Research Article

A Regularity Criterion for the 3D Incompressible Magnetohydrodynamics Equations in the Multiplier Spaces

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We are concerned with the regularity criterion for weak solutions to the 3D incompressible MHD equations in this paper. We show that if some partial derivatives of the velocity components and magnetic components belong to the multiplier spaces, then the solution actually is smooth on $(0, T)$.

1. Introduction

We consider the 3D incompressible magnetohydrodynamics (MHD) equations.

(MHD)

$$\begin{aligned} \frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u &= -\nabla p - \frac{1}{2} \nabla b^2 + b \cdot \nabla b, \\ \frac{\partial b}{\partial t} - \eta \Delta b + u \cdot \nabla b &= b \cdot \nabla u, \\ \nabla \cdot u &= \nabla \cdot b = 0, \\ u(0, x) &= u_0(x), \\ b(0, x) &= b_0(x). \end{aligned} \quad (1)$$

Here u, b describe the flow velocity vector and the magnetic field vector, respectively, p is a scalar pressure, $\nu > 0$ is the kinematic viscosity, and $\eta > 0$ is the magnetic diffusivity, while u_0 and b_0 are the given initial velocity and initial magnetic field, respectively, with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. If $\nu = \eta = 0$, (1) is called the ideal MHD equations.

Employing the standard energy method, it is easy to prove that, for given initial data $(u_0, b_0) \in H^s(\mathbb{R}^3)$ with $s > 1/2$, there exist a positive time $T = T(\|(u_0, b_0)\|_{H^s})$ and

a unique smooth solution $(u(t, x), b(t, x))$ on $[0, T)$ to the MHD equations satisfying

$$\begin{aligned} (u, b) &\in C([0, T]; H^s) \cap C^1((0, T); H^s) \\ &\cap C((0, T); H^{s+2}). \end{aligned} \quad (2)$$

Whether smooth solutions of (1) on $[0, T)$ will lead to a singularity at $t = T$ is an outstanding open problem; see Sermange and Temam [1].

However, the solution regularity can be derived when certain growth conditions are satisfied. For regularity of the weak solutions to the 3D MHD equations (1), some numerical experiments [2, 3] seem to indicate that the velocity field plays a more important role than the magnetic field in the regularity theory of solutions to the MHD equations. Recently, inspired by Constantin and Fefferman initial work [4] where the regularity condition of the direction of vorticity was used to describe the regularity criterion to the Navier-Stokes equations, He and Xin [5] extended it to the MHD equations and obtained some integrability condition of the magnitude of the only velocity u alone; that is,

$$\int_0^T \|u(t)\|_p^q dt < \infty, \quad \frac{2}{q} + \frac{3}{p} \leq 1, \quad 3 < p \leq \infty. \quad (3)$$

Later, Zhou and Gala [6] extended it to the multiplier spaces. Other important studies such as the regularity criteria can be found in [7–9] and references therein.

Recently, some regularity criteria in terms of partial velocity components and magnetic components or partial derivative of the velocity components and magnetic components were established [10–12]. However, the spaces used are not scaling invariant (in other words, not of Serrin's type). Many researchers were devoted to studying it along this direction. In 2010, Ji and Lee [13] obtained the following regularity:

$$\begin{aligned} \tilde{u} &\in L^p([0, T]; L^q(\mathbb{R}^3)), \\ \tilde{b} &\in L^l([0, T]; L^m(\mathbb{R}^3)), \\ \frac{2}{p} + \frac{3}{q} &\leq 1, \quad \frac{2}{l} + \frac{3}{m} \leq 1, \quad 3 < q, m \leq \infty. \end{aligned} \quad (4)$$

In 2012, Ni et al. [14] got some new regularity as follows:

$$\begin{aligned} \tilde{\nabla} u &\in L^p([0, T]; L^q(\mathbb{R}^3)), \\ \tilde{\nabla} b &\in L^l([0, T]; L^m(\mathbb{R}^3)), \\ \frac{2}{p} + \frac{3}{q} &\leq 2, \quad \frac{2}{l} + \frac{3}{m} \leq 2, \quad \frac{3}{2} < q, m \leq \infty. \end{aligned} \quad (5)$$

Our purpose in this paper is to obtain a new regularity criterion of weak solution for the 3D MHD equations in a sense of scaling invariant by employing a different decomposition for nonlinear terms.

Definition 1 (see [1]). A weak solution pair (u, b) of the 3D MHD equations (1) is regular in $[0, T] \times H^1(\mathbb{R}^3)$ provided that $(u, b) \in L^\infty(0, T; H^1(\mathbb{R}^3))$.

Notation 1. Throughout the paper, we use the following notations for the simplicity. The planar components of (u, b) will be denoted by $\tilde{u} = (u_1, u_2)$ and $\tilde{b} = (b_1, b_2)$ and $\tilde{\nabla}$ will be used for $\tilde{\nabla} = (\partial_1, \partial_2)$.

Now we state our result as follows.

Theorem 2. Let $(u_0, b_0) \in H^1$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. If the weak solutions (u, b) to (1) satisfy the following integrability conditions:

$$\tilde{\nabla} \tilde{u}, \tilde{\nabla} \tilde{b} \in L^{2/(2-r)}([0, T]; \dot{X}_r(\mathbb{R}^3)), \quad \text{with } r \in [0, 1), \quad (6)$$

then the solution (u, b) remains smooth on $[0, T]$.

Remark 3. Since the components of $\tilde{\nabla} \tilde{u}$ and $\tilde{\nabla} \tilde{b}$ without u_3 and b_3 are less than those of $\tilde{\nabla} u$ and $\tilde{\nabla} b$, therefore, our result improves that in [14].

Remark 4. Since

$$L^{3/r}(\mathbb{R}^3) \hookrightarrow L^{3/r, \infty}(\mathbb{R}^3) \hookrightarrow \dot{X}_r(\mathbb{R}^3) \quad (7)$$

as $0 \leq r \leq 3/2$, where $L^{p, \infty}$ denotes the weak L^p -space, therefore, our result extends that in [14].

2. Preliminaries

First, we recall the definition and some properties of the multiplier space $\dot{X}_r$ introduced recently by Lemarié-Rieusset [15] (see also [16]). The space $\dot{X}_r$ of pointwise multipliers which map L^2 into $\dot{H}^{-r}$ is defined in the following way.

Definition 5. For $0 \leq r < 3/2$, we define the homogeneous space $\dot{X}_r$ by

$$\dot{X}_r = \{f \in L^2_{\text{loc}} : \forall g \in \dot{H}^r, fg \in L^2\}, \quad (8)$$

where we denote by $\dot{H}^r(\mathbb{R}^d)$ the completion of the space $\mathcal{D}(\mathbb{R}^d)$ with respect to the norm $\|u\|_{\dot{H}^r} = \|(-\Delta)^{1/2} u\|_{L^2} = \|\xi|^r \hat{u}(\xi)\|_{L^2}$, where $\hat{u}(\xi)$ denotes the Fourier transform of u .

The norm of $\dot{X}_r$ is given by the operator norm of pointwise multiplication

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{\dot{H}^r} \leq 1} \|fg\|_{L^2} < \infty. \quad (9)$$

It is easy to check that

$$\|f(\lambda \cdot)\|_{\dot{X}_r} \leq \frac{1}{\lambda^r} \|f\|_{\dot{X}_r}, \quad \lambda > 0, \quad (10)$$

$$\|f(\cdot + x_0)\|_{\dot{X}_r} = \|f\|_{\dot{X}_r}.$$

Hence, for any function $f(x, t)$ defined for both spatial and time variables,

$$\|f_\lambda\|_{L^2([0, T]; \dot{X}_r)} = \|f\|_{L^2([0, T]; \dot{X}_r)}, \quad (11)$$

for any $\lambda > 0$, with $f_\lambda(x, t) = \lambda f(\lambda x, \lambda^2 t)$. Therefore, if (u, b) solves the MHD equations, then so does (u_λ, b_λ) . This is the so-called scaling dimension zero property.

Lemma 6. Assume $0 \leq r \leq 1$. Then the following inequality holds:

$$\|u\|_{\dot{H}^r} \leq \|u\|_{L^2}^{1-r} \|\nabla u\|_{L^2}^r. \quad (12)$$

The proof of this lemma can be easily obtained by Parseval's equality and Hölder's inequality, and we omit it.

3. Proof of Theorem 2

By Definition 1, we need to estimate $(u, b) \in L^\infty(0, T; H^1(\mathbb{R}^3))$. Thus, we will divide the proof of Theorem 2 into two steps as follows.

Step 1 (L^2 -estimates). Multiplying the first equation and the second equation in (1) by u, b , respectively, and integrating the resulting equations by parts over $\mathbb{R}^3$, we obtain after adding them together from the incompressibility condition $\nabla \cdot u = 0$ and $\nabla \cdot b = 0$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int (|u|^2 + |b|^2) dx + \int (|\nabla u|^2 + |\nabla b|^2) dx \\ = \langle (-u \cdot \nabla u + b \cdot \nabla b), u \rangle + \langle (-u \cdot \nabla b + b \cdot \nabla u), b \rangle \\ = 0, \end{aligned} \quad (13)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner-product in $L^2(\mathbb{R}^3)$. Integrating from 0 to t for the above inequality, we have

$$(u, b) \in L^\infty([0, T]; L^2) \cap L^2((0, T); \dot{H}^1). \quad (14)$$

Step 2 ($\dot{H}^1$ -estimates). Differentiating the first equation and the second equation of (1) with respect to x_k ($1 \leq k \leq 3$) and multiplying the first and the second equations of (1) by $\partial u / \partial x_k = \partial_k u$ and $\partial b / \partial x_k = \partial_k b$, respectively, and then, by integrating by parts over $\mathbb{R}^3$, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_k u\|_{L^2}^2 + \|\nabla \partial_k u\|_{L^2}^2 \\ &= - \int \partial_k [(u \cdot \nabla) u] \cdot \partial_k u \, dx - \int \partial_k \nabla p \cdot \partial_k u \, dx \end{aligned} \quad (15)$$

$$\begin{aligned} & + \int \partial_k [(b \cdot \nabla) b] \cdot \partial_k u \, dx, \\ & \frac{1}{2} \frac{d}{dt} \|\partial_k b\|_{L^2}^2 + \|\nabla \partial_k b\|_{L^2}^2 \\ &= - \int \partial_k [(u \cdot \nabla) b] \partial_k b \, dx \\ & + \int \partial_k [(b \cdot \nabla) u] \partial_k b \, dx. \end{aligned} \quad (16)$$

Noting the incompressibility conditions $\nabla \cdot u = 0$ and $\nabla \cdot b = 0$, since

$$\begin{aligned} & \int \partial_k [(u \cdot \nabla) u] \cdot \partial_k u \, dx = \int (\partial_k u \cdot \nabla) u \cdot \partial_k u \, dx, \\ & \int \partial_k [(b \cdot \nabla) b] \cdot \partial_k u \, dx \\ &= \int (\partial_k b \cdot \nabla) b \cdot \partial_k u \, dx + \int (b \cdot \nabla) \partial_k b \cdot \partial_k u \, dx, \\ & \int \partial_k \nabla p \cdot \partial_k u \, dx = 0, \\ & \int \partial_k [(u \cdot \nabla) b] \partial_k b \, dx = \int (\partial_k u \cdot \nabla) b \partial_k b \, dx, \\ & \int \partial_k [(b \cdot \nabla) u] \partial_k b \, dx \\ &= \int (\partial_k b \cdot \nabla) u \partial_k b \, dx + \int (b \cdot \nabla) \partial_k u \partial_k b \, dx, \end{aligned} \quad (17)$$

then (15) and (16) can be rewritten as

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\partial_k u\|_{L^2}^2 + \|\nabla \partial_k u\|_{L^2}^2 \\ &= - \int (\partial_k u \cdot \nabla) u \cdot \partial_k u \, dx + \int (\partial_k b \cdot \nabla) b \cdot \partial_k u \, dx \\ & + \int (b \cdot \nabla) \partial_k b \cdot \partial_k u \, dx, \\ & \frac{1}{2} \frac{d}{dt} \|\partial_k b\|_{L^2}^2 + \|\nabla \partial_k b\|_{L^2}^2 \end{aligned}$$

$$\begin{aligned} &= - \int (\partial_k u \cdot \nabla) b \partial_k b \, dx + \int (\partial_k b \cdot \nabla) u \partial_k b \, dx \\ & + \int (b \cdot \nabla) \partial_k u \partial_k b \, dx. \end{aligned} \quad (18)$$

Since $\int (b \cdot \nabla) \partial_k u \partial_k b \, dx + \int (b \cdot \nabla) \partial_k b \cdot \partial_k u \, dx = 0$, adding up (18), we have by summing up over k ($1 \leq k \leq 3$)

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2 \\ &= - \sum_{k=1}^3 \int (\partial_k u \cdot \nabla) u \cdot \partial_k u \, dx \\ & + \sum_{k=1}^3 \int (\partial_k b \cdot \nabla) b \cdot \partial_k u \, dx \\ & - \sum_{k=1}^3 \int (\partial_k u \cdot \nabla) b \partial_k b \, dx \\ & + \sum_{k=1}^3 \int (\partial_k b \cdot \nabla) u \partial_k b \, dx \triangleq I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (19)$$

First, we rewrite I_1 into seven parts as follows:

$$\begin{aligned} I_1 &= - \sum_{i,j,k=1}^3 \int \partial_k u_i \partial_i u_j \partial_k u_j \, dx \\ &= - \sum_{i,j,k=1}^2 \int \partial_k u_i \partial_i u_j \partial_k u_j \, dx \\ & - \sum_{j,k=1}^2 \int \partial_k u_3 \partial_3 u_j \partial_k u_j \, dx \\ & - \sum_{i,k=1}^2 \int \partial_k u_i \partial_i u_3 \partial_k u_3 \, dx \\ & - \sum_{i,j=1}^2 \int \partial_3 u_i \partial_i u_j \partial_3 u_j \, dx \\ & - \sum_{k=1}^3 \int \partial_k u_3 \partial_3 u_3 \partial_k u_j \, dx - \sum_{i=1}^2 \int \partial_3 u_i \partial_i u_3 \partial_3 u_3 \, dx \\ & - \sum_{j=1}^2 \int \partial_3 u_3 \partial_3 u_j \partial_3 u_j \, dx \triangleq \sum_{m=1}^7 I_{1m}. \end{aligned} \quad (20)$$

According to the definition of $\tilde{\nabla} \tilde{u}$, the former four terms of the right-hand side of (20) are bounded by

$$\left| \sum_{m=1}^4 I_{1m} \right| \leq C \int |\tilde{\nabla} \tilde{u}| \|\nabla u\| \|\nabla u\| \, dx. \quad (21)$$

And the latter three terms can be estimated, noting that $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$ from the incompressible condition

$$\begin{aligned} \left| \sum_{m=5}^7 I_{1m} \right| &\leq C \int |-\partial_1 u_1 - \partial_2 u_2| \|\nabla u\| |\nabla u| dx \\ &\leq C \int |\tilde{\nabla} \tilde{u}| \|\nabla u\| |\nabla u| dx. \end{aligned} \quad (22)$$

Thus, using Hölder's inequality and Young's inequality, we have

$$\begin{aligned} I_1 &\leq C \int |\tilde{\nabla} \tilde{u}| \|\nabla u\| |\nabla u| dx \leq C \|\tilde{\nabla} \tilde{u}\|_{L^2} \|\nabla u\|_{L^2} \\ &\leq C \|\tilde{\nabla} \tilde{u}\|_{\dot{X}^r} \|\nabla u\|_{\dot{H}^r} \|\nabla u\|_{L^2} \\ &\leq C \|\tilde{\nabla} \tilde{u}\|_{\dot{X}^r} \|\nabla u\|_{L^2}^{1-r} \|\nabla^2 u\|_{L^2}^r \|\nabla u\|_{L^2} \\ &\leq C \|\tilde{\nabla} \tilde{u}\|_{\dot{X}^r}^{2/(2-r)} \|\nabla u\|_{L^2}^2 + \frac{1}{2} \|\nabla^2 u\|_{L^2}^2. \end{aligned} \quad (23)$$

Next, we will bound the term I_2 . We can get, by splitting I_2 into seven parts,

$$\begin{aligned} I_2 &= \sum_{k=1}^3 \int (\partial_k b \cdot \nabla) b \cdot \partial_k u dx \\ &= \sum_{i,j,k=1}^3 \int \partial_k b_i \partial_i b_j \partial_k u_j dx \\ &= \sum_{i,j,k=1}^2 \int \partial_k b_i \partial_i b_j \partial_k u_j dx + \sum_{j,k=1}^2 \int \partial_k b_3 \partial_3 b_j \partial_k u_j dx \\ &\quad + \sum_{i,k=1}^2 \int \partial_k b_i \partial_i b_3 \partial_k u_3 dx + \sum_{i,j=1}^2 \int \partial_3 b_i \partial_i b_j \partial_3 u_j dx \\ &\quad + \sum_{k=1}^3 \int \partial_k b_3 \partial_3 b_3 \partial_k u_j dx + \sum_{i=1}^2 \int \partial_3 b_i \partial_i b_3 \partial_3 u_3 dx \\ &\quad + \sum_{j=1}^2 \int \partial_3 b_3 \partial_3 b_j \partial_3 u_j dx \triangleq \sum_{m=1}^7 I_{2m}. \end{aligned} \quad (24)$$

Obviously, the former four terms of the right-hand side of (24) are bounded by

$$\begin{aligned} \left| \sum_{m=1}^4 I_{2m} \right| &\leq C \left(\int |\tilde{\nabla} \tilde{u}| \|\nabla u\| |\nabla b| dx \right. \\ &\quad \left. + \int |\tilde{\nabla} \tilde{u}| \|\nabla b\| |\nabla b| dx + \int |\tilde{\nabla} \tilde{b}| \|\nabla u\| |\nabla b| dx \right). \end{aligned} \quad (25)$$

Noting that $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$ and $\partial_3 b_3 = -\partial_1 b_1 - \partial_2 b_2$ from the incompressible conditions, the latter three terms can be estimated as follows:

$$\begin{aligned} \left| \sum_{m=5}^7 I_{2m} \right| &\leq C \left(\int |-\partial_1 u_1 - \partial_2 u_2| \|\nabla b\| |\nabla b| dx \right. \\ &\quad \left. + \int |-\partial_1 b_1 - \partial_2 b_2| \|\nabla u\| |\nabla b| dx \right) \\ &\leq C \left(\int |\tilde{\nabla} \tilde{u}| \|\nabla b\| |\nabla b| dx + \int |\tilde{\nabla} \tilde{b}| \|\nabla u\| |\nabla b| dx \right). \end{aligned} \quad (26)$$

So, by Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned} I_2 &\leq C \left(\int |\tilde{\nabla} \tilde{u}| \|\nabla u\| |\nabla b| dx + \int |\tilde{\nabla} \tilde{u}| \|\nabla b\| |\nabla b| dx \right. \\ &\quad \left. + \int |\tilde{\nabla} \tilde{b}| \|\nabla u\| |\nabla b| dx \right) \leq C \left(\|\tilde{\nabla} \tilde{u}\|_{L^2} \|\nabla b\|_{L^2} \right. \\ &\quad \left. + \|\tilde{\nabla} \tilde{u}\|_{L^2} \|\nabla b\|_{L^2} + \|\tilde{\nabla} \tilde{b}\|_{L^2} \|\nabla u\|_{L^2} \right) \\ &\leq C \left(\|\tilde{\nabla} \tilde{u}\|_{\dot{X}^r} \|\nabla u\|_{\dot{H}^r} \|\nabla b\|_{L^2} \right. \\ &\quad \left. + \|\tilde{\nabla} \tilde{u}\|_{\dot{X}^r} \|\nabla b\|_{\dot{H}^r} \|\nabla b\|_{L^2} + \|\tilde{\nabla} \tilde{b}\|_{\dot{X}^r} \|\nabla u\|_{\dot{H}^r} \|\nabla b\|_{L^2} \right) \\ &\leq C \left(\|\tilde{\nabla} \tilde{u}\|_{\dot{X}^r} \|\nabla u\|_{L^2}^{1-r} \|\nabla^2 u\|_{L^2}^r \|\nabla b\|_{L^2} \right. \\ &\quad \left. + \|\tilde{\nabla} \tilde{u}\|_{\dot{X}^r} \|\nabla b\|_{L^2}^{1-r} \|\nabla^2 b\|_{L^2}^r \|\nabla b\|_{L^2} \right. \\ &\quad \left. + \|\tilde{\nabla} \tilde{b}\|_{\dot{X}^r} \|\nabla u\|_{L^2}^{1-r} \|\nabla^2 u\|_{L^2}^r \|\nabla b\|_{L^2} \right) \\ &\leq C \left(\|\tilde{\nabla} \tilde{u}\|_{\dot{X}^r}^{2/(2-r)} + \|\tilde{\nabla} \tilde{b}\|_{\dot{X}^r}^{2/(2-r)} \right) \left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right) \\ &\quad + \frac{1}{2} \left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2 \right). \end{aligned} \quad (27)$$

Similar to the estimate of I_2 , for $I_3 - I_4$, we have

$$\begin{aligned} I_3 + I_4 &\leq C \left(\|\tilde{\nabla} \tilde{u}\|_{\dot{X}^r}^{2/(2-r)} + \|\tilde{\nabla} \tilde{b}\|_{\dot{X}^r}^{2/(2-r)} \right) \left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right) \\ &\quad + \frac{1}{2} \left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2 \right). \end{aligned} \quad (28)$$

Combining the above estimates (23), (27), and (28) into (19), we deduce

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right) &+ \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2 \\ &\leq C \left(\|\tilde{\nabla} \tilde{u}\|_{\dot{X}^r}^{2/(2-r)} + \|\tilde{\nabla} \tilde{b}\|_{\dot{X}^r}^{2/(2-r)} \right) \left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right) \\ &\quad + \frac{1}{2} \left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2 \right). \end{aligned} \quad (29)$$

Therefore, we have

$$\begin{aligned} \frac{d}{dt} \left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right) &+ \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2 \\ &\leq C \left(\|\tilde{\nabla} \tilde{u}\|_{\dot{X}^r}^{2/(2-r)} + \|\tilde{\nabla} \tilde{b}\|_{\dot{X}^r}^{2/(2-r)} \right) \left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right). \end{aligned} \quad (30)$$

Due to Gronwall's inequality, it follows from (30) that

$$\begin{aligned} & \sup_{t \in [0, T]} \left(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \right) \\ & + \int_0^T \left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2 \right) dt \\ & \leq \left(\|\nabla u_0\|_{L^2}^2 + \|\nabla b_0\|_{L^2}^2 \right) \exp^C \int_0^T \left(\|\tilde{\nabla} \tilde{u}\|_{\dot{X}^r}^{2/(2-r)} + \|\tilde{\nabla} \tilde{b}\|_{\dot{X}^r}^{2/(2-r)} \right) dt. \end{aligned} \quad (31)$$

Thanks to $\tilde{\nabla} \tilde{u}, \tilde{\nabla} \tilde{b} \in L^{2/(2-r)}([0, T]; \dot{X}_r(\mathbb{R}^3))$, we have

$$u, b \in L^\infty([0, T]; \dot{H}^1) \cap L^2([0, T]; H^2). \quad (32)$$

Combining (14) with (32), we have

$$u, b \in L^\infty([0, T]; H^1). \quad (33)$$

By the standard arguments of continuation of local solutions [1], we can draw the conclusion that the smooth solution (u, b) remains smooth at $t = T$. This completes the proof of Theorem 2.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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