Research Article

Boundedness for Commutators of Bilinear $\theta$-Type Calderón-Zygmund Operators on Nonhomogeneous Metric Measure Spaces

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Let $(X, d, \mu)$ be a nonhomogeneous metric measure space. In this paper, the boundedness for commutators generated by bilinear $\theta$-type Calderón-Zygmund operators and RBMO($\mu$) functions on $(X, d, \mu)$ is obtained.

1. Introduction

As we know, Hytönen [1] introduced nonhomogeneous metric measure spaces, which include both spaces of homogeneous type and nondoubling measure spaces as special cases. From then on, many results on nonhomogeneous metric measure spaces are obtained by many researchers. Hytönen et al. [2] and Bui and Duong [3] introduced independently the atomic Hardy space $H^1(\mu)$ and proved that the dual space of $H^1(\mu)$ is RBMO($\mu$). The authors in [3] also proved that Calderón-Zygmund operator and commutators are bounded in $L^p(\mu)$ for $1 < p < \infty$. Recently, some equivalent characterizations were established by Liu et al. [4] for the boundedness of Calderón-Zygmund operators on $L^p(\mu)$ for $1 < p < \infty$. Fu et al. [5, 6] established boundedness of multilinear commutators of Calderón-Zygmund operators and generalized fractional integrals on Orlicz spaces, respectively. For more results, one can refer to [2, 4, 7–15] and the references therein.

$\theta$-type Calderón-Zygmund operator was firstly introduced by Yabuta [16] in 1985. Later, many researchers further studied the properties of this operator. We [17] obtained the boundedness of $\theta$-type Calderón-Zygmund operator and commutators on nondoubling measure spaces. Ri et al. [18, 19] researched the boundedness of $\theta$-type Calderón-Zygmund operator on Hardy spaces with nondoubling measures and nonhomogeneous metric measure spaces, respectively. Zheng et al. [20, 21] studied the bounded properties for bilinear $\theta$-type Calderón-Zygmund operator and maximal bilinear $\theta$-type Calderón-Zygmund operator on nonhomogeneous metric measure spaces, respectively.

In this paper, we prepare to study the boundedness for commutators generated by bilinear $\theta$-type Calderón-Zygmund operators and RBMO($\mu$) functions on $(X, d, \mu)$ and obtain that these commutators are bounded on Lebesgue spaces, provided that bilinear $\theta$-type Calderón-Zygmund operator is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{1/2,\infty}(\mu)$, where $L^p(\mu)$ and $L^{p,\infty}(\mu)$ denote the Lebesgue spaces and weak Lebesgue spaces, respectively. This result includes the corresponding results on both spaces of homogeneous type and nondoubling measure spaces. It is even new in the settings of spaces of homogeneous type and nondoubling measure spaces.

Throughout this paper, $L^{\infty}(\mu)$ denotes the set of all $L^{\infty}(\mu)$ functions with compact support. $C$ always denotes a positive constant independent of the main parameters involved, but it may vary from line to line. And $p'$ is the conjugate index of $p$; namely, $1/p + 1/p' = 1$. Now, let us recall some definitions and terminologies.
Definition 1 (see [1]). A metric space \((X, d)\) is geometrically doubling if there exists some \(N_0 \in \mathbb{N}\) such that, for every ball \(B(x, r) \subset X\), there exists a finite ball covering \([B(x, r/2)]\), of \(B(x, r)\) such that the cardinality of this covering is at most \(N_0\).

Definition 2 (see [1]). A metric measure space \((X, d, \mu)\) is upper doubling if \(\mu\) is a Borel measure on \(X\) and there exists a function \(\lambda: X \times (0, +\infty) \to (0, +\infty)\) and a constant \(C_\lambda > 0\) such that, for every \(x \in X, r \mapsto (x, r)\) is nondecreasing, and for any \(x \in X, r > 0\),

\[
\mu(B(x, r)) \leq \lambda(x, r) \leq C_\lambda \lambda(x, r/2).
\]

(1)

Remark 3. (i) Spaces of homogeneous type are upper doubling space, if we take \(\lambda(x, r) = \mu(B(x, r))\). Also, non-doubling measure space, which satisfies the following polynomial growth condition:

\[
\mu(B(x, r)) \leq C r^n
\]

for all \(x \in \mathbb{R}^d\) and \(r > 0\), is also upper doubling measure space if we take \(\lambda(x, r) = C r^n\).

(ii) The authors [11] showed that there exists another function \(\tilde{\lambda}\) such that, for any \(x, y \in X, d(x, y) \leq r\),

\[
\tilde{\lambda}(x, r) \leq C \tilde{\lambda}(y, r).
\]

(3)

Thus, one assumes that \(\lambda\) always satisfies (3) in this paper. As the singularity of commutators is stronger than that of bilinear operators, by [22], we suppose that there exists \(0 < m < +\infty\), such that, for any \(a, r > 0, x \in X\),

\[
\lambda(x, ar) \geq a^m \lambda(x, r).
\]

(4)

Let \(\alpha, \beta \in (1, +\infty);\) a ball \(B \subset X\) is \((\alpha, \beta)\)-doubling if \(\mu(\alpha B) \leq \beta \mu(B)\). As pointed in Lemma 2.3 of [3], there exist plenty of doubling balls with small radii and with large radii. In this paper, unless \(\alpha\) and \(\beta\) are specified otherwise, one means \((\alpha, \beta)\) doubling ball is \((6,\beta_0)\)-doubling with a fixed number \(\beta_0 > \max\{C_\lambda^{-\log_6 \beta_0}\},\) where \(n = \log_2 N_0\) is the geometric dimension of the space.

Definition 4 (see [3]). For two balls \(B \subset Q\), let \(N_{BQ}\) be the smallest integer such that \(6^{N_{BQ}r_B} \geq r_Q;\) we denote

\[
K_{BQ} = 1 + \sum_{k=1}^{N_{BQ}} \mu\left(6^kB\right)/\lambda(6^k r_B).
\]

(5)

Let \(\theta\) be a nonnegative nondecreasing function on \((0, +\infty)\) satisfying

\[
\int_0^\theta \frac{\theta(t)}{t} |\log t| \, dt < \infty.
\]

(6)

Definition 5. A kernel \(K(, , \cdot) \in L^1_{\text{loc}}(X^2 \setminus \{(x, y_1, y_2) : x = y_1 = y_2\})\) is called the bilinear \(\theta\)-type Calderón-Zygmund kernel if it satisfies the following:

\[
|K(x, y_1, y_2)| \leq C \left[\sum_{j=1}^2 \lambda\left(x, d\left(x, y_j\right)\right)\right]^{-2}
\]

for all \((x, y_1, y_2) \in X^3\) with \(x \neq y_j\) for \(j \in \{1, 2\}\).

(ii)

\[
|K(x, y_1, y_2) - K\left(x', y_1, y_2\right)|
\]

\[
\leq C \theta\left(\frac{d\left(x, x'\right)}{\sum_{j=1}^2 \lambda\left(x, d\left(x, y_j\right)\right)}\right) \left[\sum_{j=1}^2 \lambda\left(x, d\left(x, y_j\right)\right)\right]^{-2}
\]

(8)

provided that \(Cd(x, x') \leq \max_{1 \leq j \leq 2} d(x, y_j)\).

A bilinear operator \(T_\theta\) is called bilinear \(\theta\)-type Calderón-Zygmund operator with the above kernel \(K\) if for \(f_1, f_2 \in L^1_{\text{loc}}\) and \(x \notin \bigcap_{j=1}^2 \text{supp} f_j\),

\[
T_\theta(f_1, f_2)(x)
\]

\[
= \int_{X^2} K(x, y_1, y_2) f_1(y_1) f_2(y_2) \, d\mu(y_1) \, d\mu(y_2).
\]

(9)

Remark 6. As \(\max_{1 \leq j \leq 2} d(x, y_j) \leq \sum_{j=1}^2 d(x, y_j) \leq 2 \max_{1 \leq j \leq 2} d(x, y_j)\) (ii) in Definition 5 is equivalent to (ii)' in the following statement:

\[
|K(x, y_1, y_2) - K\left(x', y_1, y_2\right)|
\]

\[
\leq C \theta\left(\frac{d\left(x, x'\right)}{\max_{1 \leq j \leq 2} d\left(x, y_j\right)}\right) \left[\sum_{j=1}^2 \lambda\left(x, d\left(x, y_j\right)\right)\right]^{-2}
\]

(10)

provided that \(Cd(x, x') \leq \max_{1 \leq j \leq 2} d(x, y_j)\).

Remark 7. (i) In [20, 21], the term \([\sum_{j=1}^2 \lambda(x, d(x, y_j))]^{-2}\) in (7) and (8) of this paper is substituted by \(\min_{j \in \{1, 2\}} [\lambda(x, d(x, y_j))]^{-2}\). In fact, as

\[
\min_{j \in \{1, 2\}} \left[\lambda\left(x, d\left(x, y_j\right)\right)\right]^{-2}
\]

(11)

and \(\lambda(x, \max_{j \in \{1, 2\}} d(x, y_j)) \leq \sum_{j=1}^2 \lambda(x, d(x, y_j)) \leq 2\lambda(x, \max_{j \in \{1, 2\}} d(x, y_j))\), Definition 5 in this paper is equivalent to Definition 1.4 in [20] or Definition 1.3 in [21]. Therefore, we can directly quote the result of Theorem 1.5 in [20] as Lemma 17 below in this paper.

(ii) Because we assume that \(T_\theta\) is bounded from \(L^1(\mu) \times L^1(\mu)\) to \(L^{1/2\infty}(\mu)\), it is enough to assume that \(K\) satisfies the regularity condition on the first variable, that is, (8) in this paper for getting the result of Theorem 10 below. For more details, one can refer to Remark 1.1 in [9].
Definition 8. The commutator generated by bilinear $\theta$-type Calderón-Zygmund operator $T_\theta$ and $b_1, b_2 \in \text{RBMO}(\mu)$ is defined by

$$ [b_1, b_2, T_\theta] (f_1, f_2) (x) = b_1 (x) b_2 (x) T_\theta (f_1, f_2) (x) - b_1 (x) T_\theta (b_1 f_1, f_2) (x) - b_2 (x) T_\theta (b_1 f_1, b_2 f_2) (x). $$

Also, $[b_1, T_\theta]$ and $[b_2, T_\theta]$ are defined as follows, respectively:

$$ [b_1, T_\theta] (f_1, f_2) (x) = b_1 (x) T_\theta (f_1, f_2) (x), $$

$$ [b_2, T_\theta] (f_1, f_2) (x) = b_2 (x) T_\theta (f_1, f_2) (x). $$

Definition 9 (see [2]). Let $\rho > 1$ be some fixed constant. A function $b \in L^1_{\text{loc}}(\mu)$ is said to belong to $\text{RBMO}(\mu)$ if there exists a constant $C > 0$ such that, for any ball $B$,

$$ \frac{1}{|B|} \int_B |b(x) - m_B(b)| \, d\mu(x) \leq C, $$

and for any two doubling balls $B \subset Q$,

$$ |m_B(b) - m_Q(b)| \leq CK_{BQ}, $$

where $B$ is the smallest $(\alpha, \beta)$-doubling ball of the form $\delta^k B$ with $k \in \mathbb{N} \cup \{0\}$, and $m_B(b)$ is the mean value of $b$ on $B$; namely,

$$ m_B(b) = \frac{1}{|B|} \int_B b(x) \, d\mu(x). $$

The minimal constant $C$ in (15) and (16) is the $\text{RBMO}(\mu)$ norm of $b$, which is denoted by $\|b\|_{\text{RBMO}}$.

Theorem 10. Let $1 < p_1, p_2 < +\infty$, $1/q = 1/p_1 + 1/p_2$, and $b \in \text{RBMO}(\mu)$. Assume that $f_1 \in L^p(\mu)$, $f_2 \in L^{p/2}(\mu)$ with $\int_X T_\theta (f_1, f_2) (x) \, d\mu(x) = 0$. Then, there exists a constant $C > 0$ such that

$$ \left\| [b_1, b_2, T_\theta] (f_1, f_2) \right\|_{L^q(\mu)} \leq C \left\| f_1 \right\|_{L^p(\mu)} \left\| f_2 \right\|_{L^{p/2}(\mu)}. $$

Remark 1. The result of Theorem 10 is still valid for commutators of multilinear $\theta$-type Calderón-Zygmund operators with $\text{RBMO}(\mu)$ functions.

2. Preliminaries

For any $f \in L^1_{\text{loc}}(\mu)$, the noncentered doubling maximal operator is defined by

$$ Nf(x) = \frac{1}{|B|} \int_B |f(y)| \, d\mu(y), $$

and the sharp maximal operator $M^\sharp$ is denoted by

$$ M^\sharp f(x) = \frac{1}{|B|} \int_B |f(y) - m_B(f)| \, d\mu(y) + \frac{1}{K_{BQ}} \int_B |m_B(f) - m_Q(f)|, $$

where $\Delta = ((B, Q) : x \in B \subset Q$ and $B, Q$ are doubling balls).

Let $\rho > 1$, $p \in (1, \infty)$, and $s \in (1, p)$; the noncentered maximal operator $M_{s(p)} f$ is defined by

$$ M_{s(p)} f(x) = \frac{1}{\mu(B)} \int_B |f(y)|^{s(p)} \, d\mu(y)^{1/s(p)}. $$

When $s = 1$, we simply write $M_{1(p)} f(x)$ as $M_{s(p)} f$.

Lemma 12 (see [3, 11]). (i) For any $f \in L^1_{\text{loc}}(\mu)$ and $\mu - a.e. x \in X$,

$$ |f(x)| \leq N_\rho f(x). $$

(ii) If $\rho > 5$, then the operator $M_{s(p)} f$ is bounded on $L^p(\mu)$ for $p > 1$ and $M_{s(p)} f$ is bounded on $L^p(\mu)$ for $p > s > 1$.

Lemma 13 (see [3, 5]). Assume that $f \in L^1_{\text{loc}}(\mu)$ with $\int_X f(x) \, d\mu(x) = 0$ if $\|\mu\| < \infty$. For $1 < p < \infty$ and $0 < \eta < 1$, if $\text{inf}(1, N_\eta f) \in L^p(\mu)$, then there exists a constant $C > 0$ such that

$$ \left\| N_\eta f \right\|_{L^p(\mu)} \leq C \left\| M^\sharp f \right\|_{L^p(\mu)}. $$

Lemma 14 (see [5, 23]). Suppose that $1 \leq p < \infty$ and $1 < \rho < \infty$. Then $b \in \text{RBMO}(\mu)$ if and only if for any ball $B \subset X$,

$$ \left\{ \frac{1}{\mu(B)} \int_B |b_B - m_B(b)|^p \, d\mu(x) \right\}^{1/p} \leq C \|b\|_{\text{RBMO}}, $$

and for any two doubling balls $B \subset Q$,

$$ |m_B(b) - m_Q(b)| \leq CK_{BQ} \|b\|_{\text{RBMO}}. $$

Lemma 15 (see [51]). For any $k \in \mathbb{N}^*$,

$$ \left| m_{k\delta^k(B)} f - m_B(b) \right| \leq CK \|b\|_{\text{RBMO}}. $$

Lemma 16 (Kolmogorov's theorem). Let $(X, \mu)$ be a probability measure space and let $0 < p < \infty$; then there exists a constant $C > 0$ such that $\|f\|_{L^p(\mu)} \leq C \|f\|_{L^p(\mu)}$ for any measurable function $f$.

Lemma 17 (see [20]). Let $1 < p_1, p_2 < +\infty$, $1/q = 1/p_1 + 1/p_2$, $f_1 \in L^p(\mu)$, and $f_2 \in L^{p/2}(\mu)$. If $T_\theta f$ is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{1/2, \infty}(\mu)$, then there exists a constant $C > 0$ such that

$$ \left\| T_\theta (f_1, f_2) \right\|_{L^{1/2, \infty}(\mu)} \leq C \left\| f_1 \right\|_{L^p(\mu)} \left\| f_2 \right\|_{L^{p/2}(\mu)}. $$
3. Proof of Main Result

Lemma 18. Suppose that \( 0 < \eta < 1/2, 1 < p_1, p_2, q < \infty, 1 < s < q \) and \( b_1, b_2 \in \text{RBO}^{(s)}(\mu) \). If \( T_\theta \) is bounded from \( L^p(\mu) \times L^q(\mu) \) to \( L^{(1/s, \infty)}(\mu) \), then there exists a constant \( C > 0 \) such that, for any \( x \in \mathcal{F} \), \( f_1 \in L^{p_1}(\mu) \), and \( f_2 \in L^{p_2}(\mu) \),

\[
M_{s,\eta}^b [b_1, b_2, T_\theta] (f_1, f_2) (x) \\
\leq C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(T_\theta (f_1, f_2)) (x) \\
+ C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(M_{p_1,5}(f_1) M_{p_2,5} f_2) (x) \\
+ C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(M_{p_1,5}(f_1) M_{p_2,5} f_2) (x) \\
+ C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(M_{p_1,5}(f_1) M_{p_2,5} f_2) (x) \\
\leq C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(T_\theta (f_1, f_2)) (x) \\
+ C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(M_{p_1,5}(f_1) M_{p_2,5} f_2) (x) \\
\leq C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(T_\theta (f_1, f_2)) (x) \\
+ C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(M_{p_1,5}(f_1) M_{p_2,5} f_2) (x) \\
\leq C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(T_\theta (f_1, f_2)) (x) \\
+ C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(M_{p_1,5}(f_1) M_{p_2,5} f_2) (x)
\]

(29) \begin{align*}
M_{s,\eta}^b [b_1, b_2, T_\theta] (f_1, f_2) (x) \\
\leq C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(T_\theta (f_1, f_2)) (x) \\
+ C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(M_{p_1,5}(f_1) M_{p_2,5} f_2) (x) \\
\leq C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(T_\theta (f_1, f_2)) (x) \\
+ C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(M_{p_1,5}(f_1) M_{p_2,5} f_2) (x)
\end{align*}

(29)

(30)

\[
M_{s,\eta}^b [b_1, b_2, T_\theta] (f_1, f_2) (x) \\
\leq C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(T_\theta (f_1, f_2)) (x) \\
+ C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(M_{p_1,5}(f_1) M_{p_2,5} f_2) (x)
\]

(30)

\[
M_{s,\eta}^b [b_1, b_2, T_\theta] (f_1, f_2) (x) \\
\leq C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(T_\theta (f_1, f_2)) (x) \\
+ C \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(M_{p_1,5}(f_1) M_{p_2,5} f_2) (x)
\]

(31)

Proof. Because \( L^{(1/s, \infty)}(\mu) \) is dense in \( L^{p}(\mu) \) for \( 1 < p < \infty \), we only consider the situation of \( f_1, f_2 \in L^{(1/s, \infty)}(\mu) \). Also, by Lemma 3.11 in [5], we can assume that \( b_1, b_2 \in L^{s}(\mu) \). As it has the similar method to estimate (29), (30), and (31), here we only estimate (29) for compactly.

To obtain (29), with the similar way to prove Theorem 9.1 in [24], it suffices to show that

\[
\left( \frac{1}{\mu (6\mathcal{B})} \right) \int_{\mathcal{B}} \| [b_1, b_2, T_\theta] (f_1, f_2) (z) \|^p \, d\mu (z) \\
\leq C \left( \frac{1}{\mu (6\mathcal{B})} \right) \int_{\mathcal{B}} \| [b_1, b_2, T_\theta] (f_1, f_2) (z) \|^p \, d\mu (z) \right)^{1/p}
\]

(32)

for any \( x \in \mathcal{B} \), and

\[
|h_B - h_Q| \\
\leq C K^2 \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(T_\theta (f_1, f_2)) (x) \\
+ \|b_1\|_s \|b_2\|_s \, M_{s,\eta}(M_{p_1,5}(f_1) M_{p_2,5} f_2) (x)
\]

(33)
For $I$. Let $s_1, s_2 > 1$ such that $1/s_1 + 1/s_2 + 1/s = 1/\eta$. By Hölder’s inequality and Lemma 14,

\[
I \leq C \left( \frac{1}{\mu(6B)} \right) \int_B \left| b_i (z) - m_{\bar{b}} (b_i) \right|^s d\mu(z) \right)^{1/s_1} \\
\times \left( \frac{1}{\mu(6B)} \right) \int_B \left| b_2 (z) - m_{\bar{b}} (b_2) \right|^s d\mu(z) \right)^{1/s_2} \\
\times \left( \frac{1}{\mu(6B)} \right) \int_B \left| T_\theta (f_1, f_2) \right|^s d\mu(z) \right)^{1/s} \\
\leq C \| b_1 \|_s, M_{s,(6)} \left( f_1, f_2 \right) (x) .
\]  

(37)

Let us estimate $II$, let $t > 1$ such that $1/t + 1/s = 1/\eta$, and then

\[
II \leq C \left( \frac{1}{\mu(6B)} \right) \int_B \left| b_1 (z) - m_{\bar{b}} (b_1) \right|^t d\mu(z) \right)^{1/t} \\
\times \left( \frac{1}{\mu(6B)} \right) \int_B \left| b_2 (z) - m_{\bar{b}} (b_2) \right|^t d\mu(z) \right)^{1/t} \\
\times \left( \frac{1}{\mu(6B)} \right) \int_B \left| T_\theta (f_1, f_2) \right|^t d\mu(z) \right)^{1/t} \\
\leq C \| b_1 \|_s, M_{s,(6)} \left( f_1, f_2 \right) (x) .
\]  

(38)

Similar to estimate $II$,

\[
III \leq C \| b_2 \|_s, M_{s,(6)} \left( f_1, f_2 \right) (x) .
\]  

(39)

Let us turn to estimate $IV$. Let $f_j^i = f_j \chi_{(6/5)B} - f_j^{-1} = f_j f_j^{-1}$ for $j \in \{1, 2\}$; then

\[
IV \leq C \left( \frac{1}{\mu(6B)} \right) \\
\cdot \int_B \left| T_\theta \left( \left( b_1 - m_{\bar{b}} (b_1) \right) f_1^i (z), (b_2 - m_{\bar{b}} (b_2)) f_2^i \right) \right|^s d\mu(z) \\
\cdot \left( z \right)^s d\mu(z) \right)^{1/\eta} + \left( \frac{1}{\mu(6B)} \right) \\
\cdot \int_B \left| T_\theta \left( \left( b_1 - m_{\bar{b}} (b_1) \right) f_1^i (z), (b_2 - m_{\bar{b}} (b_2)) f_2^i \right) \right|^s d\mu(z) \\
\cdot \left( z \right)^s d\mu(z) \right)^{1/\eta} + \left( \frac{1}{\mu(6B)} \right) \\
\cdot \int_B \left| T_\theta \left( \left( b_1 - m_{\bar{b}} (b_1) \right) f_1^i (z), (b_2 - m_{\bar{b}} (b_2)) f_2^i \right) \right|^s d\mu(z) \\
\cdot \left( z \right)^s d\mu(z) \right)^{1/\eta} + \left( \frac{1}{\mu(6B)} \right) \\
\cdot \int_B \left| T_\theta \left( \left( b_1 - m_{\bar{b}} (b_1) \right) f_1^i (z), (b_2 - m_{\bar{b}} (b_2)) f_2^i \right) \right|^s d\mu(z) \\
\cdot \left( z \right)^s d\mu(z) \right)^{1/\eta} + \left( \frac{1}{\mu(6B)} \right) \\
\cdot \int_B \left| T_\theta \left( \left( b_1 - m_{\bar{b}} (b_1) \right) f_1^i (z), (b_2 - m_{\bar{b}} (b_2)) f_2^i \right) \right|^s d\mu(z) \\
\cdot \left( z \right)^s d\mu(z) \right)^{1/\eta} + \left( \frac{1}{\mu(6B)} \right) \\
\cdot \int_B \left| T_\theta \left( \left( b_1 - m_{\bar{b}} (b_1) \right) f_1^i (z), (b_2 - m_{\bar{b}} (b_2)) f_2^i \right) \right|^s d\mu(z) \\
\cdot \left( z \right)^s d\mu(z) \right)^{1/\eta} = IV_1 + IV_2 + IV_3 + IV_4 .
\]  

For $IV_1$, let $p = \eta$ and $q = 1/2$ such that $0 < \eta < 1/2$. Using Kolmogorov’s theorem, Hölder’s inequality, Lemma 14, and the boundedness from $L^1(\mu) \times L^1(\mu)$ to $L^{1/2, \infty}(\mu)$ of $T_\theta$,

\[
IV_1 \leq C \cdot \frac{1}{\mu(6B)} \left( \left| b_1 - m_{\bar{b}} (b_1) \right| \right| f_1 (z) \right| d\mu(z) \\
\times \left( \left| b_2 - m_{\bar{b}} (b_2) \right| \right| f_2 (z) \right| d\mu(z) \right) \leq C
\]  

(41)

To estimate $IV_2$, by Definition 5, Lemmas 14 and 15, Hölder’s inequality, and some properties of $\lambda$,

\[
IV_2 \leq C \left( \frac{1}{\mu(6B)} \right) \\
\cdot \int_B \left[ \left( \left( b_1 - m_{\bar{b}} (b_1) \right) \right| \left| f_1 (z) \right| \right] d\mu(z) \\
\cdot \int_B \left( \left| b_2 - m_{\bar{b}} (b_2) \right| \right| f_2 (z) \right| d\mu(z) \right) \leq C \| b_1 \|_s, M_{s,(5)} \left( f_1 \right) (x) .
\]
\[
\times \int_{\phi^{(1)}(6/5)B} \left| b_2 (y_2) - m_{\phi^{(6/5)B}} (b_2) \right|^{p_2} \, d\mu (y_2) \right)^{1/p_2} \\
\times \left( \frac{1}{\mu (5 \times 6^{k} (6/5) B)} \right) \\
\cdot \int_{\phi^{(6/5)B}} \left| f_2 (y_2) \right| \, d\mu (y_2) \right)^{1/p_2} + C k \left\| b_2 \right\|_\ast
\]

\[
\cdot \frac{1}{\mu (5 \times 6^{k} (6/5) B)} \int_{\phi^{(6/5)B}} \left| f_2 (y_2) \right| \, d\mu (y_2) \right) \\
\leq C \left\| b_2 \right\|_\ast \left\| b_2 \right\|_\ast M_{p,\ast (5)} f_1 (x) M_{p,\ast (5)} f_2 (x) .
\]

(42)

Similarly, we obtain

\[
IV_3 \leq C \left\| b_2 \right\|_\ast \left\| b_2 \right\|_\ast M_{p,\ast (5)} f_1 (x) M_{p,\ast (5)} f_2 (x) .
\]

(43)

For \( IV_4 \), by (ii) of Definition 5 and some properties of \( \lambda \),

\[
\left| T_\delta \left( (b_1 - m_{\tilde{B}} (b_1)) f_1^2 \right) + (b_2 - m_{\tilde{B}} (b_2)) f_2^2 \right| (z) - T_\delta \left( (b_1 - m_{\tilde{B}} (b_1)) f_1^2 \right) + (b_2 - m_{\tilde{B}} (b_2)) f_2^2 \right) (z_0) \\
\leq C \int_{\chi (6/5)B} \left| K (z, y_1, y_2) - K (z_0, y_1, y_2) \right| d\mu (y_1) \, d\mu (y_2) \\
\leq C \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \int_{\phi^{(6/5)B} \cap \phi^{(6/5)B} \cap \phi^{(6/5)B}} \theta \left( \frac{d (z, z_0)}{d (z, y_1)} \right) \left| (b_1 (y_1) - m_{\tilde{B}} (b_1)) f_1 (y_1) \right| \\
\times \int_{\phi^{(6/5)B} \setminus \phi^{(6/5)B}} \frac{\left| (b_2 (y_2) - m_{\tilde{B}} (b_2)) f_2 (y_2) \right|}{\left[ \lambda (z, d (z, y_2)) \right]^2} \, d\mu (y_2) \, d\mu (y_1) \\
+ C \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \int_{\phi^{(6/5)B} \cap \phi^{(6/5)B} \cap \phi^{(6/5)B}} \theta \left( \frac{d (z, z_0)}{d (z, y_2)} \right) \left| (b_2 (y_2) - m_{\tilde{B}} (b_2)) f_2 (y_2) \right| \\
\times \int_{\phi^{(6/5)B} \setminus \phi^{(6/5)B}} \frac{\left| (b_1 (y_1) - m_{\tilde{B}} (b_1)) f_1 (y_1) \right|}{\left[ \lambda (z, d (z, y_1)) \right]^2} \, d\mu (y_1) \, d\mu (y_2) = IV_{41} + IV_{42} .
\]

Let us estimate \( IV_{41} \). With the help of the fact that by Lemmas 14 and 15 and Hölder's inequality, we have

\[
\int_{0}^{1} \frac{\theta (t)}{t} | \log t | \, dt \geq \sum_{k=1}^{\infty} \int_{6^{-k}}^{6^{-k+1}} \frac{\theta \left( 6^{-k} \right)}{6^{-k}} \, | \log 6^{-k} | \, dt \\
\geq C \sum_{k=1}^{\infty} k \theta \left( 6^{-k} \right) ,
\]

(45)

\[
IV_{41} \leq C \sum_{k=1}^{\infty} \frac{1}{\left[ \lambda (z, d (6^{k} (6/5) r_{\tilde{B}})) \right]^2} \int_{\phi^{(6/5)B} \setminus \phi^{(6/5)B}} \theta \left( 6^{-k} \right) \left| (b_1 (y_1) - m_{\tilde{B}} (b_1)) f_1 (y_1) \right| \, d\mu (y_1) \\
\times \sum_{j=1}^{k} \int_{\phi^{(6/5)B} \setminus \phi^{(6/5)B} \setminus \phi^{(6/5)B}} \left| (b_2 (y_2) - m_{\tilde{B}} (b_2)) f_2 (y_2) \right| \, d\mu (y_2) \leq C \sum_{k=1}^{\infty} \theta \left( 6^{-k} \right) \frac{1}{\left[ \lambda (z, d (6^{k} (6/5) r_{\tilde{B}})) \right]^2} \\
\cdot \int_{\phi^{(6/5)B} \setminus \phi^{(6/5)B}} \left| (b_1 (y_1) - m_{\tilde{B}} (b_1)) f_1 (y_1) \right| \, d\mu (y_1) \times \int_{\phi^{(6/5)B} \setminus \phi^{(6/5)B}} \left| (b_2 (y_2) - m_{\tilde{B}} (b_2)) f_2 (y_2) \right| \, d\mu (y_2) \leq C \sum_{k=1}^{\infty} \theta \left( 6^{-k} \right) \\
\cdot \frac{2}{\mu (5 \times 6^{k} (6/5) B)} \int_{\phi^{(6/5)B}} \left| b_1 (y_1) - m_{\tilde{B}} (b_1) \right| \, d\mu (y_1) \right)^{1/p_1} \\
\times \left( \frac{1}{\mu (5 \times 6^{k} (6/5) B)} \int_{\phi^{(6/5)B}} \left| f_1 (y_1) \right| \, d\mu (y_1) \right)^{1/p_1} 
\]
With the same method to estimate $IV_{41}$,

$$IV_{42} \leq C \|b_1\|_s \|b_2\|_s M_{p,(5)}f_1(x) M_{p,(5)}f_2(x).$$

Thus, taking the mean over $z_0 \in B$, we have

$$IV_4 \leq C \|b_1\|_s \|b_2\|_s M_{p,(5)}f_1(x) M_{p,(5)}f_2(x).$$

(48)

So (32) can be obtained.

Next we prove (33). Denote $N = N_{RQ} + 1$; then

$$\left| T_0 \left( ((b_1 - m_Q(b_1)) f_1)_{X^{(6/5)}B}, (b_2 - m_B(b_2)) f_2 \right) \right|$$

$$\leq \left| T_0 \left( ((b_1 - m_b(b_1)) f_1)_{X^{(6/5)}B}, (b_2 - m_B(b_2)) \right) \right|$$

$$\cdot \left| (z) - T_0 \left( ((b_1 - m_b(b_1)) f_1)_{X^{(6/5)}B}, (b_2 - m_B(b_2)) \right) \right|$$

$$\cdot \left| (z) \mu(Q) \right|$$

Thus, for all $z_0 \in B$, we have

$$T_0 \left( ((b_1 - m_Q(b_1)) f_1)_{X^{(6/5)}B}, (b_2 - m_B(b_2)) f_2 \right)$$

$$\leq C \|b_1\|_s \|b_2\|_s M_{p,(5)}f_1(x) M_{p,(5)}f_2(x).$$

(46)
\[ + T_\theta ((b_1 - m_Q (b_1)) f_1 \chi_{6 \mathbb{B}(6/5) Q}, f_2 \chi_{6 \mathbb{B}(6/5) Q}) \]
\[ \cdot (z) = E_1 + E_2 + E_3 + E_4 + E_5. \]  
(53)

Let us estimate \( E_1 \) firstly. Since \( Q \) is a doubling ball, we have
\[
\frac{1}{\mu (Q)} \int_Q |T_\theta ((b_1 - b_1 (z)) f_1, f_2) (z)| d\mu (z) 
\leq CM_{s,\delta} \left( \left\{ b_1, T_\theta \right\} f_1, f_2 \right) (x),  
\]  
(54)
\[
\frac{1}{\mu (Q)} \int_Q |(b_1 (z) - m_q (b_1)) T_\theta (f_1, f_2) (z)| d\mu (z) 
\leq C \left\| b_1 \right\|_s M_{s,\delta} \left( T_\theta (f_1, f_2) \right) (x). 
\]

Thus,
\[
|m_Q (E_1)| \leq |m_Q (T_\theta (\{ b_1 - b_1 (z) \} f_1, f_2))| 
+ |m_Q (T_\theta (b_1 (z) - m_q (b_1)) T_\theta (f_1, f_2))|  
\leq CM_{s,\delta} \left( \left\{ b_1, T_\theta \right\} f_1, f_2 \right) (x) 
+ \left\| b_1 \right\|_s M_{s,\delta} (T_\theta (f_1, f_2)) (x).  
\]  
(55)

For \( E_2 \), let \( \nu > 1 \) and \( 1 < s_1 < p_1 \) such that \( 1/\nu = 1/s_1 + 1/p_2 \); denote \( 1/s_1 = 1/s_2 + 1/p_1 \). Note that \( Q \) is a doubling ball. By Lemmas 16 and 14 and Hölder’s inequality,
\[
|m_Q (E_2)| \leq C \left\| E_2 \right\|_{L^{s_1} (\mu (d \mu (z) / \mu (Q))}) \leq C \left( \frac{1}{\mu (Q)} \right) \]
\[
\cdot \int_{6 \mathbb{B}(6/5) Q} |(b_1 (z) - m_q (b_1)) f_1 (z)|^{s_1} d\mu (z) \]
\[
\leq C \left( \frac{1}{\mu (6 \mathbb{B} Q)} \right) \]
\[
\cdot \int_{6 \mathbb{B}(6/5) Q} |f_1 (z)|^{p_1} d\mu (z) \]
\[
\leq C \left( \frac{1}{\mu (6 \mathbb{B} Q)} \right) \]
\[
\cdot \int_{6 \mathbb{B}(6/5) Q} |f_2 (z)|^{p_2} d\mu (z) \]
\[
\leq \left\| b_1 \right\| \left\| M_{P_2,1} f_1 \right\| (x) \]
\[
\cdot M_{P_1,1} f_2 \left( x \right).  
\]

For \( E_3 \), by (i) of Definition 5, Lemma 14, and Hölder’s inequality,
\[
|E_3| \leq C \int_{6 \mathbb{B}} \int_{\mathbb{X} (6/5) Q} \frac{|b_1 (y_1) - m_q (b_1) \left| f_1 (y_1) \right| f_2 (y_2) \right| d\mu (y_1) d\mu (y_2)}{\left[ \sum_{j=1}^{\infty} \lambda (z, d (x, y_j)) \right]^2} \leq C \int_{6 \mathbb{B}} \left| f_2 (y_2) \right| d\mu (y_2) \]
\[
\cdot \sum_{k=1}^{\infty} \int_{6 \mathbb{B}(6/5) Q} \frac{|b_1 (y_1) - m_q (b_1) \left| f_1 (y_1) \right| d\mu (y_1)}{\lambda (z, 5 \mathcal{B}(6/5) r_Q)} \leq C \int_{6 \mathbb{B}} \left| f_2 (y_2) \right| d\mu (y_2) \sum_{k=1}^{\infty} 6^{-km} \]
\[
\cdot \int_{6 \mathbb{B}(6/5) Q} \frac{1}{\lambda (z, 5 \mathcal{B}(6/5) r_Q)} \left| b_1 (y_1) - m_q (b_1) \right| \left| f_1 (y_1) \right| d\mu (y_1) \]
\[
\leq C \frac{1}{\lambda (z, 6 \mathbb{B})} \int_{6 \mathbb{B}} \left| f_2 (y_2) \right| d\mu (y_2) \sum_{k=1}^{\infty} 6^{-km} \]
\[
\cdot \int_{6 \mathbb{B}(6/5) Q} \left( \frac{1}{\lambda (z, 5 \mathcal{B}(6/5) r_Q)} \int_{6 \mathbb{B}(6/5) Q} \left| b_1 (y_1) - m_q (b_1) \right| d\mu (y_1) \right)^{1/p_1} 
\]
\[
\times \left( \frac{1}{\lambda (z, 5 \mathcal{B}(6/5) r_Q)} \int_{6 \mathbb{B}(6/5) Q} \left| f_1 (y_1) \right|^{p_1} d\mu (y_1) \right)^{1/p_1} \]
\[
\leq C \sum_{k=1}^{N} \frac{1}{\lambda (z, 6 \mathbb{B})} \int_{6 \mathbb{B}} \left| f_2 (y_2) \right| d\mu (y_2) \left\| b_1 \right\| \left\| M_{P_2,1} f_1 \right\| (x) \leq C \sum_{k=1}^{N} \frac{\mu (5 \mathcal{B} \mathbb{B}) \lambda (z, 5 \mathcal{B} \mathbb{B})}{\mu (5 \mathcal{B} \mathbb{B})} \frac{1}{\lambda (z, 5 \mathcal{B} \mathbb{B})} \times \frac{1}{\lambda (z, 6 \mathbb{B})} \]
\[
\cdot \int_{6 \mathbb{B}} \left| f_2 (y_2) \right| d\mu (y_2) \left\| b_1 \right\| \left\| M_{P_2,1} f_1 \right\| (x) \left\| M_{P_1,1} f_2 \right\| (x).  
\]  
(57)
Thus,
\[
|m_Q (E_3)| \leq CK_{R,Q} \|b_1\|_{\mathcal{M}_{p_1(5)}} f_1 (x) M_{p_2(5)} f_2 (x).
\] (58)

Also, we have
\[
|m_Q (E_4)| + |m_Q (E_3)| 
\leq C \|b_1\|_{\mathcal{M}_{p_1(5)}} f_1 (x) M_{p_2(5)} f_2 (x).
\] (59)

By (26) in Lemma 14,
\[
J_2 \leq CK_{R,Q}^2 \{ \|b_1\|_{\mathcal{M}_{s}} \|b_2\|_{\mathcal{M}_{s}} (T_\theta (f_1, f_2)) (x)
+ \|b_1\|_{\mathcal{M}_{s}} (b_2, f_2) (f_1, f_2)) (x)
+ \|b_2\|_{\mathcal{M}_{s}} (b_1, f_2) (f_1, f_2)) (x)
+ \|b_1\|_{\mathcal{M}_{s}} f_1 (x) M_{p_2(5)} f_2 (x) \}
\] (60)

\[ J_{22} \text{ and } J_{23} \text{ also have similar estimate of } J_{21}. \text{ Therefore,} \]
\[
J_2 \leq CK_{R,Q}^2 \{ \|b_1\|_{\mathcal{M}_{s}} \|b_2\|_{\mathcal{M}_{s}} (T_\theta (f_1, f_2)) (x)
+ \|b_1\|_{\mathcal{M}_{s}} (b_2, f_2) (f_1, f_2)) (x)
+ \|b_2\|_{\mathcal{M}_{s}} (b_1, f_2) (f_1, f_2)) (x)
+ \|b_1\|_{\mathcal{M}_{s}} f_1 (x) M_{p_2(5)} f_2 (x) \}
\] (61)

Using the similar method to estimate \(IV_2\),
\[
J_3 + J_4 + J_5 + J_6
\leq C \|b_1\|_{\mathcal{M}_{s}} \|b_2\|_{\mathcal{M}_{s}} f_1 (x) M_{p_2(5)} f_2 (x).
\] (62)

Hence, (33) is proved. Thus, the result of Lemma 18 is proved.

**Proof of Theorem 10**. Because the proof of the result of \(\|\mu\| < \infty\) is similar to that of \(\|\mu\| = \infty\), now we only prove the result of \(\|\mu\| = \infty\). Let \(0 < \eta < 1/2, 1 < p_1, p_2, q < \infty, 1/q = 1/p_1 + 1/p_2, 1 < s < q, f_1 \in L^{p_1}(\mu), f_2 \in L^{p_2}(\mu), \) and \(b_1, b_2 \in \text{RBMO}(\mu)\). By Lemmas 12–14 and 17 and H"older’s inequality, then
\[
\begin{align*}
\| & [b_1, b_2, T_\theta] (f_1, f_2) \|_{L^q(\mu)} \\
\leq & \| N_\eta ([b_1, b_2, T_\theta] (f_1, f_2)) \|_{L^q(\mu)} \\
\leq & C \| M_{s(5)}^T ([b_1, b_2, T_\theta] (f_1, f_2)) \|_{L^q(\mu)} \\
\leq & C \|b_1\|_{\mathcal{M}_{s}} \|b_2\|_{\mathcal{M}_{s}} \|T_\theta (f_1, f_2)\|_{L^q(\mu)} \\
+ & C \|b_1\|_{\mathcal{M}_{s}} \|b_2\|_{\mathcal{M}_{s}} \|([b_2, T_\theta] (f_1, f_2)) \|_{L^q(\mu)} \\
+ & C \|b_2\|_{\mathcal{M}_{s}} \|b_1\|_{\mathcal{M}_{s}} \|([b_1, T_\theta] (f_1, f_2)) \|_{L^q(\mu)} \\
+ & C \|b_1\|_{\mathcal{M}_{s}} \|b_2\|_{\mathcal{M}_{s}} \|f_1 \|_{\mathcal{M}_{p_2(5)}} \|f_2\|_{\mathcal{M}_{p_2(5)}} \|. \quad (63)
\end{align*}
\]

Thus, the proof of Theorem 10 is finished. \(\Box\)

**Competing Interests**

The authors declare that they have no competing interests.

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Page 9


