Research Article
Isometries of Spaces of Radon Measures

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Let Ω and I denote a compact metrizable space with card(Ω) ≥ 2 and the unit interval, respectively. We prove Milutin and Cantor-Bernstein type theorems for the spaces Š(Ω) of Radon measures on compact Hausdorff spaces Ω. In particular, we obtain the following results: (1) for every infinite closed subset K of βN the spaces Š(K), Š(βN), and Š(Ω) are order-isometric; (2) for every discrete space Γ with m = card(Γ) > ℵ₀ the spaces Š(Γ) and Š(Iᵈⁿ) are order-isometric, whereas there is no linear homeomorphic injection from C(βΓ) into C(Iᵈⁿ).

1. Introduction

We use the notation from the abstract and Δ denotes the Cantor set endowed with the standard product topology. Throughout this paper, we also assume that every topological space is of cardinality at least 2. For notions and notations undefined here we refer the reader to the monographs [1–5].

Let us recall that a linear injective operator T between two Banach lattices X and Y is said to be an order isomorphism [order-isometry, resp.] if both T and T⁻¹ are order-preserving [with T an isometry, resp.]; by [1, Theorem 16.6], in the general case, T is continuous.

In this paper, we deal with surjective isometries of spaces of Radon measures defined on compact Hausdorff spaces. In the monographs and survey papers devoted to isometries on function spaces this topic is either completely overlooked [6–8] or treated marginally [9, Theorem 7 on p. 177, Exercises 4–7 on p. 229]; cf. [9, pp. 181, 226–227]. Our aim is to show that, in many cases, one can indicate pairs Ω₁, Ω₂ of “highly nonhomeomorphic” compact Hausdorff spaces such that the spaces Š(Ω₁) and Š(Ω₂) are order-isometric. It is a little surprising that here a number of results can be obtained immediately by means of a cardinality argument (Propositions A and B below), yet there are cases requiring more advanced knowledge.

In the next section, we list basic notions and results concerning Riesz spaces, that is, linear lattices, which will be applied in proofs of our main results, given in Sections 3 and 4.

2. Preliminaries

Throughout what follows Ω denotes a compact Hausdorff space with card(Ω) ≥ 2, Š(Ω) denotes the space of Radon measures on Ω, and C(Ω) stands for the space of continuous functions on Ω, that is, the predual of Š(Ω). By N we denote the discrete space of positive integers.

Let F and G be two (real or complex) Banach lattices. The lattice F is said to be an AL-lattice if its norm is additive on F⁺; that is, ∥f₁ + f₂∥ = ∥f₁∥ + ∥f₂∥ for all f₁, f₂ ≥ 0. In particular, the classical spaces L₁(µ) and Š(Ω) are typical AL-spaces [9, Chapter 6]. Moreover, every AL-space is order-isometric to L₁(µ) for some measure space (Θ, Σ, µ) [9, Theorem 3 on p. 135]. If T is an order-isometry between the real parts of two given AL-spaces F and G, then the extended operator \( \tilde{T} : F \to G \) of the form \( \tilde{T}(f₁ + if₂) = Tf₁ + itf₂ \) is an isometry, too [9, p. 139]. This allows us to restrict our considerations to real Š(Ω)–spaces and apply the theory of real linear lattices [1–3]. A Banach space X is said to be an L₁-predual space if its dual space X* is linearly isometric to an AL-space; see [9, Chapter 7].

Let F be a real Banach lattice. A linear projection P in F is said to be an order (or a band) projection if 0 ≤ Pf ≤ f for all f ∈ F⁺, and its range, P(F), is called a projection band. We
write $F \cong G$ if the Banach lattices $F$ and $G$ are order-isometric. The symbol $\beta \Omega$ denotes the Stone-Cech compactification of a discrete infinite space $\Omega$, and $2^m$ denotes $m$-copies of the two-element discrete space $\mathbb{2}$, that is, the $m$-Cantor cube endowed with the product topology; thus $\Delta = 2^{\aleph_0}$.

The following result is an immediate consequence of [10, Theorem 3.4 and Remark 3.5.(ii)].

**Lemma 1.** Let $F$ and $G$ be two real AL-lattices. If $F$ and $G$ are each order-isometric to a projection band of the other space, then $F \cong G$.

The symbol $\mathcal{B}(\Omega)$ denotes the $\sigma$-algebra of all Borel subsets of a compact space $\Omega$, and $\mathcal{B}_0(\Omega)$ denotes the Baire subalgebra of $\mathcal{B}(\Omega)$ generated by the class of $G_\delta$ subsets of $\Omega$. In particular, if $\Omega$ is metrizable, then $\mathcal{B}(\Omega) = \mathcal{B}_0(\Omega)$. Two topological spaces $U$ and $V$ are said to be Borel [Baire, resp.] isomorphic if there is a bijection $R : U \to V$ such that $R(A) : A \in \mathcal{B}(U) = \mathcal{B}(V)$ [resp., $R(A) : A \in \mathcal{B}_0(U) = \mathcal{B}_0(V)$]. If $D \in \mathcal{B}(\Omega)$, then $D \cap \mathcal{B}(\Omega)$ denotes the class $\{D \cap A : A \in \mathcal{B}\}$.

The following result is an immediate consequence of [10, Corollary 1.7.10], the spaces of continuous functions $\mathcal{C}(\Omega)$ and $\mathcal{C}(I)$ are linearly homeomorphic yet non-order-isomorphic [5, Theorem 7.8.1], in general. Thus Proposition A may be considered as a dual version of Milutin’s theorem, essentially stronger than the classical one.

Let us notice that condition $(1)$ applies also for spaces of Radon measures built on scattered spaces. The classical result [5, Corollary 19.7.7] says that if $\Omega$ is a scattered compact space (i.e., every nonempty closed subset of $\Omega$ has an isolated point), then $\mathcal{M}(\Omega)$ consists of atomic measures only; that is, $\mathcal{M}(\Omega)$ is order-isometric to $\ell_1(\Omega)$. The examples of scattered spaces are furnished, for example, by order intervals [5, pp. 151–156], Mrówka spaces [14, 15] (cf. [16, Section 3]), and Stone spaces of superatomic Boolean algebras [17, Theorem, p. 1146], [18]. Hence we obtain the following complement to part (a) of Proposition A.

**Proposition B.** Let $\Omega_1, \Omega_2$ be two infinite compact scattered spaces. Then condition $(1)$ implies that $\mathcal{M}(\Omega_1) \cong \mathcal{M}(\Omega_2)$.

In this paper, we shall prove and apply the following theorem which, by the above results, is essential in the class of compact Hausdorff spaces nonhomeomorphic either to products of metrizable spaces or to scattered spaces.

**Theorem 3.** Let $\Omega_1, \Omega_2$ be two compact Hausdorff spaces. If

(i) $\Omega_1$ and $\Omega_2$ are each homeomorphic to a closed subset of the other space, or

(ii) $\Omega_1$ and $\Omega_2$ are each continuously mapped onto the other space and they are extremally disconnected;

then $\mathcal{M}(\Omega_1) \cong \mathcal{M}(\Omega_2)$.

**Proof.** Part (i). Let $\Omega_1$ be a closed subspace of $\Omega_2$. We shall show that $\mathcal{M}(\Omega_1)$ is order-isometric to a projection band of $\mathcal{M}(\Omega_2)$. The formula

$$(P \mu)(B) = \mu(B \cap \Omega_1), \quad B \in \mathcal{B}(\Omega_2)$$

defines an order projection $P$ from $\mathcal{M}(\Omega_2)$ onto the projection band $\mathcal{M}(\Omega_1)$ of all Radon measures concentrated on $\Omega_1$. Since $\mathcal{B}(\Omega_1) = \Omega_1 \cap \mathcal{B}(\Omega_2)$, the mapping $\mathcal{M}(\Omega_1) \ni \nu \mapsto \nu|\Omega_1$ is an order-isometry from $\mathcal{M}(\Omega_1)$ onto $\mathcal{M}(\Omega_2)$. Similarly, $\mathcal{M}(\Omega_2)$ is order-isometric to a projection band in $\mathcal{M}(\Omega_1)$. By Lemma 1, we obtain that $\mathcal{M}(\Omega_1) \cong \mathcal{M}(\Omega_2)$.

Part (ii). Observe that if $\phi$ maps $\Omega_1$ onto $\Omega_2$, then there is a closed subset $V$ of $\Omega_1$ such that the restriction $\phi|_V$ is a homeomorphism from $V$ onto $\Omega_2$ (see, e.g., [5, Proposition 71.13 and Theorem 24.2.10]). Now we apply part (i).

**Corollary 4.** Let $\Omega$ be a compact space, and let $I$ be an infinite discrete space with $m = \text{card } I$. Then $\mathcal{M}(\beta I) \cong \mathcal{M}(\Omega^m) \cong \mathcal{M}(I^{2m})$.

**Proof.** This is an immediate consequence of Lemma 2 and part (b) of Proposition A.

**Corollary 5.** Let $\Omega$ be a metrizable compact space, and let $K$ denote an infinite closed subspace of $\beta \Omega$. Then $\mathcal{M}(K) \cong \mathcal{M}(\beta N) \cong \mathcal{M}(I^{2m}) \cong \mathcal{M}(I^{2m})$. 

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Corollary 4. Let \( M \) be a discrete space with cardinality \( \gamma = m > N_\circ \). From Corollary 4 we know that the spaces \( \mathcal{M}(\gamma \Gamma) \) and \( \mathcal{M}(\mathcal{I}_m) \) are order-isometric.

Example 8. Let \( \Gamma \) be a discrete space with cardinality \( m > N_\circ \). From Corollary 4 we know that the spaces \( \mathcal{M}(\mathcal{I}_m) \) and \( \mathcal{M}(\mathcal{I}_m) \) are order-isometric.

4. Examples

The examples presented in this section are motivated by the result stated in [9, Exercise 5]: If \( X \) and \( Y \) are \( L_1 \)-predual spaces such that each of them is linearly isomorphically embedded into the other space, then their dual spaces, \( X^\ast \) and \( Y^\ast \), are linearly isomorphic. We shall show by examples the possibility of \( X^\ast \) and \( Y^\ast \) being order-isomorphic with no linear isomorphic embedding of either of these spaces into the other. To this end, we shall apply the following strengthening of Pelczyński's observation [5, pp. 366-367] that if \( \Gamma \) is an uncountable set then there is no isometric embedding of \( C(\beta \Gamma) \) into \( C(\mathcal{I}_m) \) for every infinite cardinal \( m \).

Lemma 7. Let \( \Omega \) denote either one of the spaces: \( N^\ast = \beta N \setminus N \) or \( \beta \Omega \), where \( \Gamma \) is an uncountable set. Then there is a linear isomorphic embedding of \( C(\Omega) \) into \( C(\mathcal{I}_m) \), for every infinite cardinal number \( m \).

Proof. We follow an idea of the proof of the above-mentioned Pelczyński's result. By the remark of Pelczyński [5, p. 367], the space \( C(\mathcal{I}_m) \) has an equivalent strictly convex norm. By Partington's results [20, 21], in each of the either cases, \( C(\Omega) \) contains an isometric copy of \( L_\infty \). Hence, \( C(\Omega) \) cannot be embedded into \( C(\mathcal{I}_m) \).

Example 9. By Corollary 5, the spaces \( \mathcal{M}(\mathcal{N}^\ast) \) and \( \mathcal{M}(\mathcal{I}_m) \) are order-isometric, yet, by Lemma 7, \( C(\mathcal{N}^\ast) \) does not embed linearly isomorphically into \( C(\mathcal{I}_m) \).

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

References

