Research Article

Existence and Multiplicity of Nontrivial Solutions for a Class of Semilinear Fractional Schrödinger Equations

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This paper is concerned with the existence of solutions to the following fractional Schrödinger type equations: $(-\Delta)^s u + V(x)u = f(x, u)$, $x \in \mathbb{R}^N$, where the primitive of the nonlinearity f is of superquadratic growth near infinity in u and the potential V is allowed to be sign-changing. By using variant Fountain theorems, a sufficient condition is obtained for the existence of infinitely many nontrivial high energy solutions.

1. Introduction and the Main Result

In this work, under the assumptions that V satisfies some weaker conditions than those in [1] and the primitive of f satisfies a more general superquadratic condition near infinity, we study the existence of infinitely many nontrivial high energy solutions to the following fractional Schrödinger equations:

$$(-\Delta)^s u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N, \quad (1)$$

where $s \in (0, 1)$, $N > 2$, and $f: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function with some proper growth conditions. Here $(-\Delta)^s$ is the so-called fractional Laplacian operator of order $s \in (0, 1)$, which can be characterized as $(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}u)$, with $\mathcal{F}$ being the usual Fourier transform in $\mathbb{R}^N$ ($(-\Delta)^s$ is the pseudodifferential operator with symbol $|\xi|^{2s}$).

We need the following assumptions on $V(x)$:

(V₁) $V \in C(\mathbb{R}^N, \mathbb{R})$ is bounded from below.

(V₂) For any $M > 0$, there exists $r_0 > 0$ such that

$$\lim_{|y| \rightarrow \infty} \text{meas} \left(\{x \in \mathbb{R}^N : |x - y| \leq r_0, V(x) \leq M\} \right) = 0, \quad (2)$$

where meas denotes the Lebesgue measure in $\mathbb{R}^N$.

More recently, Di Nezza et al. [2] have proved that $(-\Delta)^s$ can be reduced to the standard Laplacian $-\Delta$ as $s \rightarrow 1$. When $s = 1$, problem (1) is the classical Schrödinger equations

$$(-\Delta)u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N. \quad (3)$$

Over the past several decades, the existence and multiplicity of nontrivial solutions to problem (3) have been studied extensively by numerous researches using a variety of methods and techniques; see, for instance, [3–7].

The nonlinear equations involving fractional Laplacian, which is a powerful tool for the descriptions of physics, probability, and finance, have attracted the attention of many researchers and have been successfully applied in various fields; see, for instance, [1, 8, 9] and the reference therein.

We need the following conditions on f and its primitive $F(x, u) = \int_0^u f(x, s)ds$:

(f₁) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exist constant $C > 0$ and $v \in (2, 2_s^*)$ such that

$$|f(x, u)| \leq C(|u| + |u|^{v-1}), \quad \forall (x, u) \in \mathbb{R}^N \times \mathbb{R}, \quad (4)$$

where 2_s^* denotes the critical Sobolev exponent, that is, $2_s^* = 2N/(N - 2s)$.

(f₂) $F(x, 0) \equiv 0$, $F(x, u) \geq 0$ for all $(x, u) \in \mathbb{R}^N \times \mathbb{R}$, and $\lim_{|u| \rightarrow \infty} (F(x, u)/u^2) = +\infty$ uniformly on $\mathbb{R}^N$.

(f_3) There exists a constant $\theta \geq 1$ such that

$$\theta \tilde{F}(x, u) \geq \tilde{F}(x, su), \quad \forall (x, u) \in R^N \times R, \quad s \in [0, 1], \quad (5)$$

where $\tilde{F}(x, u) := uf(x, u) - 2F(x, u)$.

(f_4) $f(x, -u) = -f(x, u)$, $\forall (x, u) \in R^N \times R$.

Our main theorem of this work reads as follows.

Theorem 1. Suppose that (V_1) , (V_2) , and (f_1) – (f_4) hold. Then problem (1) possesses infinitely many nontrivial solutions.

Remark 2. Conditions like (V_1) and (V_2) have been given in [1], but there $\inf_{R^N} V(x) > 0$ is required. As shown in [10], the condition (f_3) due to [11] is somewhat weaker than the condition that $f(x, u)/|u|$ is nondecreasing in u for all $x \in R^N$.

Remark 3. It follows from (V_1) that there exists a constant $V_0 \geq 0$ such that $\bar{V}(x) := V(x) + V_0 \geq 1$ for all $x \in R^N$. Let $\bar{f}(x, u) = f(x, u) + V_0 u$ for all $(x, u) \in R^N \times R$ and consider the following new fractional Schrödinger equations

$$(-\Delta)^s u + \bar{V}(x)u = \bar{f}(x, u), \quad x \in R^N. \quad (6)$$

Then, problem (1) and problem (6) are equivalent. Evidently, the hypotheses (V_1) , (V_2) , and (f_1) – (f_4) still hold for $\bar{V}$ and $\bar{f}$ provided that those hold for V and f . Hence, we can assume without loss of generality that $V(x) \geq 1$ for all $x \in R^N$ in (V_1) .

2. The Proof of Main Result

In this section, $s \in (0, 1)$ is a fixed number. We denote by $\|\cdot\|_p$ the usual norm of the space $L^p(R^N)$. c_i ($i = 1, 2, \dots$) or C denotes some positive constants.

In the light of finite differences, the nonhomogeneous Sobolev space $H^s(R^N)$ is defined by

$$\begin{aligned} H^s(R^N) \\ = \left\{ u \in L^2(R^N) : \frac{|u(x) - u(y)|}{|x - y|^{N/2+s}} \in L^2(R^N \times R^N) \right\}. \end{aligned} \quad (7)$$

It is a Hilbert space, when endowed with the scalar product given by

$$\begin{aligned} (u, v) \\ = \int_{R^N} uv \, dx \\ + \int_{R^N} \int_{R^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy. \end{aligned} \quad (8)$$

The corresponding norm is therefore

$$\begin{aligned} \|u\|_{H^s} \\ = \left(\int_{R^N} |u|^2 \, dx + \int_{R^N} \int_{R^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{1/2}. \end{aligned} \quad (9)$$

The space $H^s(R^N)$ is also denoted by the Fourier transform. Indeed, it is defined as follows:

$$\begin{aligned} H^s(R^N) = \left\{ u \right. \\ \left. \in L^2(R^N) : \int_{R^N} (1 + |\xi|^2)^s |\mathcal{F}u(\xi)|^2 \, d\xi < +\infty \right\}. \end{aligned} \quad (10)$$

This space has a Hilbert structure when endowed with the scalar product

$$(u, v) = \int_{R^N} (1 + |\xi|^2)^s \mathcal{F}u(\xi) \mathcal{F}v(\xi) \, d\xi, \quad (11)$$

so that the corresponding norm is

$$\|u\|_{H^s} = \left(\int_{R^N} (1 + |\xi|^2)^s |\mathcal{F}u(\xi)|^2 \, d\xi \right)^{1/2}. \quad (12)$$

To illustrate the relationship of the above two norms, let us start from the concept of Schwartz function $\mathcal{S}$ (is dense in $H^s(R^N)$), that is, the rapidly decreasing C^∞ function on R^N , which will be used later. If $u \in \mathcal{S}$, the fractional Laplacian $(-\Delta)^s$ acts on u as

$$\begin{aligned} (-\Delta)^s u(x) &= C(N, s) \text{P.V.} \int_{R^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy \\ &= C(N, s) \lim_{\epsilon \rightarrow 0^+} \int_{R^N \setminus B(0, \epsilon)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy, \end{aligned} \quad (13)$$

where the symbol P.V. represents the principle value of the integral and the constant $C(N, s)$ depends only on the space dimension N and on the order s . We can write an integral expression for $C(N, s)$ in the form

$$C(N, s) = \int_{R^N} \frac{1 - \cos \zeta}{|\zeta|^{N+2s}} \, d\zeta. \quad (14)$$

In [2], the authors have proved

$$\begin{aligned} (-\Delta)^s u &= \mathcal{F}(|\xi|^{2s} \mathcal{F}u), \\ [u]_{H^s}^2 &= \frac{2}{C(N, s)} \int_{R^N} |\xi|^{2s} |\mathcal{F}u|^2 \, d\xi, \end{aligned} \quad (15)$$

where

$$[u]_{H^s} = \left(\int_{R^N} \int_{R^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{1/2} \quad (16)$$

is the Gagliardo (semi)norm. Moreover, by the Plancherel formula in Fourier analysis, it is easy to show that

$$[u]_{H^s}^2 = \frac{2}{C(N, s)} \|(-\Delta)^{s/2} u\|_{L^2}^2. \quad (17)$$

Hence, the norms on $H^s(R^N)$ which was defined above are all equivalent.

For our problem (1), the Hilbert space H is defined by

$$H = \left\{ u \in H^s(R^N) : \int_{R^N} |\xi|^{2s} |\mathcal{F}u|^2 d\xi + \int_{R^N} V(x) |u|^2 dx < +\infty \right\}. \quad (18)$$

The inner product and the norm are defined as

$$(u, v) = \int_{R^N} V(x) uv dx + \int_{R^N} |\xi|^{2s} \mathcal{F}u(\xi) \mathcal{F}v(\xi) d\xi, \quad (19)$$

$$\|u\| = \left(\int_{R^N} |\xi|^{2s} |\mathcal{F}u|^2 d\xi + \int_{R^N} V(x) |u|^2 dx \right)^{1/2}.$$

Under the assumptions (V_1) and (V_2) , we have the following lemma due to [10].

Lemma 4. *The Hilbert space H is compactly embedded in $L^t(R^N)$ for $2 \leq t < 2_s^* = 2N/(N - 2s)$.*

Definition 5. A weak solution to problem (1) is a function $u \in H$ such that

$$\begin{aligned} & \int_{R^N} V(x) uv dx + \int_{R^N} |\xi|^{2s} \mathcal{F}u(\xi) \mathcal{F}v(\xi) d\xi \\ &= \int_{R^N} f(x, u) v dx, \quad \forall u \in H. \end{aligned} \quad (20)$$

The energy functional associated with problem (1) is defined by

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \int_{R^N} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi + \frac{1}{2} \int_{R^N} V(x) |u|^2 dx \\ &\quad - \int_{R^N} F(x, u) dx, \quad u \in H. \end{aligned} \quad (21)$$

By Lemma 4 and conditions (f_1) and (f_2) , we can prove that Φ is well defined and $\Phi \in C^1(H)$ with

$$\begin{aligned} \langle \Phi'(u), v \rangle &= \int_{R^N} V(x) uv dx \\ &\quad + \int_{R^N} |\xi|^{2s} \mathcal{F}u(\xi) \mathcal{F}v(\xi) d\xi \\ &\quad - \int_{R^N} f(x, u) v dx, \quad \forall u, v \in H. \end{aligned} \quad (22)$$

Therefore weak solutions of (1) correspond to critical points of Φ .

Let X be a Banach space with the norm $\|\cdot\|$ and let $\{X_j\}$ be a sequence of subspace of X with $\dim X_j < +\infty$ for each $j \in \mathcal{N}$. Further, $X = \overline{\bigoplus_{j \in \mathcal{N}} X_j}$, the closure of the direct sum of all X_j . Set

$$\begin{aligned} W_k &= \bigoplus_{j=0}^k X_j, \\ Z_k &= \overline{\bigoplus_{j=k}^{\infty} X_j}, \end{aligned} \quad (23)$$

$$B_k = \{u \in W_k : \|u\| \leq \rho_k\},$$

$$S_k = \{u \in Z_k : \|u\| = r_k\}$$

for $\rho_k > r_k > 0$. Consider a family of C^1 functionals $\Phi_\lambda : X \rightarrow \mathbb{R}$ defined by

$$\Phi_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1, 2]. \quad (24)$$

The following variant Fountain theorem comes from Zou [12].

Theorem 6. *Assume that the functional Φ_λ defined above satisfies the following:*

(A_1) Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$, and $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times X$.

(A_2) $B(u) \geq 0$ for all $u \in X$, $A(u) \rightarrow +\infty$ or $B(u) \rightarrow +\infty$ as $\|u\| \rightarrow +\infty$.

(A_3) There exists $\rho_k > r_k > 0$ such that

$$\begin{aligned} b_k(\lambda) &= \inf_{u \in Z_k, \|u\| = r_k} \Phi_\lambda(u) > a_k(\lambda) \\ &= \max_{u \in W_k, \|u\| = \rho_k} \Phi_\lambda(u), \quad \forall \lambda \in [1, 2]. \end{aligned} \quad (25)$$

Then

$$b_k(\lambda) \leq c_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi_\lambda(\gamma(u)), \quad \forall \lambda \in [1, 2], \quad (26)$$

where $\Gamma_k = \{\gamma \in C(B_k, X) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = \text{id}\}$. Moreover, for almost every $\lambda \in [1, 2]$, there exists a sequence $\{u_n^k(\lambda)\}$ such that

$$\begin{aligned} \sup_n \|u_n^k(\lambda)\| &< \infty, \\ \Phi'_\lambda(u_n^k(\lambda)) &\longrightarrow 0, \\ \Phi_\lambda(u_n^k(\lambda)) &\longrightarrow c_k(\lambda) \end{aligned} \quad (27)$$

as $n \rightarrow \infty$.

In order to use Theorem 6 to prove the main result, we define the functionals A , B , and Φ_λ on the working space H by

$$\begin{aligned} A(u) &= \frac{1}{2} \int_{R^N} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi \\ &\quad + \frac{1}{2} \int_{R^N} V(x) |u|^2 dx, \\ B(u) &= \int_{R^N} F(x, u) dx, \\ \Phi_\lambda(u) &= A(u) - \lambda B(u) = \frac{1}{2} \|u\|^2 - \lambda \int_{R^N} F(x, u) dx \end{aligned} \quad (28)$$

for all $u \in H$ and $\lambda \in [1, 2]$. We choose an orthonormal basis $\{e_j : j \in \mathcal{N}\}$ of H and let $X_j = \text{span}\{e_j\}$ for all $j \in \mathcal{N}$. Obviously $\Phi_1 = \Phi$. In order to complete the proof of our theorem, we need the following lemmas.

Lemma 7. Assume that (V_1) , (V_2) , and (f_1) hold. Then there exists a positive integer k_1 and a sequence $r_k \rightarrow \infty$ as $k \rightarrow \infty$ such that

$$b_k(\lambda) := \inf_{u \in Z_k, \|u\|=r_k} \Phi_\lambda(u) > 0, \quad \forall k \geq k_1, \quad (29)$$

where $Z_k = \overline{\bigoplus_{j=k}^\infty X_j} = \overline{\text{span}\{e_k, \dots\}}$ for all $k \in \mathcal{N}$.

Proof. From the definition of $\Phi_\lambda(u)$ and (f_1) , there holds

$$\begin{aligned} \Phi_\lambda(u) &\geq \frac{1}{2} \|u\|^2 - 2 \int_{R^N} F(x, u) dx \\ &\geq \frac{1}{2} \|u\|^2 - 2c_1 (\|u\|_2^2 + \|u\|_v^v), \end{aligned} \quad (30)$$

$$\forall (\lambda, u) \in [1, 2] \times H,$$

where c_1 is a constant. Let

$$\begin{aligned} l_2(k) &= \sup_{u \in Z_k, \|u\|=1} \|u\|_2, \\ l_v(k) &= \sup_{u \in Z_k, \|u\|=1} \|u\|_v, \\ &\quad \forall k \in \mathcal{N}. \end{aligned} \quad (31)$$

Since H is compactly embedded into both $L^2(R^N)$ and $L^v(R^N)$, we have

$$\begin{aligned} l_2(k) &\longrightarrow 0, \\ l_v(k) &\longrightarrow 0 \\ &\text{as } k \longrightarrow \infty. \end{aligned} \quad (32)$$

From (30) and (31), it follows that

$$\begin{aligned} \Phi_\lambda(u) &\geq \frac{1}{2} \|u\|^2 - 2c_1 l_2^2(k) \|u\|^2 - 2c_1 l_v^v(k) \|u\|^v, \\ &\quad \forall (\lambda, u) \in [1, 2] \times Z_k. \end{aligned} \quad (33)$$

From (32), there exists a positive integer k_1 such that

$$2c_1 l_2^2(k) \leq \frac{1}{8}, \quad \forall k \geq k_1. \quad (34)$$

For each $k \geq k_1$, let $r_k = (16c_1 l_v^v(k))^{1/(2-v)}$. Then $r_k \rightarrow +\infty$ as $k \rightarrow \infty$, since $v > 2$. It is immediate to check that

$$b_k(\lambda) := \inf_{u \in Z_k, \|u\|=r_k} \Phi_\lambda(u) \geq \frac{r_k^2}{4} > 0, \quad \forall k \geq k_1. \quad (35)$$

Hence, the proof of Lemma 7 is complete. $\square$

Lemma 8. Assume that (V_1) , (V_2) , (f_1) , and (f_2) hold. Then for the positive integer k_1 and the sequence $\{r_k\}$ obtained in Lemma 7, there exist $\rho_k > r_k$ for each $k \geq k_1$, such that

$$a_k(\lambda) = \max_{u \in W_k, \|u\|=\rho_k} \Phi_\lambda(u) < 0, \quad \forall k \geq k_1, \quad (36)$$

where $W_k = \bigoplus_{j=1}^k X_j = \text{span}\{e_1, \dots, e_k\}$ for all $k \in \mathcal{N}$.

Proof. To begin with, we carry out that for any finite dimensional subspace $F \in H$ there exists a constant $\varepsilon > 0$ such that

$$\begin{aligned} \text{meas} \left(\left\{ x \in R^N : |u(x)| \geq \varepsilon \|u\| \right\} \right) &\geq \varepsilon, \\ &\quad \forall u \in F \setminus \{0\}. \end{aligned} \quad (37)$$

Assume for contradiction that, for any $n \in \mathcal{N}$, there exists $u_n \in F \setminus \{0\}$ such that

$$\text{meas} \left(\left\{ x \in R^N : |u(x)| \geq \frac{\|u_n\|}{n} \right\} \right) < \frac{1}{n}. \quad (38)$$

Let $v_n = u_n / \|u_n\| \in F$ for all $n \in \mathcal{N}$. Then $\|v_n\| = 1$ for all $n \in \mathcal{N}$, and

$$\text{meas} \left(\left\{ x \in R^N : |v_n(x)| \geq \frac{1}{n} \right\} \right) < \frac{1}{n}, \quad \forall n \in \mathcal{N}. \quad (39)$$

Up to a subsequence, if necessary, we can say that $v_n \rightarrow v_0$ in H for some $v_0 \in F$ since F is of finite dimension. Evidently, $\|v_0\| = 1$. By Lemma 4 and the equivalence of any two norms on F , we conclude that

$$\int_{R^N} |v_n - v_0|^2 dx \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (40)$$

Since $v_0 \neq 0$, we see that there exists a constant $\delta_0 > 0$ satisfying

$$\text{meas} \left(\left\{ x \in R^N : |v_0(x)| \geq \delta_0 \right\} \right) \geq \delta_0. \quad (41)$$

For any $n \in \mathcal{N}$, we define the sets

$$\begin{aligned} \Lambda_n &= \left\{ x \in R^N : |v_n(x)| \leq \frac{1}{n} \right\}, \\ \Lambda_n^c &= R^N \setminus \Lambda_n = \left\{ x \in R^N : |v_n(x)| \geq \frac{1}{n} \right\}. \end{aligned} \quad (42)$$

Let $\Lambda_0 = \{x \in R^N : |v_0(x)| \geq \delta_0\}$. Then for sufficiently large n , from (39) and (41), one can easily see that

$$\begin{aligned} \text{meas}(\Lambda_n \cap \Lambda_0) &\geq \text{meas}(\Lambda_0) - \text{meas}(\Lambda_n^c) \geq \delta_0 - \frac{1}{n} \\ &\geq \frac{\delta_0}{2}. \end{aligned} \quad (43)$$

So that, for n large enough, we obtain

$$\begin{aligned} \int_{R^N} |v_n - v_0|^2 dx &\geq \int_{\Lambda_n \cap \Lambda_0} |v_n - v_0|^2 dx \\ &\geq \int_{\Lambda_n \cap \Lambda_0} (|v_n| - |v_0|)^2 dx \\ &\geq \left(\delta_0 - \frac{1}{n}\right)^2 \cdot \text{meas}(\Lambda_n \cap \Lambda_0) \\ &\geq \frac{\delta_0^3}{8} > 0. \end{aligned} \quad (44)$$

This is impossible, and (37) holds.

Since W_k is finite dimensional for each $k \in \mathcal{N}$, we deduce from this and (37) that

$$\text{meas}(\Lambda_u^k) \geq \varepsilon_k, \quad \forall u \in W_k \setminus \{0\}, \quad (45)$$

where $\Lambda_u^k = \{x \in R^N : |u(x)| \geq \varepsilon_k \|u\|\}$ for all $k \in \mathcal{N}$ and $u \in W_k \setminus \{0\}$. By (f_2) , for each $k \in \mathcal{N}$, there exists a constant $b_k > 0$ such that

$$F(x, u) \geq \frac{u^2}{\varepsilon_k^3}, \quad \forall x \in R^N, |u| \geq b_k. \quad (46)$$

So, by (45), (46), and (f_2) , for any $k \in \mathcal{N}$ and $\lambda \in [1, 2]$, we have

$$\begin{aligned} \Phi_\lambda(u) &\leq \frac{1}{2} \|u\|^2 - \int_{R^N} F(x, u) dx \\ &\leq \frac{1}{2} \|u\|^2 - \int_{\Lambda_u^k} \left(\frac{|u|^2}{\varepsilon_k^3} \right) dx \\ &\leq \frac{1}{2} \|u\|^2 - \varepsilon_k^2 \|u\|^2 \cdot \frac{\text{meas}(\Lambda_u^k)}{\varepsilon_k^3} \\ &\leq \frac{1}{2} \|u\|^2 - \|u\|^2 = -\frac{1}{2} \|u\|^2 \end{aligned} \quad (47)$$

for all $u \in W_k$ with $\|u\| \geq b_k/\varepsilon_k$. Let

$$\rho_k > \max \left\{ r_k, \frac{b_k}{\varepsilon_k} \right\}, \quad \forall k \geq k_1. \quad (48)$$

We see that

$$\begin{aligned} a_k(\lambda) &:= a_k(\lambda) = \max_{u \in W_k, \|u\|=\rho_k} \Phi_\lambda(u) \leq -\frac{\rho_k^2}{2} < 0, \\ &\forall k \geq k_1. \end{aligned} \quad (49)$$

The proof of Lemma 8 is complete. $\square$

Proof of Theorem 1. Now the assumption (f_1) and the definition of $\Phi_\lambda(u)$ imply that Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. From (f_4) , we obtain $\Phi_\lambda(u) = \Phi_\lambda(-u)$ for all $(\lambda, u) \in [1, 2] \times H$. Therefore, the condition (A_1) of Theorem 6 holds. Clearly, the condition (A_2) holds by the definitions of $A(u)$, $B(u)$ and (f_2) . Moreover, Lemmas 7 and 8 imply that the condition (A_3) holds for all $k \geq k_1$, where k_1 is given in Lemma 7. Thus, it follows from Theorem 6 that, for each $k \geq k_1$, a.e. $\lambda \in [1, 2]$, there exists a sequence $\{u_m^k(\lambda)\} \subset H$ such that

$$\begin{aligned} \sup_m \|u_m^k(\lambda)\| &< \infty, \\ \Phi'_\lambda(u_m^k(\lambda)) &\longrightarrow 0 \\ \Phi_\lambda(u_m^k(\lambda)) &\longrightarrow c_k(\lambda) \\ &\text{as } m \longrightarrow \infty, \end{aligned} \quad (50)$$

where

$$c_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi_\lambda(\gamma(u)), \quad \forall \lambda \in [1, 2], \quad (51)$$

with $B_k = \{u \in W_k : \|u\| \leq \rho_k\}$ and $\Gamma_k = \{\gamma \in C(B_k, X) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = id\}$. $\square$

Furthermore, the proof of Lemma 7 shows that

$$c_k(\lambda) \in [\bar{b}_k, \bar{c}_k], \quad \forall k \geq k_1, \lambda \in [1, 2], \quad (52)$$

where $\bar{c}_k = \max_{u \in B_k} \Phi_1(u)$ and $\bar{b}_k := r_k^2/4 \rightarrow \infty$ as $k \rightarrow \infty$.

For each $k \geq k_1$, (50) shows that we can choose $\lambda_n \rightarrow 1$ (depending on k) and can get the corresponding sequences satisfying

$$\begin{aligned} \sup_m \|u_m^k(\lambda_n)\| &< \infty, \\ \Phi'_{\lambda_n}(u_m^k(\lambda_n)) &\longrightarrow 0 \\ &\text{as } m \longrightarrow \infty. \end{aligned} \quad (53)$$

We prove the following 2 claims.

Claim 1. For each λ_n given above, the sequence $\{u_m^k(\lambda_n)\}_m^\infty$ has a convergent subsequence.

Throughout this paragraph, for the simplicity of notation, we denote $u_m = u_m^k(\lambda_n)$ by $m \in \mathcal{N}$ throughout this paragraph. Indeed, by (53), we may let

$$u_m \rightharpoonup u_0 \quad \text{as } m \longrightarrow \infty, \quad (54)$$

for some $u_0 \in H$. With the aid of Riesz Representation Theorem, $\Phi'_{\lambda_n} : H \rightarrow H^*$ and $B' : H \rightarrow H^*$ can be written as $\Phi'_{\lambda_n} : H \rightarrow H$ and $B' : H \rightarrow H$, respectively. Hence

$$u_m = \Phi'_{\lambda_n}(u_m) + \lambda_n B'(u_m), \quad \forall m \in \mathcal{N}. \quad (55)$$

From Lemma 4, (53), (54), and (55), we see immediately that the right-hand side of (55) converges strongly in H , so that $u_m \rightarrow u_0$ in H , and then the proof of Claim 1 is complete.

Therefore, by Claim 1, up to subsequences, we may assume

$$\lim_{m \rightarrow \infty} u_m^k(\lambda_n) = u_n^k, \quad \forall n \in \mathcal{N}, \quad k \geq k_1, \quad (56)$$

and hence from (50) and (52) we deduce that

$$\begin{aligned} \Phi'_{\lambda_n}(u_n^k) &= 0, \\ \Phi_{\lambda_n}(u_n^k) &\in [\bar{b}_k, \bar{c}_k], \\ \forall n \in \mathcal{N}, \quad k &\geq k_1. \end{aligned} \quad (57)$$

Claim 2. For each $k \geq k_1$, the sequence $\{u_n^k\}_{n=1}^\infty$ in (56) is bounded.

Throughout this paragraph, for notational simplicity, we will write $u_n = u_n^k$ for all $n \in \mathcal{N}$.

Otherwise, up to subsequences, we may assume

$$\begin{aligned} \|u_n\| &\rightarrow \infty, \\ w_n := \frac{u_n}{\|u_n\|} &\rightarrow w_0 \in H \end{aligned} \quad (58)$$

as $n \rightarrow \infty$.

By Lemma 4 and (58), we can prove that

$$w_n \rightarrow w_0 \quad \text{in } L^p(R^N) \quad \text{for } 2 \leq p < 2_s^*. \quad (59)$$

Up to subsequences, if necessary, we have

$$w_n(x) \rightarrow w_0(x) \quad \text{a.e. } x \in R^N. \quad (60)$$

If $w_0 = 0$, then there exists a sequence $\{t_n\} \subset [0, 1]$ such that

$$\Phi_{\lambda_n}(t_n u_n) = \max_{t \in [0, 1]} \Phi_{\lambda_n}(t u_n). \quad (61)$$

For any $D > 0$, setting $\bar{w}_n = \sqrt{4D} w_n$, by (59) and (f₂), we obtain that

$$\begin{aligned} 0 &\leq \int_{R^N} F(x, \bar{w}_n) dx \leq C \int_{R^N} (|\bar{w}_n|^2 + |\bar{w}_n|^v) dx \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (62)$$

Thus, for n large enough, there holds

$$\begin{aligned} \Phi_{\lambda_n}(t_n u_n) &\geq \Phi_{\lambda_n}(\bar{w}_n) \\ &= \frac{1}{2} \|\bar{w}_n\|^2 - \lambda_n \int_{R^N} F(x, \bar{w}_n) dx \\ &= 2D - \lambda_n \int_{R^N} F(x, \bar{w}_n) dx \geq D. \end{aligned} \quad (63)$$

These yield immediately $\lim_{n \rightarrow \infty} \Phi_{\lambda_n}(t_n u_n) = +\infty$. Since $\Phi_{\lambda_n}(0) = 0$ and $\Phi_{\lambda_n}(u_n) \in [\bar{b}_k, \bar{c}_k]$ in (57), we obtain $t_n \in (0, 1)$ in (61) for n large enough, which shows that

$$0 = \frac{d}{dt} \Big|_{t=t_n} \Phi'_{\lambda_n}(t u_n) = \langle \Phi'_{\lambda_n}(t u_n), t_n u_n \rangle. \quad (64)$$

Hence, from (64) and (f₃), we deduce that

$$\begin{aligned} \Phi_{\lambda_n}(u_n) &= \Phi_{\lambda_n}(u_n) - \frac{1}{2} \langle \Phi'_{\lambda_n}(u_n), u_n \rangle \\ &= \frac{\lambda_n}{2} \int_{R^N} \tilde{F}(x, u_n) dx \geq \frac{\lambda_n}{2\theta} \tilde{F}(x, t_n u_n) dx \\ &= \frac{1}{\theta} \Phi_{\lambda_n}(t_n u_n) - \frac{1}{2\theta} \langle \Phi'_{\lambda_n}(t_n u_n), t_n u_n \rangle \\ &= \frac{1}{\theta} \Phi_{\lambda_n}(t_n u_n) \rightarrow +\infty \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (65)$$

where θ is given in (f₃). By (57), this is impossible.

Since $w_0 \neq 0$, of course $\Omega := \{x \in R^N : w_0(x) \neq 0\}$ has positive measure. Equation (58) shows that

$$u_n(x) \rightarrow \infty, \quad x \in \Omega. \quad (66)$$

From (58), (66), (f₂), and Fatou's lemma, it is easy to see that

$$\begin{aligned} \frac{1}{2} - \frac{\Phi_{\lambda_n} u_n}{\|u_n\|^2} &= \lambda_n \int_{R^N} \frac{F(x, u_n)}{\|u_n\|^2} dx \\ &\geq \lambda_n \int_{\Omega} |w_n|^2 \frac{F(x, u_n)}{|u_n|^2} dx \rightarrow +\infty \end{aligned} \quad (67)$$

as $n \rightarrow \infty$.

This contradiction shows that $w_0 \neq 0$ cannot hold and concludes the proof of Claim 2.

For each $k \geq k_1$, using the same arguments in the proof of Claim 1, one can also prove that the sequence $\{u_n^k\}_{n=1}^\infty$ has a strong convergent subsequence with the limit u^k being just the critical point of $\Phi = \Phi_1$ by Claim 2 and (57). Clearly, $\Phi(u^k) \in [\bar{b}_k, \bar{c}_k]$ for all $k \geq k_1$. These imply that Φ have infinitely many nontrivial critical points of since $\bar{b}_k \rightarrow +\infty$ as $k \rightarrow \infty$. Therefore, problem (1) admits infinitely many nontrivial solutions and the proof of Theorem 1 is complete.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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