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Research Article

Precompact Sets, Boundedness, and Compactness of Commutators for Singular Integrals in Variable Morrey Spaces

Wei Wang[1] and Jingshi Xu[2]

[1] *School of Sciences, Central South University of Forestry and Technology, Changsha 410004, China*
[2] *Department of Mathematics, Hainan Normal University, Haikou 571158, China*

Correspondence should be addressed to Jingshi Xu; jingshixu@126.com

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We give sufficient conditions for subsets to be precompact sets in variable Morrey spaces. Then we obtain the boundedness of the commutator generated by a singular integral operator and a BMO function on the variable Morrey spaces. Finally, we discuss the compactness of the commutator generated by a singular integral operator and a BMO function on the variable Morrey spaces.

1. Introduction

Let the Calderón-Zygmund singular integral operator T be defined by

$$Tf(x) := \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy, \quad \forall x \in \mathbb{R}^n, \tag{1}$$

where Ω is a measurable function on $\mathbb{R}^n$ and satisfies the following conditions:

(i) Ω is a homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$; that is,

$$\Omega(\mu x) = \Omega(x) \quad \text{for any } \mu > 0,\ x \in \mathbb{R}^n \setminus \{0\}. \tag{2}$$

(ii) Ω has mean zero on S^{n-1}; that is,

$$\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0. \tag{3}$$

Here $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere in $\mathbb{R}^n$ and $d\sigma$ is the area measure on it.

For a function $b \in L_{\text{loc}}(\mathbb{R}^n)$ (the set of all locally integrable functions on $\mathbb{R}^n$), let M_b be the corresponding multiplication operator defined by $M_b f = bf$ for a measurable function f. Then the commutator between T and M_b is denoted by

$$\begin{aligned}[b,T] &:= M_b T - T M_b \\ &= \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} (b(x) - b(y)) f(y)\,dy,\end{aligned} \tag{4}$$

for suitable functions f. Denote the bounded mean oscillation function space by

$$\begin{aligned}&\text{BMO}(\mathbb{R}^n) \\ &\quad := \left\{ b \in L_{\text{loc}}(\mathbb{R}^n) : \|b\|_* := \sup_{\text{cube}:Q\subset\mathbb{R}^n} M_{b,Q} < \infty \right\},\end{aligned} \tag{5}$$

here and in the sequel

$$\begin{aligned}M_{b,Q} &:= \frac{1}{|Q|} \int_Q |b(x) - b_Q|\,dx, \\ b_Q &:= \frac{1}{|Q|} \int_Q b(y)\,dy.\end{aligned} \tag{6}$$

It is well known that commutators play a very important role in harmonic analysis and PDEs. Indeed, Coifman et al. [1] characterized the L^p-boundedness of $[b, R_j]$, where

R_j $(j = 1, \ldots, n)$ are the Reisz transforms and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Using this characterization, the authors of [1] obtained a decomposition theorem of the real Hardy spaces $H^1(\mathbb{R}^n)$. Uchiyama [2] and Janson [3] showed that the Riesz transform R_j may be replaced by the Calderón-Zygmund singular integral operator T as in (1). Coifman et al. generalized the boundedness results of $[b, T]$ to Hardy spaces and gave important applications to some nonlinear PDEs in [4]. The characterization of L^p-compactness of $[b, T]$ was obtained by Uchiyama [2]. We remark that the interest in the compactness of $[b, T]$ in complex analysis is from the connection between the commutators and the Hankel-type operators; see [5]. In recent years, Chen et al. have considered the compactness of commutators in [6–8]. Specially, the results in [2] were generalized to Morrey spaces in [8]. The Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ was introduced by Morrey in 1938 and it is connected to certain problems in elliptic PDE [9]. After that the Morrey spaces were found to have many important applications to the Navier-Stokes equations (see [10]), the Schrödinger equations (see [11]), and potential theory (see [12–14]).

During last three decades, the theory of variable function spaces has developed quickly; see [15–40]. We claim that the list is not exhaust. The boundedness in variable function spaces of many classical operators from harmonic analysis has been obtained; see [18, 19, 41–44]. Motivated by these works, we will consider analogous results in [8] to variable exponent situation. The structure of this paper is as follows. In Section 2, we give sufficient conditions for a set to be a precompact set in a variable Morrey space. In Section 3, we obtain the boundedness of singular integrals and their commutator in variable Morrey spaces. In Section 4, we discuss compactness of commutators in variable Morrey spaces. The remainder of this section is some notions.

Let E be a measurable subset in $\mathbb{R}^n$ with $|E| > 0$, where as usual $|E|$ is the Lebesgue measure of E. Let $p(\cdot)$ be a measurable function on E with range in $[1, \infty)$. The variable exponent modulus is defined for measurable functions f on E by

$$\rho_{p(\cdot)}(f) = \int_E |f(x)|^{p(x)}\, dx < \infty. \tag{7}$$

$L^{p(\cdot)}(E)$ denotes the set of measurable functions f on E such that $\rho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$. The set becomes a Banach function space when equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 \colon \rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \le 1\right\}. \tag{8}$$

These spaces are the so-called variable Lebesgue spaces. Denote by $\mathscr{P}^1(E)$ the set of measurable functions $p(\cdot)$ on E with range in $[1, \infty)$ such that

$$\begin{gathered} 1 < p^- := \operatorname*{ess\,inf}_{x\in E} p(x), \\ \operatorname*{ess\,sup}_{x\in E} p(x) =: p^+ < \infty. \end{gathered} \tag{9}$$

For $p(\cdot) \in \mathscr{P}^1(\mathbb{R}^n)$ and $0 < \lambda(x) < n$ for $x \in \mathbb{R}^n$, the variable Morrey space $L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ is defined as the set of integrable functions f on $\mathbb{R}^n$ with the finite norm

$$\|f\|_{L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{v_1 r^{\lambda(x)/p(x)}} \|f\chi_{B(x,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \tag{10}$$

where $B(x, r)$ denotes a ball centered at x with radius r and v_1 is the volume of the unit ball in $\mathbb{R}^n$.

2. Precompact Sets in Variable Morrey Spaces

In this section, we give a compactness criterion in variable Morrey spaces. We remark here that a compactness criterion for variable exponent Lebesgue spaces was given in [45].

Theorem 1. *Let $p(\cdot) \in \mathscr{P}^1(\mathbb{R}^n)$ and $0 < \lambda(x) < n$ for $x \in \mathbb{R}^n$. Suppose $\mathscr{W}$ is a subset in $L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ satisfying the following conditions:*

(i) *Norm boundedness uniformly is*

$$\sup_{f\in\mathscr{W}} \|f\|_{L^{p(\cdot),\lambda(\cdot)}} < \infty. \tag{11}$$

(ii) *Translation continuity uniformly is*

$$\lim_{y\to 0} \|f(\cdot + y) - f(\cdot)\|_{L^{p(\cdot),\lambda(\cdot)}} = 0 \quad \text{for any } f \in \mathscr{W}. \tag{12}$$

(iii) *Uniformly convergence at infinity is*

$$\lim_{\alpha\to\infty} \|f\chi_{E_\alpha}\|_{L^{p(\cdot),\lambda(\cdot)}} = 0 \quad \text{for any } f \in \mathscr{W}, \tag{13}$$

where $E_\alpha = \{x \in \mathbb{R}^n \colon |x| > \alpha\}$.

Then $\mathscr{W}$ is a precompact set in $L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$.

To prove Theorem 1, we need the following two lemmas, which are well known; for example, see [46].

Lemma 2. *A set E is precompact in a Banach space X if and only if it is totally bounded which means for every positive number ϵ there is a finite subset N_ϵ of points of X such that*

$$E \subset \bigcup_{y\in N_\epsilon} B_\epsilon(y), \tag{14}$$

where $B_\epsilon(y)$ denotes a ball centered at y with radius ϵ. The set N_ϵ is called an ϵ-net of E.

Lemma 3 (the Ascoli-Arzela theorem). *Let E be a bounded domain in $\mathbb{R}^n$. A subset F of $C(\overline{\Omega})$ is precompact in $C(\overline{\Omega})$ if the following two conditions hold:*

(i) *There exists a constant M such that $|u(x)| \leqslant M$ holds for every $u \in F$ and $x \in E$.*

(ii) *For every $\epsilon > 0$, there exists $\delta > 0$ such that $|u(x) - u(y)| < \epsilon$ for $u \in F$, $x, y \in E$, and $|x - y| < \delta$.*

Now there is a position to prove Theorem 1.

Proof of Theorem 1. Let $h > 0$, we denote the mean of f on $B(x,h)$ by

$$M_h f(x) := \frac{1}{v_1 h^n} \int_{y\in B(x,h)} f(x+y)\,\mathrm{d}y, \quad \text{for } x \in \mathbb{R}^n. \tag{15}$$

For $f \in L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$

$$\begin{aligned} M_h f(x) - f(x) &= \frac{1}{v_1 h^n} \int_{y\in B(x,h)} [f(x+y) - f(x)]\,\mathrm{d}y. \end{aligned} \tag{16}$$

Then by Hölder's inequality and Fubini's Theorem, for any $t \in \mathbb{R}^n$ and $r > 0$,

$$\begin{aligned} \|M_h f - f\|_{L^{p(\cdot)}(B(t,r))} &\leqslant C \sup_{\|g\|_{L^{p'(\cdot)}(B(t,r))}\leqslant 1} \int_{x\in B(t,r)} \int_{y\in B(x,h)} \frac{1}{v_1 h^n} (f(x+y) - f(x))\,\mathrm{d}y\, g(x)\,\mathrm{d}x \\ &= C \sup_{\|g\|_{L^{p'(\cdot)}(B(t,r))}\leqslant 1} \int_{y\in B(x,h)} \frac{1}{v_1 h^n} \int_{x\in B(t,r)} (f(x+y) - f(x))\, g(x)\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant C \sup_{|y|\leqslant h} \|f(\cdot + y) - f(\cdot)\|_{L^{p(\cdot)}(B(t,r))}. \end{aligned} \tag{17}$$

Thus,

$$\|M_h f - f\|_{L^{p(\cdot),\lambda(\cdot)}} \leqslant C \sup_{|y|\leqslant h} \|f(\cdot + y) - f(\cdot)\|_{L^{p(\cdot),\lambda(\cdot)}}. \tag{18}$$

Therefore, to prove that $\mathscr{W}$ is a precompact set, we need only to prove that $M_h\mathscr{W}$ is precompact for small h. By Lemma 2, it suffices to show that $M_h\mathscr{W}$ has finite ϵ-net for any $\epsilon > 0$. To do so, firstly, by Lemma 3, we show that $M_h\mathscr{W}$ is precompact in $C(E_\alpha^c)$ for each $\alpha > 0$, where $E_\alpha^c = \{x \in \mathbb{R}^n : |x| \leqslant \alpha\}$. For any $x \in \mathbb{R}^n$, by Hölder's inequality,

$$\begin{aligned} |M_h f(x)| &\leqslant \frac{1}{v_1 h^n} \int_{|y|\leqslant h} |f(x+y)|\,\mathrm{d}y \\ &\leqslant C \frac{1}{v_1 h^n} \|f\|_{L^{p(\cdot)}(B(x,h))} \|1\|_{L^{p'(\cdot)}(B(x,h))} \\ &\leqslant C \frac{h^{\lambda(x)/p(x)}}{h^n} \frac{1}{v_1 h^{\lambda(x)/p(x)}} \|f\|_{L^{p(\cdot)}(B(x,h))} \|1\|_{L^{p'(\cdot)}(B(x,h))} \\ &\leqslant C \frac{h^{\lambda(x)/p(x)}}{h^n} \|1\|_{L^{p'(\cdot)}(B(x,h))} \|f\|_{L^{p(\cdot),\lambda(\cdot)}} \\ &\leqslant C(h, \lambda(\cdot), p(\cdot)) \|f\|_{L^{p(\cdot),\lambda(\cdot)}}, \end{aligned} \tag{19}$$

since $\|1\|_{L^{p'(\cdot)}(B(x,h))} \leqslant \max\{(v_1 h)^{1/p'^-}, (v_1 h)^{1/p'^+}\}$, and $h^{\lambda(x)/p(x)} \leqslant \max\{1, h^{n/p^-}\}$. Therefore, $M_h\mathscr{W}$ are uniformly bounded functions by Condition (i). Then for $x_1, x_2 \in \mathbb{R}^n$,

$$\begin{aligned} |M_h f(x_1) - M_h f(x_2)| &\leqslant \frac{1}{v_1 h^n} \\ &\cdot \int_{|y|\leqslant h} |f(x_1+y) - f(x_2+y)|\,\mathrm{d}y \leqslant C \\ &\cdot \frac{1}{v_1 h^n} \|f(x_1 - x_2 + \cdot) - f(\cdot)\|_{L^{p(\cdot)}(B(x_2,h))} \\ &\cdot \|1\|_{L^{p'(\cdot)}(B(x,h))} \leqslant C(h, \lambda(\cdot), p(\cdot)) \\ &\cdot \|f(x_1 - x_2 + \cdot) - f(\cdot)\|_{L^{p(\cdot),\lambda(\cdot)}}. \end{aligned} \tag{20}$$

Thus, by Condition (ii) and Lemma 3, $M_h\mathscr{W}$ is precompact in $C(E_\alpha^c)$.

Finally, we verify that $M_h\mathscr{W}$ has finite 2ϵ-net for each small positive ϵ. For $0 < \epsilon < 1$, there exist $N > 0$ and $\alpha > 0$ such that $1 < \epsilon^{-N}/4 < \alpha^{n/p^-} < \epsilon^{-N}/2$ and for each $f \in \mathscr{W}$

$$\|f_{E_\alpha}\|_{L^{p(\cdot),\lambda(\cdot)}} < \frac{\epsilon}{8}. \tag{21}$$

By Lemma 3, there exist $\{f_1, f_2, \ldots, f_m\} \subset \mathscr{W}$ such that $\{M_h f_1, M_h f_2, \ldots, M_h f_m\}$ is a finite ϵ^{N+1}-net in $M_h\mathscr{W}$ in the norm of $C(E_\alpha^c)$. Below we verify that $\{M_h f_1, M_h f_2, \ldots, M_h f_m\}$ is a finite ϵ-net in $M_h\mathscr{W}$ in the norm of $L^{p(\cdot),\lambda(\cdot)}$.

To finish the proof, we only need to show that, for $f \in \mathscr{W}$, there is a f_j ($j \in \{1, 2, \ldots, m\}$) such that for $r > 0$, $t \in \mathbb{R}^n$

$$I := \frac{1}{v_1 r^{\lambda(t)/p(t)}} \|M_h f - M_h f_j\|_{L^{p(\cdot)}(B(t,r))} < \epsilon. \tag{22}$$

Now, we choose f_j, $j \in \{1, 2, \ldots, m\}$ such that

$$\sup_{y\in E_\alpha^c} |M_h f(y) - M_h f_j(y)| < \epsilon^{N+1}. \tag{23}$$

To show (22), we consider $B(t,r)$ into three cases.

Case 1. $B(t,r) \subset E_\alpha^c$.

If $r \leqslant 1$,

$$I \leqslant \frac{1}{v_1 r^{\lambda(t)/p(t)}} \epsilon^{N+1} r^{n/p^+} \leqslant \epsilon^{N+1} r^{(n-\lambda(t))/p^+} < \epsilon. \tag{24}$$

If $r > 1$,

$$I \leqslant \|M_h f - M_h f_j\|_{L^{p(\cdot)}(E_\alpha^c)} \leqslant \epsilon^{N+1} \alpha^{n/p^-} < \epsilon. \tag{25}$$

Case 2. $B(t,r) \subset E_\alpha$. Thus, we have

$$\begin{aligned} I &= \frac{1}{v_1 r^{\lambda(t)/p(t)}} \left\| \left(M_h f - M_h f_j\right) \chi_{E_\alpha} \right\|_{L^{p(\cdot)}(B(t,r))} \\ &\leqslant \frac{1}{v_1 r^{\lambda(t)/p(t)}} \left(\left\| \left(M_h f - f\right) \chi_{E_\alpha} \right\|_{L^{p(\cdot)}(B(t,r))} \right. \\ &\quad + \left\| \left(f - f_j\right) \chi_{E_\alpha} \right\|_{L^{p(\cdot)}(B(t,r))} \\ &\quad \left. + \left\| \left(f_j - M_h f_j\right) \chi_{E_\alpha} \right\|_{L^{p(\cdot)}(B(t,r))} \right) \leqslant \left\| M_h f \right. \\ &\quad \left. - f \right\|_{L^{p(\cdot),\lambda(\cdot)}} + \left\| f \chi_{E_\alpha} \right\|_{L^{p(\cdot),\lambda(\cdot)}} + \left\| f_j \chi_{E_\alpha} \right\|_{L^{p(\cdot),\lambda(\cdot)}} + \left\| f_j \right. \\ &\quad \left. - M_h f_j \right\|_{L^{p(\cdot),\lambda(\cdot)}} < \epsilon. \end{aligned} \tag{26}$$

Case 3. $B(t,r) \cap E_\alpha^c \neq \emptyset$, and $B(t,r) \cap E_\alpha \neq \emptyset$. Hence,

$$\begin{aligned} I &\leqslant \frac{1}{v_1 r^{\lambda(t)/p(t)}} \left\| \left(M_h f - M_h f_j\right) \chi_{E_\alpha} \right\|_{L^{p(\cdot)}(B(t,r))} \\ &\quad + \frac{1}{v_1 r^{\lambda(t)/p(t)}} \left\| \left(M_h f - M_h f_j\right) \chi_{E_\alpha^c} \right\|_{L^{p(\cdot)}(B(t,r))} \\ &:= I_1 + I_2. \end{aligned} \tag{27}$$

Here, I_1, I_2 can be estimated that $I_1, I_2 < \epsilon$ by Cases 1 and 2, respectively.

Therefore, $M_h \mathscr{W}$ has a finite 2ϵ-net in $L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$. This completes the proof. □

3. Boundedness of Singular Integrals and Their Commutators

To consider the boundedness of singular integrals, a fundamental operator is the Hardy-Littlewood maximal operator. Given a function f, the maximal function Mf is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy, \quad \forall x \in \mathbb{R}^n, \tag{28}$$

where the supremum is taken over all cubes containing x. It is well known that M is bounded on L^p, $1 < p < \infty$. However, for any $p(\cdot) \in \mathscr{P}^1(\mathbb{R}^n)$, M need not be bounded in $L^{p(\cdot)}(\mathbb{R}^n)$. Let $\mathscr{B}(\mathbb{R}^n)$ be the set of $p(\cdot) \in \mathscr{P}^1(\mathbb{R}^n)$ such that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. For the set $\mathscr{B}(\mathbb{R}^n)$, we refer the reader to [19, 41] for details. If $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, we will use the following results.

Lemma 4 (see [22]). *Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then there exist $0 < \delta < 1$ depending only on $p(\cdot)$ and n such that for balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$*

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leqslant C \left(\frac{|S|}{|B|} \right)^\delta. \tag{29}$$

Lemma 5 (see [22]). *Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then there exists a positive constant $C > 0$ such that for balls B in $\mathbb{R}^n$,*

$$\frac{1}{|B|} \|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leqslant C, \tag{30}$$

where and what follows $p'(\cdot)$ is the conjugate exponent of $p(\cdot)$, which means $p'(x) = p(x)/(p(x) - 1)$.

Lemma 6 (see [23, 24]). *Suppose $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, and then there exists a positive constant C such that for each $b \in BMO(\mathbb{R}^n)$*

$$\begin{aligned} C^{-1} \|b\|_{BMO(\mathbb{R}^n)} &\leq \sup_Q \frac{\|(b - b_Q)\chi_Q\|_{L^{p(\cdot)}}}{\|\chi_Q\|_{L^{p(\cdot)}}} \\ &\leq C \|b\|_{BMO(\mathbb{R}^n)}. \end{aligned} \tag{31}$$

Theorem 7. *Let $0 < \lambda(x) < n$ for $x \in \mathbb{R}^n$. Suppose S is a linear or sublinear operator satisfying*

$$|Sf(x)| \leqslant C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^n} dy, \quad \text{for } x \notin \operatorname{supp} f. \tag{32}$$

If $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, $\lambda^+ < n\delta p^-$, where δ is as in Lemma 4 and the operator S is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then S is also bounded on $L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$. That means

$$\|Sf\|_{L^{p(\cdot),\lambda(\cdot)}} \leqslant C \|f\|_{L^{p(\cdot),\lambda(\cdot)}}, \tag{33}$$

where the constant C is independent of f.

Proof. Let $f \in L^{p(\cdot),\lambda(\cdot)}$, pick any $t \in \mathbb{R}^n$, and write $f(x) = f_0(x) + \sum_{i=1}^\infty f_i(x)$, where $f_0 = \chi_{B(t,2r)} f$, $f_i = \chi_{B(t,2^{i+1}r) \setminus B(t,2^i r)} f$.

Firstly, we estimate Sf_0 on $B(t,r)$. By the boundedness of S on $L^{p(\cdot)}(\mathbb{R}^n)$, we have

$$\begin{aligned} \|Sf_0\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leqslant C \|f_0\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C v_1 r^{\lambda(t)/p(t)} \|f\|_{L^{p(\cdot),\lambda(\cdot)}}. \end{aligned} \tag{34}$$

Thus,

$$\begin{aligned} \|Sf_0\|_{L^{p(\cdot)}(B(t,r))} &\leqslant \|Sf_0\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C v_1 r^{\lambda(t)/p(t)} \|f\|_{L^{p(\cdot),\lambda(\cdot)}}. \end{aligned} \tag{35}$$

Hence, we obtain

$$\|Sf_0\|_{L^{p(\cdot),\lambda(\cdot)}} \leqslant C \|f\|_{L^{p(\cdot),\lambda(\cdot)}}. \tag{36}$$

It remains to estimate $S(\sum_{i=1}^\infty f_i)(x)$ on $B(t,r)$. By the size estimate of S, we have

$$|Sf_i(x)| \leqslant C \left(2^i r\right)^{-n} \int_{\mathbb{R}^n} |f_i(y)| \, dy, \quad \text{for } x \in B(t,r). \tag{37}$$

Thus, by Lemmas 4 and 5 and Hölder's inequality, we have

$$\begin{aligned} \left\| S\left(\sum_{i=1}^\infty f_i\right) \right\|_{L^{p(\cdot)}(B(t,r))} &\leqslant \left\| \sum_{i=1}^\infty Sf_i \right\|_{L^{p(\cdot)}(B(t,r))} \\ &\leqslant C \sum_{i=1}^\infty \left\| \left(2^i r\right)^{-n} \int_{\mathbb{R}^n} |f_i(y)| \, dy \right\|_{L^{p(\cdot)}(B(t,r))} \\ &\leqslant C \sum_{i=1}^\infty \left(2^i r\right)^{-n} \|1\|_{L^{p(\cdot)}(B(t,r))} \int_{B(t,2^{i+1}r)} |f(y)| \, dy \end{aligned}$$

$$\begin{aligned}
&\leqslant C\sum_{i=1}^{\infty}\left(2^{i}r\right)^{-n}\|1\|_{L^{p(\cdot)}(B(t,r))}\|1\|_{L^{p'(\cdot)}(B(t,2^{i+1}r))}\\
&\cdot\|f\|_{L^{p(\cdot)}(B(t,2^{i+1}r))}\\
&\leqslant C\sum_{i=1}^{\infty}\left(2^{i}r\right)^{-n}\frac{\|1\|_{L^{p(\cdot)}(B(t,r))}}{\|1\|_{L^{p(\cdot)}(B(t,2^{i+1}r))}}\|1\|_{L^{p(\cdot)}(B(t,2^{i+1}r))}\\
&\times\|1\|_{L^{p'(\cdot)}(B(t,2^{i+1}r))}\nu_1\left(2^{i+1}r\right)^{\lambda(t)/p(t)}\|f\|_{L^{p(\cdot),\lambda(\cdot)}}\\
&\leqslant C\sum_{i=1}^{\infty}2^{-n\delta(i+1)}\nu_1\left(2^{i+1}r\right)^{\lambda(t)/p(t)}\|f\|_{L^{p(\cdot),\lambda(\cdot)}}.
\end{aligned}\tag{38}$$

Thus,

$$\begin{aligned}
\left\|S\left(\sum_{i=1}^{\infty}f_i\right)\right\|_{L^{p(\cdot),\lambda(\cdot)}}&\leqslant C\sum_{i=1}^{\infty}2^{-(i+1)(n\delta-\lambda^{+}/p^{-})}\|f\|_{L^{p(\cdot),\lambda(\cdot)}}\\
&\leqslant C\|f\|_{L^{p(\cdot),\lambda(\cdot)}}.
\end{aligned}\tag{39}$$

This finishes the proof of Theorem 7. □

Next we turn to the boundedness of commutators in variable Morrey spaces. Many authors have studied it; see [44], but they considered that it restricts the underlying space with finite measure.

Theorem 8. *Suppose S is a linear operator satisfying*

$$Sf(x)=\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^n}f(y)\,dy,\quad \text{for } x\notin \operatorname{supp} f,\tag{40}$$

where Ω is a bounded measurable function. Let $0<\lambda(x)<n$, and $b\in BMO(\mathbb{R}^n)$ and $p(\cdot)\in\mathscr{B}(\mathbb{R}^n)$. If the commutator $[b,S]$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then $[b,S]$ is also bounded on $L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$.

Proof. For any $t\in\mathbb{R}^n$ and $r>0$, let $B=B(t,r)$ and write f as in the proof of Theorem 7. By the $L^{p(\cdot)}$-boundedness of $[b,S]$, we obtain

$$\begin{aligned}
\|[b,S]f_0(x)\chi_B\|_{L^{p(\cdot)}}&\leq C\|f_0\|_{L^{p(\cdot)}}\\
&\leq C\nu_1(2r)^{\lambda(\cdot)/p(x)}\|f\|_{L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}.
\end{aligned}\tag{41}$$

For $k>0$ and $x\in B$, we write

$$\begin{aligned}
|[b,S]f_k(x)|&\leq\frac{C}{(2^kr)^n}\\
&\cdot\int_{2^{k+1}B}|b(x)-b_{2r}||\Omega(x-y)f_k(y)|\,dy+\frac{C}{(2^kr)^n}\\
&\cdot\int_{2^{k+1}B}|b_{2r}-b_{2^{k+1}r}||\Omega(x-y)f_k(y)|\,dy+\frac{C}{(2^kr)^n}\\
&\cdot\int_{2^{k+1}B}|b(y)-b_{2^{k+1}r}||\Omega(x-y)f_k(y)|\,dy\\
&:=I_{1k}(x)+I_{2k}(x)+I_{3k}(x),
\end{aligned}\tag{42}$$

where and in what follows, for $k\geqslant 0$, $b_{2^{k+1}r}$ is defined by

$$b_{2^{k+1}r}=\frac{1}{|B(t,2^{k+1}r)|}\int_{B(t,2^{k+1}r)}b(y)\,dy.\tag{43}$$

By the well-known fact that, for any $r>0$ and $k\in\mathbb{N}$, $|b_{2^{k+1}r}-b_{2r}|\leq C_nk\|b\|_*$ (see [47, Proposition 7.1.5(i)]), we obtain

$$\begin{aligned}
I_{2k}(x)&\leq C(k+1)\|b\|_*\left(2^kr\right)^{-n}\int_{2^{k+1}B}|f_k(y)|\,dy\\
&\leq C(k+1)\left(2^kr\right)^{-n}\|b\|_*\|\chi_{2^{k+1}B}\|_{L^{p'(\cdot)}}\|f\|_{L^{p(\cdot)}(2^{k+1}B)}.
\end{aligned}\tag{44}$$

Hence, for $k>0$, using Lemmas 4 and 5 we have

$$\begin{aligned}
\|I_{2k}\chi_B\|_{L^{p(\cdot)}}&\leq C(k+1)\left(2^kr\right)^{-n}\|b\|_*\|\chi_B\|_{L^{p(\cdot)}}\\
&\cdot\|\chi_{2^{k+1}B}\|_{L^{p'(\cdot)}}\|f\|_{L^{p(\cdot)}(2^{k+1}B)}\leq C(k+1)\|b\|_*\\
&\cdot 2^{-n\delta(k+1)}\nu_1\left(2^{k+1}r\right)^{\lambda(t)/p(t)}\|f\|_{L^{p(\cdot),\lambda(\cdot)}}\leq C(k+1)\\
&\cdot\|b\|_*2^{-(k+1)(n\delta-\lambda^{+}/p^{-})}\nu_1(2r)^{\lambda(t)/p(t)}\|f\|_{L^{p(\cdot),\lambda(\cdot)}}.
\end{aligned}\tag{45}$$

For $I_{3k}(X)$, by Hölder's inequality and Lemma 6, we have

$$\begin{aligned}
&I_{3k}(x)\\
&\quad\leq C\frac{1}{(2^kr)^n}\|(b-b_{2^{k+1}r})\chi_{2^{k+1}B}\|_{L^{p'(\cdot)}}\|f\|_{L^{p(\cdot)}(2^{k+1}B)}\\
&\quad\leq C\frac{1}{(2^kr)^n}\|b\|_*\|\chi_{2^{k+1}B}\|_{L^{p'(\cdot)}}\|f\|_{L^{p(\cdot)}(2^{k+1}B)}.
\end{aligned}\tag{46}$$

Therefore, as the argument as for I_{2k}, we have

$$\begin{aligned}
&\|I_{3k}\chi_B\|_{L^{p(\cdot)}}\\
&\quad\leq C\|b\|_*2^{-(k+1)(n\delta-\lambda^{+}/p^{-})}\nu_1(2r)^{\lambda(t)/p(t)}\|f\|_{L^{p(\cdot),\lambda(\cdot)}}.
\end{aligned}\tag{47}$$

Finally, for $I_{1k}(x)$ by Hölder's inequality, we have

$$\begin{aligned}
&I_{1k}(x)\\
&\quad\leq C|b(x)-b_r|\frac{1}{(2^kr)^n}\|\chi_{2^{k+1}B}\|_{L^{p'(\cdot)}}\|f\|_{L^{p(\cdot)}(2^{k+1}B)}.
\end{aligned}\tag{48}$$

Thus, as the argument as before, we obtain that

$$\begin{aligned}
&\|I_{1k}\chi_B\|_{L^{p(\cdot)}}\\
&\quad\leq C\|b\|_*\frac{1}{(2^kr)^n}\|\chi_B\|_{L^{p(\cdot)}}\|\chi_{2^{k+1}B}\|_{L^{p'(\cdot)}}\|f\|_{L^{p(\cdot)}(2^{k+1}B)}\\
&\quad\leq C\|b\|_*2^{-(k+1)(n\delta-\lambda^{+}/p^{-})}\nu_1(2r)^{\lambda(t)/p(t)}\|f\|_{L^{p(\cdot),\lambda(\cdot)}}.
\end{aligned}\tag{49}$$

From (45), (47), and (49), we get

$$\begin{aligned}
\left\|[b,S]\sum_{k=1}^{\infty}f_k\chi_B\right\|_{L^{p(\cdot)}}&\leq C\sum_{k=1}^{\infty}(k+1)\|b\|_*\\
&\cdot 2^{-(k+1)(n\delta-\lambda^{+}/p^{-})}\nu_1(2r)^{\lambda(t)/p(t)}\|f\|_{L^{p(\cdot),\lambda(\cdot)}}\leq C\|b\|_*\\
&\cdot\nu_1(2r)^{\lambda(t)/p(t)}\|f\|_{L^{p(\cdot),\lambda(\cdot)}}.
\end{aligned}\tag{50}$$

Hence,

$$\left\|[b,S]\,f\right\|_{L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)} \le C\,\|b\|_*\,\|f\|_{L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)}. \tag{51}$$

This finishes the proof of Theorem 8. □

Corollary 9. *Let $0 < \lambda(x) < n$. Suppose that Ω is a bounded homogeneous function of degree 0 and satisfies conditions (2) and*

$$\int_0^1 \frac{w(\delta)}{\delta}\,\mathrm{d}\delta < \infty, \tag{52}$$

where

$$w(\delta) = \sup_{x',y'\in S^{n-1},|x'-y'|<\delta} \left|\Omega\left(x'\right) - \Omega\left(y'\right)\right|. \tag{53}$$

If $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, then the Calderón-Zygmund singular integral operator T defined by (1) and its commutator $[b,T]$ with $b \in BMO(\mathbb{R}^n)$ are both bounded on $L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$.

Proof. Corollary 9 is the result of the following lemmas. Indeed, the boundedness of T on $L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ is the direct result of Theorem 7, Lemmas 12 and 13. For the commutator, if $w \in A_p$, $p \in (1,\infty)$, then by Corollary 9.2.6 in [47] there exists $\epsilon > 0$ such that $w \in A_{p-\epsilon}$ and $p-\epsilon > 1$. Then we choose $s = p/(p-\epsilon) < p$ in Lemma 11; for $w \in A_p$ by Lemma 10 we obtain

$$\int_{\mathbb{R}^n} \left|[b,T]\,f(x)\right|^p w(x)\,\mathrm{d}x \leqslant C \int_{\mathbb{R}^n} \left|f(x)\right|^p w(x)\,\mathrm{d}x, \tag{54}$$

for bounded functions b and compactly supported functions f. Finally, using the similar argument for Theorem 7.5.6 in [47], we obtain that the last inequality holds for any $b \in \mathrm{BMO}(\mathbb{R}^n)$. Thus, by Lemma 13, we obtain that $[b,T]$ is bounded on $L^p(\cdot)(\mathbb{R}^n)$. Consequently, by Theorem 8, $[b,T]$ is bounded on $L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$. □

We remark here that the boundedness in variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ of commutator $[b,T]$ has been proved in [44] by another method when Ω is an infinitely differentiable function on S^{n-1}.

Lemma 10 (see Lemma 2.1 in [48]). *Let $1 < p < \infty$ and $w \in A_p$ (Muckenhoupt weight); then there exists a positive constant C such that*

$$\int_{\mathbb{R}^n} Mf(x)^p\,w(x)\,\mathrm{d}x \le C \int_{\mathbb{R}^n} M^{\#}f(x)^p\,w(x)\,\mathrm{d}x, \tag{55}$$

for functions f such that the left-hand side is finite.

Here we say $w \in A_p$, $1 < p < \infty$ if for every cube Q

$$\left(\frac{1}{|Q|}\int_Q w(x)\,\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\,\mathrm{d}x\right)^{p-1} \le C < \infty. \tag{56}$$

For properties of A_p, we refer the reader to [47].

Lemma 11 (see Lemma 2.4.1 in [49]). *Let $b \in BMO(\mathbb{R}^n)$ and $p \in (1,\infty)$. Then for any $s \in (1,p)$, there exists a constant C, independent of b and f, such that*

$$M^{\#}[b,T](f)(x) \le C\,\|b\|_* \left[\left(M\left(|Tf|^s\right)(x)\right)^{1/s} + \left(M\left(|f|^s\right)(x)\right)^{1/s}\right], \quad \forall x \in \mathbb{R}^n. \tag{57}$$

Lemma 12 (see Theorem 2.1.6 in [49]). *Suppose that Ω is a bounded homogeneous function of degree 0 and satisfies conditions (1) and (52). If $p \in (1,\infty)$ and $w \in A_p$, then there exists a constant C, independent of f, such that*

$$\int_{\mathbb{R}^n} \left|Tf(x)\right|^p w(x)\,\mathrm{d}x \le C \int_{\mathbb{R}^n} \left|f(x)\right|^p w(x)\,\mathrm{d}x. \tag{58}$$

Lemma 13 (see Corollary 1.11 in [18]). *Given that $\mathscr{F}$ denotes a family of ordered pairs of nonnegative, measurable functions (f,g) on $\mathbb{R}^n$, assume that*

$$\int_{\mathbb{R}^n} f(x)^{p_0}\,w(x)\,\mathrm{d}x \leqslant C_0 \int_{\mathbb{R}^n} g(x)^{p_0}\,w(x)\,\mathrm{d}x, \quad (f,g) \in \mathscr{F}, \tag{59}$$

holds for some $1 < p_0 < \infty$, for every $\omega \in A_{p_0}$ and for all $(f,g) \in \mathscr{F}$. Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then

$$\|f\|_{L^{p(\cdot)}} \leqslant C\,\|g\|_{L^{p(\cdot)}}. \tag{60}$$

4. Compactness of Commutators

Now we obtain sufficient conditions for the commutator $[b,T]$ to be a compact operator on $L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$.

Theorem 14. *Let $0 < \lambda(x) < n$ for $x \in \mathbb{R}^n$ and $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ such that $\lambda(\cdot)/p(\cdot)$ is a constant function. Suppose that Ω is a bounded homogeneous function of degree of 0 and satisfies (2) and for some $q \in (1,p^-)$*

$$\int_0^1 \frac{\omega_q(\delta)}{\delta}\left(1 + \left|\log\delta\right|\right)\mathrm{d}\delta < \infty, \tag{61}$$

where $\omega_q(\delta)$ denotes the integral modulus of continuity of order q of Ω defined by

$$\omega_q(\delta) = \sup_{\|\rho\|<\delta} \left(\int_{S^{n-1}} \left|\Omega\left(\rho x'\right) - \Omega\left(x'\right)\right|^q \mathrm{d}\sigma\left(x'\right)\right)^{1/q}, \tag{62}$$

and ρ is a rotation in $\mathbb{R}^n$ and $\|\rho\| = \sup_{x'\in S^{n-1}}|\rho x' - x'|$. If $b \in VMO(\mathbb{R}^n)$ (the closure of the set of compactly supported infinite differential functions in $BMO(\mathbb{R}^n)$), then the commutator $[b,T]$ is a compact operator on $L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$.

Lemma 15. *Let $0 < \lambda(\cdot) < n$. Suppose that Ω is a bounded homogeneous function of degree of 0 and satisfies (2) and (52). For $\eta > 0$, let*

$$T_\eta f(x) = \int_{|x-y|>\eta} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,\mathrm{d}y. \tag{63}$$

Then for $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, $\|T_\eta\|_{L^{p(\cdot),\lambda(\cdot)}} \leqslant C\|f\|_{L^{p(\cdot),\lambda(\cdot)}}$, where C is independent of η and f.

Proof. Lemma 15 is a direct consequence of Theorem 7. In fact, by Theorem 2.1.8 in [49], given that $1 < p < \infty$, if $\omega \in A_p$, then

$$\int_\Omega T_\eta f(x)^p \omega(x)\,\mathrm{d}x \leqslant C_0 \int_\Omega f(x)^p \omega(x)\,\mathrm{d}x \tag{64}$$

holds uniformly in η. Using Lemma 13, we can get that T_η is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ uniformly in η. Now all conditions in Theorem 7 are fulfilled. □

Lemma 16 (see Lemma 2.2 in [8]). *Suppose that $0 < \beta < n$, Ω satisfies (2), and $\Omega \in L^q(S^{n-1})$, where $q > 1$. Then there exist $C > 0$ such that for an $R > 0$ and $x \in \mathbb{R}^n$ with $|x| < R/2$*

$$\begin{aligned}&\left(\int_{R<|y|<2R}\left|\frac{\Omega(y-x)}{|y-x|^{n-\beta}}-\frac{\Omega(y)}{|y|^{n-\beta}}\right|^q \mathrm{d}y\right)^{1/q}\\ &\quad\leqslant CR^{n/q-(n-\beta)}\left\{\frac{|x|}{R}+\int_{|x|/2R}^{|x|/R}\frac{w_q(\delta)}{\delta}\mathrm{d}\delta\right\}.\end{aligned} \tag{65}$$

Proof of Theorem 14. We will use the method in [8]. Let $\mathscr{F}$ be the unit ball in $L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$. By density, we only need to prove that when $b \in C_c^\infty(\mathbb{R}^n)$, the set $\mathscr{G} = \{[b,T]f : f \in \mathscr{F}\}$ is a precompact in $L^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$. By Theorem 1, it is sufficient to show that (11)–(13) hold uniformly in $\mathscr{G}$.

Notice that $b \in C_c^\infty(\mathbb{R}^n)$. Applying Corollary 9, we have

$$\begin{aligned}\sup_{f\in\mathscr{F}}\|[b,T]f\|_{L^{p(\cdot),\lambda(\cdot)}} &\leqslant C\|b\|_* \sup_{f\in\mathscr{F}}\|f\|_{L^{p(\cdot),\lambda(\cdot)}} \leqslant C\|b\|_*\\ &< \infty.\end{aligned} \tag{66}$$

This shows that (11) holds.

Next we show that (13) holds. To do so, we suppose that $\beta > 1$ taken so large that $\operatorname{supp} b \subset \{x : |x| \leq \beta\}$. For any $0 < \epsilon < 1$, we take $\alpha > \beta$ such that $(\alpha-\beta)^{n(1-q)} < \epsilon^q$. Below we show that for every $t \in \mathbb{R}^n$ and $r > 0$, $q > 1$

$$\frac{1}{\nu_1 r^{\lambda(t)/p(t)}}\left\|[b,T]f\chi_{E_\alpha}\right\|_{L^{p(\cdot)}B(t,r)} < C\epsilon\|\Omega\|_{L^q(S^{n-1})}. \tag{67}$$

In fact, for any $x \in E_\alpha = \{x \in \mathbb{R}^n : |x| > \alpha\}$ and every $f \in \mathscr{F}$, by Hölder's inequality we have

$$\begin{aligned}&\left|[b,T]f(x)\right|\\ &\quad=\left|\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^n}(b(x)-b(y))f(y)\,\mathrm{d}y\right|\\ &\quad\leq C\|b\|_\infty\int_{|y|<\beta}\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|\,\mathrm{d}y\\ &\quad\leqslant C\left(\int_{|x-y|\leqslant\beta}\frac{|\Omega(y)|^q}{|y|^{nq}}|f(x-y)|^q\,\mathrm{d}y\right)^{1/q}.\end{aligned} \tag{68}$$

Then for every $t \in \mathbb{R}^n$ and $r > 0$, by the Minkowski inequality and the choice of α, we get

$$\begin{aligned}&\frac{1}{\nu_1 r^{\lambda(t)/p(t)}}\left\|[b,T]f\chi_{E_\alpha}\right\|_{L^{p(\cdot)}(B(t,r))} \leqslant C\\ &\quad\cdot\frac{1}{\nu_1 r^{\lambda(t)/p(t)}}\left\|\left(\int_{|\cdot-y|\leq\beta}\frac{|\Omega(y)|^q}{|y|^{nq}}|f(\cdot-y)|^q\,\mathrm{d}y\right)^{1/q}\right.\\ &\quad\left.\cdot\chi_{E_\alpha}\right\|_{L^{p(\cdot)}(B(t,r))} \leqslant C\\ &\quad\cdot\frac{1}{\nu_1 r^{\lambda(t)/p(t)}}\left\|\left(\int_{|\cdot-y|\leq\beta}\frac{|\Omega(y)|^q}{|y|^{nq}}|f(\cdot-y)|^q\,\mathrm{d}y\right)\right.\\ &\quad\left.\cdot\chi_{E_\alpha}\right\|^{1/q}_{L^{p(\cdot)/q}(B(t,r))} \leqslant C\frac{1}{r^{\lambda(t)/p(t)-\lambda(t-y)/p(t-y)}}\\ &\quad\cdot\sup_{|x-y|\leqslant\beta}\frac{1}{\nu_1 r^{\lambda(t-y)/p(t-y)}}\|f\|_{L^{p(\cdot)}B(t-y,r)}\\ &\quad\cdot\left\{\int_{|y|>\alpha-\beta}\frac{|\Omega(y)|^q}{|y|^{nq}}\mathrm{d}y\right)^{1/q} \leqslant C\epsilon\|\Omega\|_{L^q(S^{n-1})}\\ &\quad\cdot\|f\|_{L^{p(\cdot),\lambda(\cdot)}}.\end{aligned} \tag{69}$$

Thus, we get (67), which shows that (13) holds for $[b,T]$ in $\mathscr{G}$ uniformly.

Finally, we show that the translation continuity condition (12) holds for the commutator $[b,T]$ in $\mathscr{G}$ uniformly. We need to prove that, for any $0 < \epsilon < 1/2$, if $|z|$ is sufficiently small depending only on ϵ, then for every $f \in \mathscr{F}$

$$\left\|[b,T]f(\cdot)-[b,T]f(\cdot+z)\right\|_{L^{p(\cdot),\lambda(\cdot)}} \leqslant C\epsilon. \tag{70}$$

Now for $z \in \mathbb{R}^n$, we write

$$\begin{aligned}&[b,T]f(x+z)-[b,T]f(x)\\ &\quad=\int_{|x-y|>e^{1/\epsilon}|z|}\frac{\Omega(x-y)}{|x-y|^n}[b(x+z)-b(x)]\\ &\quad\cdot f(y)\,\mathrm{d}y\\ &\quad+\int_{|x-y|>e^{1/\epsilon}|z|}\left(\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x+z-y)}{|x+z-y|^n}\right)\\ &\quad\cdot[b(y)-b(x+z)]f(y)\,\mathrm{d}y\\ &\quad+\int_{|x-y|\leqslant e^{1/\epsilon}|z|}\frac{\Omega(x-y)}{|x-y|^n}[b(y)-b(x)]f(y)\,\mathrm{d}y\\ &\quad-\int_{|x-y|\leqslant e^{1/\epsilon}|z|}\frac{\Omega(x+z-y)}{|x-y|^n}[b(y)-b(x+z)]\\ &\quad\cdot f(y)\,\mathrm{d}y := J_1+J_2+J_3-J_4.\end{aligned} \tag{71}$$

Since $b \in C_c^\infty(\mathbb{R}^n)$, we have $|b(x)-b(x+z)| \leqslant C\|\nabla b\|_\infty|z|$. Since $\Omega \in L^q(S^{n-1})$ and applying Lemma 15, we get

$$\|J_1\|_{L^{p(\cdot),\lambda(\cdot)}} \leqslant C|z|\,\|f\|_{L^{p(\cdot),\lambda(\cdot)}} < C|z|. \tag{72}$$

As for J_2, for every $t \in \mathbb{R}^n$ and $r > 0$, using Lemma 16 and the Minkowski inequality, we get

$$\begin{aligned}
&\frac{1}{v_1 r^{\lambda(t)/p(t)}} \|J_2\|_{L^{p(\cdot)}(B(t,r))} \leqslant 2\|b\|_\infty \frac{1}{v_1 r^{\lambda(t)/p(t)}} \\
&\quad \cdot \sup_{|x-y|>e^{1/\epsilon}|z|} \|f\|_{L^{p(\cdot)}(B(t-y,r))} \int_{|y|>e^{1/\epsilon}|z|} \left| \frac{\Omega(y)}{|y|^n} \right. \\
&\quad \left. - \frac{\Omega(y+z)}{|y+z|^n} \right| \mathrm{d}y \leqslant C \\
&\quad \cdot \frac{1}{r^{\lambda(t)/p(t)-\lambda(t-y)/p(t-y)}} \|f\|_{L^{p(\cdot),\lambda(\cdot)}} \sum_{k=0}^{\infty} \left\{ \frac{|z|}{2^k e^{1/\epsilon}|z|} \right. \\
&\quad \left. + \int_{|z|/2^{k+1}e^{1/\epsilon}|z|}^{|z|/2^k e^{1/\epsilon}|z|} \frac{\omega(\delta)}{\delta} \mathrm{d}\delta \right\} \leqslant C\|f\|_{L^{p(\cdot),\lambda(\cdot)}} \sum_{k=0}^{\infty} \left\{ \frac{1}{2^k e^{1/\epsilon}} \right. \\
&\quad \left. + \frac{1}{1+k+1/\epsilon} \int_{|z|/2^{k+1}e^{1/\epsilon}|z|}^{|z|/2^k e^{1/\epsilon}|z|} \frac{\omega(\delta)}{\delta} \left(1+|\log\delta|\right) \mathrm{d}\delta \right\} \\
&\quad \leqslant C\left(e^{-1/\epsilon}+\epsilon\right)\|f\|_{L^{p(\cdot),\lambda(\cdot)}}.
\end{aligned} \tag{73}$$

Thus, we have

$$\|J_2\|_{L^{p(\cdot),\lambda(\cdot)}} \leqslant C\epsilon. \tag{74}$$

Regarding J_3, we have $|b(x)-b(y)| \leqslant C\|\nabla b\|_\infty |x-y|$ by $b \in C_c^\infty(\mathbb{R}^n)$. Thus,

$$J_3 \leqslant C \int_{|x-y|\leqslant e^{1/\epsilon}|z|} |\Omega(x-y)| \, |x-y|^{-n+1} |f(y)| \, \mathrm{d}y. \tag{75}$$

By the Minkowski inequality, for every $t \in \mathbb{R}^n$ and $r > 0$, we have

$$\begin{aligned}
&\frac{1}{v_1 r^{\lambda(t)/p(t)}} \|J_3\|_{L^{p(\cdot)}(B(t,r))} \leqslant C \frac{1}{v_1 r^{\lambda(t)/p(t)}} \\
&\quad \cdot \sup_{|x-y|\leqslant e^{1/\epsilon}|z|} \|f\|_{L^{p(\cdot)}(B(t-y,r))} \\
&\quad \cdot \int_{|y|\leqslant e^{1/\epsilon}|z|} |\Omega(y)| \, |y|^{-n+1} \mathrm{d}y \leqslant C \\
&\quad \cdot \frac{1}{r^{\lambda(t)/p(t)-\lambda(t-y)/p(t-y)}} e^{1/\epsilon}|z| \, \|f\|_{L^{p(\cdot),\lambda(\cdot)}} \leqslant Ce^{1/\epsilon}|z| \\
&\quad \cdot \|f\|_{L^{p(\cdot),\lambda(\cdot)}}.
\end{aligned} \tag{76}$$

Thus,

$$\|J_3\|_{L^{p(\cdot),\lambda(\cdot)}} \leqslant Ce^{1/\epsilon}|z|. \tag{77}$$

Finally, by $|b(x+z)-b(y)| \leqslant C\|\nabla b\|_\infty |x+z-y|$, we have

$$J_4 \leqslant C \int_{|x-y|\leqslant e^{1/\epsilon}|z|} |\Omega(x+z-y)| \, |x+z-y|^{-n+1} \mathrm{d}y. \tag{78}$$

Using the same argument for J_3, it is easy to check that

$$\|J_4\|_{L^{p(\cdot),\lambda(\cdot)}} \leqslant C\left(e^{1/\epsilon}|z|+|z|\right). \tag{79}$$

From (72), (74), (77), and (79), and taking $|z|$ to be sufficiently small, we can get

$$\begin{aligned}
&\|[b,T]f(\cdot)-[b,T]f(\cdot+z)\|_{L^{p(\cdot),\lambda(\cdot)}} \\
&\quad \leqslant \|J_1\|_{L^{p(\cdot),\lambda(\cdot)}} + \|J_2\|_{L^{p(\cdot),\lambda(\cdot)}} + \|J_3\|_{L^{p(\cdot),\lambda(\cdot)}} + \|J_4\|_{L^{p(\cdot),\lambda(\cdot)}} \\
&\quad \leqslant C\epsilon.
\end{aligned} \tag{80}$$

Therefore, we show that the translation continuity (12) holds for the commuator $[b,T]$ in $\mathcal{G}$ uniformly and this completes the proof of Theorem 14. □

We remark that in Theorem 14 the condition $b \in \mathrm{VMO}(\mathbb{R}^n)$ is necessary by Theorem 1.2 in [8].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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