Research Article

A Compact Difference Scheme for Solving Fractional Neutral Parabolic Differential Equation with Proportional Delay

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A linearized compact finite difference scheme is constructed for solving the fractional neutral parabolic differential equation with proportional delay. By the energy method, the unconditional stability of the scheme is proved, and the convergence order of the scheme is proved to be $O(\tau^{2-\alpha} + h^4)$. A numerical test is also conducted to validate the accuracy and efficiency of the numerical algorithm.

1. Introduction

In the past few years, more and more scholars have been attracted to the research of delay partial differential equations (DPDEs) [1–3]. However, most DPDEs have no exact solutions. Constructing efficient numerical methods for DPDEs is of great importance [4–7]. For details on numerically solving neutral delay parabolic differential equations (NDPDEs), the reader is referred to [5, 8]. Recently, fractional delay partial differential equations have been of great interest due to their application in automatic control, population dynamics, economics, and so forth [9, 10]. For details on numerical solutions to fractional delay partial differential equations, we refer the reader to [11, 12]. The work in [11] considers the numerical method without theoretical analysis, and the work in [12] considers the numerical method for a type of semilinear fractional partial differential equation with time delay.

In this paper, we consider the following fractional neutral parabolic differential equation with proportional delay:

$$\begin{aligned} \frac{\partial^\alpha w}{\partial t^\alpha} &= D \left(\frac{\partial^2 w(x, t)}{\partial x^2} + \frac{\partial^2 w(x, pt)}{\partial x^2} \right) \\ &+ f_w(w(x, t), w(x, pt), x, t), \\ (x, t) &\in (0, 1) \times [0, T], \end{aligned}$$

$$\begin{aligned} w(0, t) &= \gamma_w(t), \\ w(1, t) &= \beta_w(t), \\ t &\in (0, T], \end{aligned} \quad (1)$$

where $D > 0$ is a constant, $0 < p < 1$.

Let $u(x, t) = w(x, e^t)$ for $t \geq t_0 + \ln(p)$, where $t_0 \geq 0$. Then, $u(x, t)$ satisfies the following equation:

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= D \left(\frac{\partial^2 u(x, t)}{\partial x^2} + \frac{\partial^2 u(x, t-s)}{\partial x^2} \right) \\ &+ f_w(u(x, t), u(x, t-s), x, e^t), \\ (x, t) &\in (0, 1) \times (t_0, T], \\ u(x, t) &= \phi_w(x, e^t), \quad x \in [0, 1], \quad t \in [-s, t_0], \\ u(0, t) &= \gamma_w(e^t), \\ u(1, t) &= \beta_w(e^t), \\ t &\in (t_0, T], \end{aligned} \quad (2)$$

where $s = -\ln(p)$.

For simplicity, we consider the following fractional neutral parabolic differential equation with delay instead of (1):

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= D \left(\frac{\partial^2 u(x, t)}{\partial x^2} + \frac{\partial^2 u(x, t-s)}{\partial x^2} \right) \\ &+ f(u(x, t), u(x, t-s), x, t), \end{aligned} \quad (3)$$

$$(x, t) \in (0, 1) \times (0, T],$$

$$u(x, t) = \phi(x, t), \quad x \in [0, 1], \quad t \in [-s, 0], \quad (4)$$

$$\begin{aligned} u(0, t) &= \gamma(t), \\ u(1, t) &= \beta(t), \end{aligned} \quad (5)$$

$$t \in (0, T],$$

where $s > 0$ is a constant delay term. Time fractional partial derivative $(\partial^\alpha u / \partial t^\alpha)$ ($0 < \alpha < 1$) is defined in the Caputo sense by the following:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha} \frac{\partial u(x, \xi)}{\partial \xi} d\xi, \quad (6)$$

where $\Gamma(\cdot)$ is the Gamma function.

In this paper, a linearized compact finite difference scheme is constructed for solving (3)–(5). By the energy method, the unconditional stability of the scheme is then proved, and the convergence order of the scheme is proved to be $O(\tau^{2-\alpha} + h^4)$. A numerical test is also conducted to validate the accuracy and efficiency of the numerical algorithm.

The rest of the paper is organized as follows. In Section 2, a compact difference scheme is constructed to solve (3)–(5). Section 3 considers the solvability, convergence, and stability of the provided difference scheme. In Section 4, a numerical test is presented to illustrate the validity of the theoretical results. Section 5 gives a brief conclusion of this paper.

2. The Construction of the Compact Difference Scheme

Throughout this paper, assume $u(x, t) \in C^{6,2}([0, 1] \times (0, T])$. Function $f(u(x, t), u(x, t-s), x, t)$ is sufficiently smooth and satisfies

$$\begin{aligned} &|f(\mu + \epsilon_1, \nu + \epsilon_2, x, t) - f(\mu, \nu, x, t)| \\ &\leq c_0 |\epsilon_1| + c_1 |\epsilon_2|, \end{aligned} \quad (7)$$

where ϵ_1, ϵ_2 are arbitrary real numbers and c_0 and c_1 are positive constants.

First, let M and N be two positive integers; then, we take $h = 1/M$, $\tau = s/n$ ($n > 0$ is a positive integer), $x_i = ih$, $t_k = k\tau$. Define $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$, where $\Omega_h = \{x_i \mid 0 \leq i \leq M\}$, $\Omega_\tau = \{t_k \mid -n \leq k \leq N\}$, $N = [T/\tau]$. Denote $U_i^k = u(x_i, t_k)$, $0 \leq i \leq M$, $-n \leq k \leq N$ throughout this paper. Let

$$\mathcal{W} = \{v_i^k \mid 0 \leq i \leq M, \quad -n \leq k \leq N\} \quad (8)$$

be the grid function space defined on $\Omega_{h\tau}$. The following notations are used:

$$\begin{aligned} \delta_x v_{i+1/2}^k &= \frac{v_{i+1}^k - v_i^k}{h}, \\ \delta_x^2 v_i^k &= \frac{v_{i+1}^k - 2v_i^k + v_{i-1}^k}{h^2}, \\ \mathcal{A} v_i^k &= \frac{1}{12} (v_{i-1}^k + 10v_i^k + v_{i+1}^k), \\ \delta_t^\alpha v^k &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[d_0 v^k - \sum_{j=1}^{k-1} (d_{k-j-1} - d_{k-j}) v^j \right. \\ &\quad \left. - d_{k-1} v^0 \right], \end{aligned} \quad (9)$$

where $d_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, $j \geq 0$.

For the time fractional derivative, we have the following lemma.

Lemma 1 (see [13]). Suppose $0 < \alpha < 1$, $y \in C^2[0, t_k]$; it holds that

$$\begin{aligned} &\left| \frac{1}{\Gamma(1-\alpha)} \int_0^{t_k} \frac{y'(s) ds}{(t_k-s)^\alpha} - \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[d_0 y(t_k) \right. \right. \\ &\quad \left. \left. - \sum_{j=1}^{k-1} (d_{k-j-1} - d_{k-j}) y(t_j) - d_{k-1} y(t_0) \right] \right| \\ &\leq \frac{1}{\Gamma(2-\alpha)} \left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1+2^{-\alpha}) \right] \\ &\quad \cdot \max_{0 \leq t \leq t_k} |y''(t)| \tau^{2-\alpha}, \end{aligned} \quad (10)$$

and d_j satisfies the following lemma.

Lemma 2 (see [14]). Assume $0 < \alpha < 1$; then, it holds that

- (1) d_j decreases monotonically as j increases, and $0 < d_j \leq 1$;
- (2) $d_0 = 1$, $\sum_{j=1}^{k-1} (d_{k-j-1} - d_{k-j}) = d_0 - d_{k-1}$.

Lemma 3 (see [5]). Suppose $y(x) \in C^6[x_{i-1}, x_{i+1}]$; then, one has

$$\begin{aligned} &\frac{1}{12} [y''(x_{i-1}) + 10y''(x_i) + y''(x_{i+1})] \\ &\quad - \frac{1}{h^2} [y(x_{i-1}) - 2y(x_i) + y(x_{i+1})] \\ &= \frac{h^4}{240} y^{(6)}(t_i), \end{aligned} \quad (11)$$

where $t_i \in (x_{i-1}, x_{i+1})$.

Considering (3) at the point (x_i, t_k) , we have

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha}(x_i, t_k) &= D \left(\frac{\partial^2 u}{\partial x^2}(x_i, t_k) + \frac{\partial^2 u}{\partial x^2}(x_i, t_k - s) \right) \\ &\quad + f(u(x_i, t_k), u(x_i, t_k - s), x_i, t_k), \end{aligned} \quad (12)$$

$$0 \leq i \leq M, \quad 0 \leq k \leq N.$$

From Lemma 1, we obtain

$$\frac{\partial^\alpha u}{\partial t^\alpha}(x_i, t_k) = \delta_t^\alpha U_i^k + r_i^k, \quad (13)$$

where

$$\begin{aligned} r_i^k &= \frac{1}{\Gamma(2-\alpha)} \left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1+2^{-\alpha}) \right] \\ &\quad \cdot \max_{0 \leq t \leq t_k} \left| \frac{\partial^2 u(x_i, t_k)}{\partial t^2} \right| \tau^{2-\alpha}. \end{aligned} \quad (14)$$

From Taylor expansion, we have

$$\begin{aligned} &f(u(x_i, t_k), u(x_i, t_k - s), x_i, t_k) \\ &= f(2U_i^{k-1} - U_i^{k-2}, U_i^{k-n}, x_i, t_k) \\ &\quad + \tau^2 \frac{\partial^2 u}{\partial t^2}(x_i, \eta^k) f_\mu(\varrho_i^k, U_i^{k-n}, x_i, t_k), \end{aligned} \quad (15)$$

where $\eta^k \in (t_{k-2}, t_k)$, ϱ_i^k in between $u(x_i, t_k)$ and $2U_i^{k-1} - U_i^{k-2}$.

Substituting (13) and (15) into (12) and applying the operator $\mathcal{A}$ on both sides of (12), we obtain

$$\begin{aligned} \mathcal{A} \delta_t^\alpha U_i^k &= D \left(\mathcal{A} \frac{\partial^2 u}{\partial x^2}(x_i, t_k) + \mathcal{A} \frac{\partial^2 u}{\partial x^2}(x_i, t_k - s) \right) \\ &\quad + \mathcal{A} f(2U_i^{k-1} - U_i^{k-2}, U_i^{k-n}, x_i, t_k) + \mathcal{A} R_{0i}^k, \end{aligned} \quad (16)$$

where

$$R_{0i}^k = r_i^k + \tau^2 \frac{\partial^2 u}{\partial t^2}(x_i, \eta^k) f_\mu(\varrho_i^k, U_i^{k-n}, x_i, t_k). \quad (17)$$

From Lemma 3 and Taylor expansion, we have

$$\begin{aligned} \mathcal{A} \frac{\partial^2 u}{\partial x^2}(x_i, t_k) &= \delta_x^2 U_i^k + \frac{h^4}{240} \frac{\partial^6 u}{\partial x^6}(\theta_i^k, t_k), \\ \theta_i^k &\in (x_{i-1}, x_{i+1}), \\ \mathcal{A} \frac{\partial^2 u}{\partial x^2}(x_i, t_k - s) &= \delta_x^2 U_i^{k-n} + \frac{h^4}{240} \frac{\partial^6 u}{\partial x^6}(\theta_i^{k-n}, t_{k-n}), \\ \theta_i^{k-n} &\in (x_{i-1}, x_{i+1}). \end{aligned} \quad (18)$$

Substituting (18) into (16), we have

$$\begin{aligned} \mathcal{A} \delta_t^\alpha U_i^k &= D(\delta_x^2 U_i^k + \delta_x^2 U_i^{k-n}) \\ &\quad + \mathcal{A} f(2U_i^{k-1} - U_i^{k-2}, U_i^{k-n}, x_i, t_k) + R_i^k, \end{aligned} \quad (19)$$

where

$$\begin{aligned} R_i^k &= \mathcal{A} R_{0i}^k + \frac{h^4}{240} \frac{\partial^6 u}{\partial x^6}(\theta_i^k, t_k) \\ &\quad + \frac{h^4}{240} \frac{\partial^6 u}{\partial x^6}(\theta_i^{k-n}, t_{k-n}). \end{aligned} \quad (20)$$

Noticing $u(x, t) \in C^{6,2}([0, 1] \times (0, T])$ and (7), we can easily obtain

$$|R_i^k| \leq C_R (\tau^{2-\alpha} + h^4), \quad 0 \leq i \leq M, \quad 1 \leq k \leq N. \quad (21)$$

Discretizing the initial and boundary conditions of (4) and (5), we obtain

$$U_i^k = \phi(x_i, t_k), \quad 0 \leq i \leq M, \quad -n \leq k \leq 0, \quad (22)$$

$$U_0^k = \gamma(t_k),$$

$$\begin{aligned} U_M^k &= \beta(t_k), \\ 1 &\leq k \leq N. \end{aligned} \quad (23)$$

Replacing U_i^k by u_i^k in (19), (22), and (23) and omitting R_i^k , we can obtain the following compact difference scheme:

$$\begin{aligned} \mathcal{A} \delta_t^\alpha u_i^k &= D(\delta_x^2 u_i^k + \delta_x^2 u_i^{k-n}) \\ &\quad + \mathcal{A} f(2u_i^{k-1} - u_i^{k-2}, u_i^{k-n}, x_i, t_k), \end{aligned} \quad (24)$$

$$u_i^k = \phi(x_i, t_k), \quad 0 \leq i \leq M, \quad -n \leq k \leq 0, \quad (25)$$

$$u_0^k = \gamma(t_k),$$

$$\begin{aligned} u_M^k &= \beta(t_k), \\ 1 &\leq k \leq N. \end{aligned} \quad (26)$$

3. The Solvability, Convergence, and Stability of the Difference Scheme

Define the following grid function space on Ω_h :

$$\mathcal{V}_{h,0} = \{u \mid u = (u_0, u_1, \dots, u_M), \quad u_0 = u_M = 0\}. \quad (27)$$

If $u, v \in \mathcal{V}_{h,0}$, we introduce the following inner products and corresponding norms:

$$\begin{aligned} (u, v) &= h \sum_{i=1}^{M-1} u_i v_i, \\ \|u\| &= \sqrt{(u, u)}, \quad \|u\|_\infty = \max_{1 \leq i \leq M-1} |u_i|, \end{aligned} \quad (28)$$

$$\langle \delta_x u, \delta_x v \rangle = h \sum_{i=0}^{M-1} (\delta_x u_{i+1/2}) (\delta_x v_{i+1/2}),$$

$$|\delta_x u|_1 = \sqrt{\langle \delta_x u, \delta_x u \rangle}.$$

It is easy to obtain the following lemma.

Lemma 4 (see [12]). $\forall u \in \mathcal{V}_{h,0}$, one has $\|\mathcal{A}u\|^2 \leq \|u\|^2$.

Lemma 5 (see [15]). $\forall u \in \mathcal{V}_{h,0}$, one has

$$\begin{aligned} \|u\|_\infty &\leq \frac{|\delta_x u|_1}{2}, \\ \|u\| &\leq \frac{|\delta_x u|_1}{\sqrt{6}}. \end{aligned} \quad (29)$$

The following lemma will be used in the proof of the stability and convergence analysis.

Lemma 6 (see [15]). Assume that $\{F^k \mid k \geq 0\}$ is a nonnegative sequence and satisfies

$$F^{k+1} \leq A + B\tau \sum_{i=1}^k F^i, \quad k = 0, 1, \dots; \quad (30)$$

then,

$$F^{k+1} \leq A \exp(Bk\tau), \quad k = 0, 1, 2, \dots, \quad (31)$$

where A and B are nonnegative constants.

Theorem 7. The difference scheme (24)–(26) has a unique solution.

Proof. Denote $u^k = [u_1^k, u_2^k, \dots, u_{M-1}^k]^T$; the difference scheme (24)–(26) is a linear tridiagonal system $\mathcal{M}u^k = d$, where d only depends on u^j ($j \leq k-1$), u_0^k , and u_M^k and is independent of u^k .

$$\mathcal{M} = \text{tridiag} \left(\frac{1}{12\lambda} - \frac{D}{h^2}, \frac{10}{12\lambda} + \frac{2D}{h^2}, \frac{1}{12\lambda} - \frac{D}{h^2} \right), \quad (32)$$

where $\lambda = \tau^\alpha \Gamma(2 - \alpha)$. We can see that $\mathcal{M}$ is a strictly diagonally dominant coefficient matrix. Thus, scheme (24)–(26) has a unique solution. $\square$

To discuss the stability of the difference scheme (24)–(26), we consider the following problem:

$$\begin{aligned} \mathcal{A} \frac{\partial^\alpha v}{\partial t^\alpha} &= D \left(\frac{\partial^2 v(x, t)}{\partial x^2} + \frac{\partial^2 v(x, t-s)}{\partial x^2} \right) \\ &\quad + \mathcal{A} f(v(x, t), v(x, t-s), x, t), \\ &\quad (x, t) \in (0, 1) \times (0, T], \\ v(x, t) &= \phi(x, t) + \psi(x, t), \quad x \in [0, 1], \quad t \in [-s, 0], \end{aligned} \quad (33)$$

$$v(0, t) = \gamma(t),$$

$$u(1, t) = \beta(t),$$

$$t \in (0, T],$$

where $\psi(x, t)$ is the perturbation caused by $\phi(x, t)$. The following difference scheme can be obtained for solving (33):

$$\begin{aligned} \mathcal{A} \delta_t^\alpha v_i^k &= D (\delta_x^2 v_i^k + \delta_x^2 v_i^{k-n}) \\ &\quad + \mathcal{A} f(2v_i^{k-1} - v_i^{k-2}, v_i^{k-n}, x_i, t_k), \end{aligned} \quad (34)$$

$$v_i^k = \phi(x_i, t_k) + \psi_i^k, \quad 0 \leq i \leq M, \quad -n \leq k \leq 0, \quad (35)$$

$$v_0^k = \gamma(t_k),$$

$$v_M^k = \beta(t_k), \quad (36)$$

$$1 \leq k \leq N.$$

Denote

$$\omega_i^k = v_i^k - u_i^k, \quad 0 \leq i \leq M, \quad -n \leq k \leq N. \quad (37)$$

Definition 8. Assume that u_i^k satisfies (24)–(26) and v_i^k satisfies (34)–(36); then, a numerical scheme for (3)–(5) is stable if one has

$$\|\omega^k\|_\infty \leq \bar{C} \max_{-n \leq j \leq 0} |\psi^j|_1, \quad (38)$$

where $\bar{C}$ is a bounded constant independent of h and τ .

Theorem 9. Assume $u(x, t) \in C^{6,2}([0, 1] \times (-s, T])$ is the solution of (3)–(5); the difference scheme (24)–(26) is stable with respect to the initial perturbation of ψ_i^k ; that is, $\|\omega^k\|_\infty \leq \widehat{C} \max_{-n \leq j \leq 0} |\psi^j|_1$, where $\widehat{C}$ is a positive constant independent of h and τ .

Proof. Subtracting (34)–(36) from (24)–(26), respectively, we can obtain the following equations:

$$\mathcal{A} \delta_t^\alpha \omega_i^k = D (\delta_x^2 \omega_i^k + \delta_x^2 \omega_i^{k-n}) + \mathcal{A} q_i^k, \quad (39)$$

$$\omega_i^k = \psi_i^k, \quad 0 \leq i \leq M, \quad -n \leq k \leq 0, \quad (40)$$

$$\omega_0^k = 0,$$

$$\omega_M^k = 0, \quad (41)$$

$$1 \leq k \leq N,$$

where $q_i^k = f(2v_i^{k-1} - v_i^{k-2}, v_i^{k-n}, x_i, t_k) - f(2u_i^{k-1} - u_i^{k-2}, u_i^{k-n}, x_i, t_k)$.

Multiplying $h\delta_t^\alpha \omega_i^k$ on both sides of (39) and summing up for i from 1 to $M-1$, we obtain

$$\begin{aligned} &h \sum_{i=1}^{M-1} (\mathcal{A} \delta_t^\alpha \omega_i^k) (\delta_t^\alpha \omega_i^k) \\ &= Dh \sum_{i=1}^{M-1} (\delta_x^2 \omega_i^k + \delta_x^2 \omega_i^{k-n}) (\delta_t^\alpha \omega_i^k) \\ &\quad + h \sum_{i=1}^{M-1} (\mathcal{A} q_i^k) (\delta_t^\alpha \omega_i^k), \end{aligned} \quad (42)$$

$$1 \leq i \leq M-1, \quad 0 \leq k \leq N-1.$$

Then, each term of (42) will be estimated. From the discrete Green formula and inequality $-(a+b)^2 \geq -2(a^2+b^2)$, we have

$$\begin{aligned} & h \sum_{i=1}^{M-1} (\mathcal{A} \delta_t^\alpha \omega_i^k) (\delta_t^\alpha \omega_i^k) \\ &= \|\delta_t^\alpha \omega^k\|^2 + \frac{h^2}{12} \sum_{i=1}^{M-1} (\delta_x^2 (\delta_t^\alpha \omega_i^k)) (\delta_t^\alpha \omega_i^k) \quad (43) \\ &= \|\delta_t^\alpha \omega^k\|^2 - \frac{h^2}{12} \|\delta_x \delta_t^\alpha \omega^k\|^2 \geq \frac{2}{3} \|\delta_t^\alpha \omega^k\|^2. \end{aligned}$$

From the discrete Green formula and inequality $-ab \geq -(a^2 + b^2)/2$, we have

$$\begin{aligned} & Dh \sum_{i=1}^{M-1} (\delta_x^2 \omega_i^k) (\delta_t^\alpha \omega_i^k) \\ &= -Dh \sum_{i=0}^{M-1} (\delta_x \omega_{i+1/2}^k) (\delta_t^\alpha \delta_x \omega_{i+1/2}^k) = -\frac{D}{\lambda} \\ & \cdot h \sum_{i=0}^{M-1} (\delta_x \omega_{i+1/2}^k) \left[\delta_x \omega_{i+1/2}^k \right. \\ & \left. - \sum_{j=1}^{k-1} (d_{k-j-1} - d_{k-j}) \delta_x \omega_{i+1/2}^j - d_{k-1} \delta_x \omega_{i+1/2}^0 \right] \\ &= -\frac{D}{\lambda} \left\{ |\delta_x \omega^k|_1^2 \right. \\ & \left. - \sum_{j=1}^{k-1} (d_{k-j-1} - d_{k-j}) \langle \delta_x \omega^k, \delta_x \omega^j \rangle \right. \\ & \left. - d_{k-1} \langle \delta_x \omega^k, \delta_x \omega^0 \rangle \right\} \leq -\frac{D}{\lambda} \left\{ |\delta_x \omega^k|_1^2 \right. \\ & \left. - \sum_{j=1}^{k-1} (d_{k-j-1} - d_{k-j}) \frac{|\delta_x \omega^k|_1^2 + |\delta_x \omega^j|_1^2}{2} \right. \\ & \left. - d_{k-1} \frac{|\delta_x \omega^k|_1^2 + |\delta_x \omega^0|_1^2}{2} \right\} = -\frac{D}{\lambda} \left\{ \frac{|\delta_x \omega^k|_1^2}{2} \right. \\ & \left. - \sum_{j=1}^{k-1} (d_{k-j-1} - d_{k-j}) \frac{|\delta_x \omega^j|_1^2}{2} - d_{k-1} \frac{|\delta_x \omega^0|_1^2}{2} \right\}, \\ & Dh \sum_{i=1}^{M-1} (\delta_x^2 \omega_i^{k-n}) (\delta_t^\alpha \omega_i^k) \\ &= -Dh \sum_{i=0}^{M-1} (\delta_x \omega_{i+1/2}^{k-n}) (\delta_t^\alpha \delta_x \omega_{i+1/2}^k) = -\frac{D}{\lambda} \end{aligned}$$

$$\begin{aligned} & \cdot h \sum_{i=0}^{M-1} (\delta_x \omega_{i+1/2}^{k-n}) \left[\delta_x \omega_{i+1/2}^k \right. \\ & \left. - \sum_{j=1}^{k-1} (d_{k-j-1} - d_{k-j}) \delta_x \omega_{i+1/2}^j - d_{k-1} \delta_x \omega_{i+1/2}^0 \right] \\ & \leq \frac{D}{\lambda} \left\{ \frac{1}{4} |\delta_x \omega^k|_1^2 + |\delta_x \omega^{k-n}|_1^2 \right. \\ & \left. + \sum_{j=1}^{k-1} (d_{k-j-1} - d_{k-j}) \frac{|\delta_x \omega^{k-n}|_1^2 + |\delta_x \omega^j|_1^2}{2} \right. \\ & \left. + d_{k-1} \frac{|\delta_x \omega^{k-n}|_1^2 + |\delta_x \omega^0|_1^2}{2} \right\} = \frac{D}{\lambda} \left\{ \frac{|\delta_x \omega^k|_1^2}{4} \right. \\ & \left. + \frac{3|\delta_x \omega^{k-n}|_1^2}{2} + \sum_{j=1}^{k-1} (d_{k-j-1} - d_{k-j}) \frac{|\delta_x \omega^j|_1^2}{2} \right. \\ & \left. + d_{k-1} \frac{|\delta_x \omega^0|_1^2}{2} \right\}. \quad (44) \end{aligned}$$

From the Cauchy-Schwarz inequality, Lemma 4, and (7), we have

$$\begin{aligned} & h \sum_{i=1}^{M-1} (\mathcal{A} q_i^k) (\delta_t^\alpha \omega_i^k) \\ & \leq \frac{1}{2\varepsilon} \|\delta_t^\alpha \omega^k\|^2 \\ & \quad + \frac{\varepsilon}{2} h \sum_{i=1}^{M-1} (\mathcal{A} [c_0 |2\omega_i^{k-1} - \omega_i^{k-2}| + c_1 |\omega_i^{k-n}|])^2 \\ & \leq \frac{1}{2\varepsilon} \|\delta_t^\alpha \omega^k\|^2 \\ & \quad + \frac{\varepsilon}{2} h \sum_{i=1}^{M-1} (c_0 |2\omega_i^{k-1} - \omega_i^{k-2}| + c_1 |\omega_i^{k-n}|)^2 \quad (45) \\ & \leq \frac{1}{2\varepsilon} \|\delta_t^\alpha \omega^k\|^2 + \varepsilon c_0^2 h \sum_{i=1}^{M-1} (2\omega_i^{k-1} - \omega_i^{k-2})^2 \\ & \quad + \varepsilon c_1^2 h \sum_{i=1}^{M-1} (\omega_i^{k-n})^2 \\ & \leq \frac{1}{2\varepsilon} \|\delta_t^\alpha \omega^k\|^2 + 8\varepsilon c_0^2 (\|\omega^{k-1}\|^2 + \|\omega^{k-2}\|^2) \\ & \quad + \varepsilon c_1^2 \|\omega^{k-n}\|^2. \end{aligned}$$

Substituting (43)–(45) into (42) and taking $\varepsilon = 3/4$ in (45), we obtain

$$\begin{aligned}
\frac{D}{4\lambda} |\delta_x \omega^k|_1^2 &\leq \frac{3D}{2\lambda} |\delta_x \omega^{k-n}|_1^2 \\
&+ \frac{D}{\lambda} \sum_{j=1}^{k-1} (d_{k-j-1} - d_{k-j}) |\delta_x \omega^j|_1^2 \\
&+ \frac{D}{\lambda} d_{k-1} |\delta_x \omega^0|_1^2 \\
&+ 6c_0^2 (\|\omega^{k-1}\|^2 + \|\omega^{k-2}\|^2) \\
&+ \frac{3}{4} c_1^2 \|\omega^{k-n}\|^2 \\
&\leq \frac{D}{\lambda} \sum_{j=1}^{k-1} (d_{k-j-1} - d_{k-j}) |\delta_x \omega^j|_1^2 \\
&+ \frac{D}{\lambda} d_{k-1} |\delta_x \omega^0|_1^2 \\
&+ c_0^2 (|\delta_x \omega^{k-1}|_1^2 + |\delta_x \omega^{k-2}|_1^2) \\
&+ \left(\frac{3D}{2\lambda} + \frac{1}{8} c_1^2 \right) |\delta_x \omega^{k-n}|_1^2,
\end{aligned} \tag{46}$$

where Lemma 5 has been used.

Multiplying (46) by $4\lambda/D$, we have

$$\begin{aligned}
|\delta_x \omega^k|_1^2 &\leq 4 \sum_{j=1}^{k-1} (d_{k-j-1} - d_{k-j}) |\delta_x \omega^j|_1^2 + 4 |\delta_x \omega^0|_1^2 \\
&+ \frac{4\lambda}{D} c_0^2 (|\delta_x \omega^{k-1}|_1^2 + |\delta_x \omega^{k-2}|_1^2) \\
&+ \left(6 + \frac{\lambda}{2D} c_1^2 \right) |\delta_x \omega^{k-n}|_1^2.
\end{aligned} \tag{47}$$

Denote

$$C_k = \max \left\{ \frac{4\Gamma(2-\alpha)}{D} c_0^2, \frac{\Gamma(2-\alpha)}{2D} c_1^2 + 6 \right\} > 0. \tag{48}$$

Noticing $\lambda = \tau^\alpha \Gamma(2-\alpha)$ for $0 < \tau < 1$, we have $\tau^\alpha < 1$. Then, from (47), we obtain

$$\begin{aligned}
|\delta_x \omega^k|_1^2 &\leq 4 \sum_{j=1}^{k-1} (d_{k-j-1} - d_{k-j}) |\delta_x \omega^j|_1^2 + 4 |\delta_x \omega^0|_1^2 \\
&+ C_k (|\delta_x \omega^{k-1}|_1^2 + |\delta_x \omega^{k-2}|_1^2 + |\delta_x \omega^{k-n}|_1^2).
\end{aligned} \tag{49}$$

From Lemmas 2 and 6, we have

$$\begin{aligned}
|\delta_x \omega^k|_1^2 &\leq (3C_k + 4) \exp(3C_k + 4(1 - d_{k-1})) \max_{-n \leq j \leq 0} |\delta_x \omega^j|_1^2 \\
&\leq (3C_k + 4) \exp(3C_k + 4) \max_{-n \leq j \leq 0} |\delta_x \psi^j|_1^2.
\end{aligned} \tag{50}$$

From Lemma 5, we have

$$\|\omega^k\|_\infty \leq \widehat{C} \max_{-n \leq j \leq 0} |\psi^j|_1. \tag{51}$$

The proof is completed. $\square$

Denote $e_i^k = U_i^k - u_i^k$, $0 \leq i \leq M$, $-n \leq k \leq N$; by subtracting (24)–(26) from (19), (22), and (23), respectively, the following error equations can be obtained:

$$\begin{aligned}
\mathcal{A} \delta_t^\alpha e_i^k &= D (\delta_x^2 e_i^k + \delta_x^2 e_i^{k-n}) + \mathcal{A} p_i^k + R_i^k, \\
e_i^k &= 0, \quad 0 \leq i \leq M, \quad -n \leq k \leq 0, \\
e_0^k &= 0, \\
e_M^k &= 0,
\end{aligned} \tag{52}$$

$1 \leq k \leq N$,

where $p_i^k = f(2U_i^{k-1} - U_i^{k-2}, U_i^{k-n}, x_i, t_k) - f(2u_i^{k-1} - u_i^{k-2}, u_i^{k-n}, x_i, t_k)$.

Similar to the proof of Theorem 9, the following convergence result can be obtained.

Theorem 10. Assume $u(x, t) \in C^{6,2}([0, 1] \times (-s, T])$ is the solution of (3)–(5) and $\{u_i^k \mid 0 \leq i \leq M, -n \leq k \leq N\}$ is the solution of (24)–(26). Then, one has

$$\|e^k\|_\infty \leq C (\tau^{2-\alpha} + h^4), \quad 1 \leq k \leq N, \tag{53}$$

where C is a positive constant independent of h and τ .

4. Numerical Test

In this section, a numerical test is used to validate the performance of scheme (24)–(26). Denote the maximum error at all grid points as

$$e(h, \tau) = \max_{1 \leq k \leq N} \|U^k - u^k\|_\infty; \tag{54}$$

the convergence order in time and space is defined, respectively, as

$$\begin{aligned}
\text{Rate}_\tau &= \frac{\log(e(h, \tau_1)/e(h, \tau_2))}{\log(\tau_1/\tau_2)}, \\
\text{Rate}_h &= \frac{\log(e(h_1, \tau)/e(h_2, \tau))}{\log(h_1/h_2)}.
\end{aligned} \tag{55}$$

For Rate_τ , we require h to be fixed and small enough, while for Rate_h , τ should be fixed and small enough.

FIGURE 1: Error planes for $\tau = h = 1/100$ for different α in Example 1.

Example 1. Consider the following problem:

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= \left(\frac{\partial^2 u(x, t)}{\partial x^2} + \frac{\partial^2 u(x, t-0.1)}{\partial x^2} \right) - u(x, t)^2 \\ &\quad + u(x, t-0.1) + G(x, t), \\ (x, t) &\in (0, 1) \times (0, 1], \\ u(x, t) &= e^x x^2 (1-x)^2 t^{2+\alpha}, \\ x &\in [0, 1], \quad t \in [-0.1, 0], \\ u(0, t) &= 0, \\ u(1, t) &= 0, \\ t &\in (0, 1]; \end{aligned} \quad (56)$$

the exact solution of (56) is $u(x, t) = e^x x^2 (1-x)^2 t^{2+\alpha}$, and

$$\begin{aligned} G(x, t) &= e^{2x} x^4 (1-x)^4 t^{4+2\alpha} - e^x x^2 (1-x)^2 \\ &\quad \cdot (t-0.1)^{2+\alpha} + \frac{\Gamma(3+\alpha)}{2} t^2 e^x x^2 (1-x)^2 \end{aligned}$$

$$\begin{aligned} &- e^x (2 - 8x + x^2 + 6x^3 + x^4) \\ &\quad \cdot (t^{2+\alpha} + (t-0.1)^{2+\alpha}). \end{aligned} \quad (57)$$

From Table 1, we can see the maximum errors between the numerical solution and the exact solution in the temporal directions for $\alpha = 0.3, 0.5, 0.8$, respectively, where the spatial step is fixed to be $h = 1/400$. The results show that the temporal convergence order matches well the theoretical convergence order of $2 - \alpha$.

Table 2 shows the maximum errors in the spatial directions for $\alpha = 0.2$ when the temporal step is fixed at $\tau = 1/2000$. From the results, we can see that the spatial convergence order is 4, which coincides with the theoretical result.

Figure 1 gives the error plane for $\alpha = 0.2, 0.4, 0.6, 0.8$, respectively. From this figure, we can see that the error becomes larger when a larger α is taken.

5. Conclusion

This paper presents a compact finite difference scheme for solving the fractional neutral parabolic differential equation with proportional delay. The unconditional stability and the

TABLE 1: Maximum errors and convergence order in temporal direction with $h = 1/400$ and $\alpha = 0.3, 0.5, 0.8$ for Example 1.

τ	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.8$	
	$e(h, \tau)$	Rate_τ	$e(h, \tau)$	Rate_τ	$e(h, \tau)$	Rate_τ
1/100	$1.083e - 006$	*	$4.502e - 006$	*	$3.800e - 005$	*
1/200	$3.234e - 007$	1.744	$1.571e - 006$	1.519	$1.651e - 005$	1.203
1/300	$1.598e - 007$	1.739	$8.501e - 007$	1.515	$1.014e - 005$	1.202
1/400	$69.694e - 008$	1.737	$5.501e - 007$	1.513	$7.176e - 006$	1.202

TABLE 2: Maximum errors and convergence order in spatial direction with $\tau = 1/2000$ and $\alpha = 0.2$ for Example 1.

h	$e(h, \tau)$	Rate_h
1/8	$7.949e - 005$	*
1/12	$1.578e - 005$	3.988
1/16	$4.976e - 006$	4.011
1/20	$2.045e - 006$	3.985

global convergence of the scheme in the maximum norm are proved. The convergence order of the considered scheme is $O(\tau^{2-\alpha} + h^4)$. A numerical experiment is presented to support the theoretical results and validate the efficiency of the difference scheme.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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