r),
$$

where $C$ is not depending on $x \in \Pi$ and $r$.

Proof. Since $|S_\alpha f(x)| \leq I_{\alpha}(f)(x)$ and thanks to Lemma 2 proof is completed.

3.2. Boundedness of $S_\alpha$ and $S_{\alpha b}$ in the Spaces $VM^{p,q}_w(w)$

Lemma 12. Let $1 \leq p < \infty$, $b \in BMO(\mathbb{R}^n)$, $w \in A_p$, $K$ is a standard C-Z kernel, and the C-Z integral operator $T_\alpha$ is type of $(L_2(\mathbb{R}^n), L_2(\mathbb{R}^n))$. Then for $1 < p < \infty$ and polynomial $P(x, y)$ the inequality

$$
\|S_b f\|_{L^p(B)} \leq C \|b\|_{L^p(w)} \|u\|_{L^p(B)}^{1/p} \cdot \int_0^{\infty} \ln \left(1 + \frac{t}{r}\right) \frac{dt}{t} \tag{53}
$$

holds for every ball $B = B(x_0, r)$ and for all $f \in L^1(B)$.

Proof. Let $1 < p < \infty$. For any $x_0 \in \mathbb{R}^n$, we split $f$ into two parts in a neighbourhood of point $x_0$ such that

$$
f(x) = f_1 + f_2,
$$

$$
f_1(x) = f(x) \chi_B(y),
$$

$$
f_2(x) = f(x) \chi_{\mathbb{R}^n \setminus B(y)}(y),$$

where $r > 0$, and by linearity of the operator $S_b$, we have

$$
\|S_b f\|_{L^p(B)} \leq \|S_b f_1\|_{L^p(B)} + \|S_b f_2\|_{L^p(B)}. \tag{55}
$$

Moreover, from the boundedness of $S_b$ in $L^p,B^{w}(\mathbb{R}^n)$ (see Lemma 3), it follows that

$$
\|S_b f_1\|_{L^p(B)} \leq \|S_b f_1\|_{L^p(B)} \leq C \|b\|_{L^p(w)} \|f_1\|_{L^p(\mathbb{R}^n)} = C \|b\|_{L^p(w)} \|f_1\|_{L^p(\mathbb{R}^n)}.
$$

where constant $C > 0$ is independent of $f$.

For $x \in B(x_0, r)$ we have

$$
|S_b f_2(x)| \leq \int_{\mathbb{R}^n} \frac{|b(y) - b(x)|}{|x - y|^r} |f_2(y)| dy
$$

$$
= \int_{\mathbb{R}^n \setminus B(y)} \frac{|b(y) - b(x)|}{|x - y|^r} |f(y)| dy.
$$

Then

$$
\|S_b f\|_{L^p(B)} \leq C \left(\int_B \left(\int_{\mathbb{R}^n \setminus B(2r)} \frac{|b(y) - b(x)|}{|x - y|^r} |f(y)| dy\right)^p \right)^{1/p} \cdot w(x) dx.
$$

Using Hölder's inequality and (25), we have

$$
I \leq \|w\|_{L^{p'}} \int_{2r}^{\infty} \left(\int_{B(x_0, r)} \left|b(y) - b_{B(x_0, r)}\right|^p \right)^{1/p'} \cdot w(y)^{1/p'} \cdot y^\alpha dy \cdot \|f\|_{L^{p'}(B(2r))}^{1/p'} \cdot \|w\|_{L^{p'}(B(2r))} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^{-\alpha} \frac{dt}{t^{n+1}}.
$$

Let us estimate $I$

$$
I = C \|w\|_{L^{p'}(B)} \int_{\mathbb{R}^n \setminus B(2r)} \frac{|b(y) - b_{B(2r)}|}{|x - y|^r} |f(y)| dy.
$$

$$
\leq C \|w\|_{L^{p'}(B)} \cdot \int_{\mathbb{R}^n \setminus B(2r)} \frac{|b(y) - b_{B(2r)}|}{|x - y|^r} |f(y)| dy \int_{2r}^{\infty} \frac{dt}{t^{n+1}} dy.
$$

$$
\leq C \|w\|_{L^{p'}(B)} \cdot \int_{2r}^{\infty} \int_{B(2r)} |b(y) - b_{B(2r)}| |f(y)| dy \frac{dt}{y^{n+1}}.
$$

Using Hölder's inequality and (25), we have

$$
I \leq \|w\|_{L^{p'}(B)} \int_{2r}^{\infty} \left(\int_{B(x_0, r)} \left|b(y) - b_{B(x_0, r)}\right|^p \right)^{1/p'} \cdot w(y)^{1/p'} \cdot y^\alpha dy \cdot \|f\|_{L^{p'}(B(2r))}^{1/p'} \cdot \|w\|_{L^{p'}(B(2r))} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right)^{-\alpha} \frac{dt}{t^{n+1}}.
$$

\[\square\]
In order to estimate II note that

\[
II = \left( \int_{B} |b(x) - b_{B} w_{B}| w(x) \, dx \right)^{1/p} \cdot \int_{\mathbb{R}^{n}} \frac{|f(y)|}{|x_{0} - y|^{n}} \, dy.
\]

By (23) we get

\[
II \leq \|b\|_{L^{1/p}(B)} \int_{\mathbb{R}^{n}} \frac{|f(y)|}{|x_{0} - y|^{n}} \, dy.
\]

Applying Hölder's inequality, we get

\[
\int_{\mathbb{R}^{n}} \frac{|f(y)|}{|x_{0} - y|^{n}} \, dy \leq \int_{2r} \|f\|_{L^{p,w}(B(x_{0},2r))} \|w^{-1/p}\|_{L^{p,w}(B(x_{0},2r))} \, dt \leq \|w\|_{L^{1/p}(B(x_{0},2r))} \|w(B(x_{0},t))^{-1/p}\|_{L^{1/p}(B(x_{0},2r))}. \]

Thus by (63) we write

\[
II \leq \|b\|_{L^{1/p}(B)} \int_{2r} \|f\|_{L^{p,w}(B(x_{0},2r))} \|w(B(x_{0},t))^{-1/p}\|_{L^{1/p}(B(x_{0},2r))} \, dt.
\]

Summing up I and II, for all \(1 < p < \infty\) we get

\[
\|S_{0} f\|_{L^{p,w}(B(x,r))} \leq \|b\|_{L^{1/p}(B)} \int_{2r} \|f\|_{L^{p,w}(B(x_{0},2r))} \|w(B(x_{0},t))^{-1/p}\|_{L^{1/p}(B(x_{0},2r))} \, dt.
\]

On the other hand,

\[
\|f\|_{L^{p,w}(B(r))} \simeq |B|^{1/p} \|f\|_{L^{p,w}(B(2r))} \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \leq |B|^{1/p} \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \leq |B|^{1/p} \int_{2r}^{\infty} \frac{dt}{t^{n}}.
\]

Thus,

\[
\|S_{0} f\|_{L^{p,w}(B(x,r))} \leq \|b\|_{L^{1/p}(B)} \|f\|_{L^{p,w}(B(2r))} \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \leq \|b\|_{L^{1/p}(B)} \|f\|_{L^{p,w}(B(2r))} \int_{2r}^{\infty} \frac{dt}{t^{n}} \leq \|b\|_{L^{1/p}(B)} \|f\|_{L^{p,w}(B(2r))} \int_{2r}^{\infty} \frac{dt}{t^{n-1}}.
\]

Theorem 13. Let \(1 < p < \infty\) and \(b \in \text{BMO}(\mathbb{R}^{n}), w \in A_{p}\) and \(\varphi, \psi \in B(\omega)\). Then \(S_{0}\) is bounded from \(VM_{\Pi}^{1,p}(\mathbb{R}^{n};w)\) to \(VM_{\Pi}^{p,\psi}(\mathbb{R}^{n};w)\) if

\[
c_{0} = \sup_{0<\rho<\delta} \int_{r}^{\infty} \ln \left( e + \frac{t}{r} \right) \sup_{x \in \mathbb{R}^{n}} \frac{\|S_{0} f\|_{L^{p,w}(B(x,r))}}{\|f\|_{L^{p,w}(B(x,r))}} \, dt < \infty
\]

for every \(\delta > 0\), and

\[
\int_{r}^{\infty} \ln \left( e + \frac{t}{r} \right) \frac{\|S_{0} f\|_{L^{p,w}(B(x,r))}}{\|f\|_{L^{p,w}(B(x,r))}} \, dt \leq c_{0} \frac{\|\psi\|_{L^{1/p}(B(x,r))}}{\|\varphi\|_{L^{1/p}(B(x,r))}}.
\]

where \(c_{0}\) does not depend on \(x \in \mathbb{R}^{n}\) and \(r > 0\).

Proof. Boundedness follows from Lemma 12 and the same procedure is argued in the proof of Theorem 10. We have to prove that

\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^{n}} \frac{1}{\psi^{1/p}(x,r)} \|S_{0} f\|_{L^{p,w}(B(x,r))} = 0 \quad \text{implies} \quad \lim_{r \to 0} \sup_{x \in \mathbb{R}^{n}} \frac{1}{\psi^{1/p}(x,r)} \|S_{0} f\|_{L^{p,w}(B(x,r))} = 0.
\]

We suppose that \(0 < r < \delta_{0}\). In view of (53) we write

\[
\frac{1}{\psi^{1/p}(x,r)} \|S_{0} f\|_{L^{p,w}(B(x,r))} \leq C \frac{\|b\|_{L^{1/p}(B(x,r))}}{\psi^{1/p}(x,r)} \int_{r}^{\infty} \ln \left( e + \frac{t}{r} \right) \frac{\|f\|_{L^{p,w}(B(x,r))}}{\|w\|_{L^{1/p}(B(x,r))}} \, dt.
\]

To show that \(\sup_{x \in \mathbb{R}^{n}} \left( \psi^{1/p}(x,r) \right) \|S_{0} f\|_{L^{p,w}(B(x,r))} < \varepsilon\) for small \(r\), we split the right side of (71)

\[
\frac{1}{\psi^{1/p}(x,r)} \|S_{0} f\|_{L^{p,w}(B(x,r))} \leq C \left[ \delta_{0}(x,r) + J_{\delta_{0}}(x,r) \right].
\]
Theorem 14. Let $1 < p < \infty$, $b \in \text{BMO}(\mathbb{R}^n)$, $0 < \alpha < n/p$, $1/q = 1/p - \alpha/n$, $P(x, y)$ be a real valued polynomial, $\varphi \in \mathfrak{B}(w^\alpha)$ and $\psi \in \mathfrak{B}(w^\beta)$. Then $S_{\alpha, b}$ is bounded from $\text{VM}^{p, q}_{n, \psi^\beta}(w)$ to $\text{VM}^{p, q}_{n, \psi^\beta}(w)$ if

\[
\sup_{x \in \mathbb{R}^n} \|w\|_{L^1_x(B(x, 1))} \leq \left( \frac{\epsilon}{2C\delta_0 \|b\|_* \|f\|_{L^{p, q}_x(w)}} \right)^p
\]

which completes the proof. \qed

Competing Interests

The authors declare that they have no competing interests.

References


