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Research Article

Isometries of Spaces of Radon Measures

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Let Ω and I denote a compact metrizable space with $\mathrm{card}(\Omega) \geq 2$ and the unit interval, respectively. We prove Milutin and Cantor-Bernstein type theorems for the spaces $\mathscr{M}(\Omega)$ of Radon measures on compact Hausdorff spaces Ω. In particular, we obtain the following results: (1) for every infinite closed subset K of $\beta\mathbf{N}$ the spaces $\mathscr{M}(K)$, $\mathscr{M}(\beta\mathbf{N})$, and $\mathscr{M}(\Omega^{2^{\aleph_0}})$ are order-isometric; (2) for every discrete space Γ with $\mathfrak{m} := \mathrm{card}(\Gamma) > \aleph_0$ the spaces $\mathscr{M}(\beta\Gamma)$ and $\mathscr{M}(I^{2^{\mathfrak{m}}})$ are order-isometric, whereas there is no linear homeomorphic injection from $C(\beta T)$ into $C(I^{2^{\mathfrak{m}}})$.

1. Introduction

We use the notation from the abstract and Δ denotes the Cantor set endowed with the standard product topology. Throughout this paper, we also assume that every topological space is of cardinality at least 2. For notions and notations undefined here we refer the reader to the monographs [1–5]. Let us recall that a linear injective operator T between two Banach lattices X and Y is said to be an order isomorphism [order-isometry, resp.] if both T and T^{-1} are order-preserving [with T an isometry, resp.]; by [1, Theorem 16.6], in the general case, T is continuous.

In this paper, we deal with surjective isometries of spaces of Radon measures defined on compact Hausdorff spaces. In the monographs and survey papers devoted to isometries on function spaces this topic is either completely overlooked [6–8] or treated marginally [9, Theorem 7 on p. 177, Exercises 4–7 on p. 229]; cf. [9, pp. 181, 226-227]. Our aim is to show that, in many cases, one can indicate pairs Ω_1, Ω_2 of "highly nonhomeomorphic" compact Hausdorff spaces such that the spaces $\mathscr{M}(\Omega_1)$ and $\mathscr{M}(\Omega_2)$ are order-isometric. It is a little surprising that here a number of results can be obtained immediately by means of a cardinality argument (Propositions A and B below), yet there are cases requiring more advanced knowledge.

In the next section, we list basic notions and results concerning Riesz spaces, that is, linear lattices, which will be applied in proofs of our main results, given in Sections 3 and 4.

2. Preliminaries

Throughout what follows Ω denotes a compact Hausdorff space with $\mathrm{card}(\Omega) \geq 2$, $\mathscr{M}(\Omega)$ denotes the space of Radon measures on Ω, and $C(\Omega)$ stands for the space of continuous functions on Ω, that is, the predual of $\mathscr{M}(\Omega)$. By $\mathbf{N}$ we denote the discrete space of positive integers.

Let F and G be two (real or complex) Banach lattices. The lattice F is said to be an AL-lattice if its norm is additive on F^+; that is, $\|f_1 + f_2\| = \|f_1\| + \|f_2\|$ for all $f_1, f_2 \geq 0$. In particular, the classical spaces $L_1(\mu)$ and $\mathscr{M}(\Omega)$ are typical AL-spaces [9, Chapter 6]. Moreover, every AL-space is order-isometric to $L_1(\mu)$ for some measure space (Θ, Σ, μ) [9, Theorem 3 on p. 135]. If T is an order-isometry between the real parts of two given AL-spaces F and G, then the extended operator $\widetilde{T} : F \to G$ of the form $\widetilde{T}(f_1 + if_2) = Tf_1 + iTf_2$ is an isometry, too [9, p. 139]. This allows us to restrict our considerations to real $\mathscr{M}(\Omega)$-spaces and apply the theory of real linear lattices [1–3]. A Banach space X is said to be an L_1-predual space if its dual space X^* is linearly isometric to an AL-space; see [9, Chapter 7].

Let F be a real Banach lattice. A linear projection P in F is said to be *an order* (or a band) *projection* if $0 \leq Pf \leq f$ for all $f \in F^+$, and its range, $P(F)$, is called *a projection band*. We

write $F \cong G$ if the Banach lattices F and G are order-isometric. The symbol $\beta\Gamma$ denotes the Stone-Čech compactification of a discrete infinite space Γ, and $\mathbf{2}^{\mathfrak{m}}$ denotes $\mathfrak{m}$-copies of the two-element discrete space $\mathbf{2}$, that is, the m-Cantor cube endowed with the product topology; thus $\Delta = \mathbf{2}^{\aleph_0}$.

The following result is an immediate consequence of [10, Theorem 3.4 and Remark 3.5.(ii)].

Lemma 1. *Let F and G be two real AL-lattices. If F and G are each order-isometric to a projection band of the other space, then $F \cong G$.*

The symbol $\mathscr{B}(\Omega)$ denotes the σ-algebra of all Borel subsets of a compact space Ω, and $\mathscr{B}_0(\Omega)$ denotes the Baire-subalgebra of $\mathscr{B}(\Omega)$ generated by the class of G_δ subsets of Ω. In particular, if Ω is metrizable, then $\mathscr{B}(\Omega) = \mathscr{B}_0(\Omega)$. Two topological spaces U and V are said to be *Borel* [*Baire*, resp.] isomorphic if there is a bijection $R : U \to V$ such that $\{R(A) : A \in \mathscr{B}(U)\} = \mathscr{B}(V)$ [resp., $\{R(A) : A \in \mathscr{B}_0(U)\} = \mathscr{B}_0(V)$]. If $D \in \mathscr{B}(\Omega)$, then $D \cap \mathscr{B}(\Omega)$ denotes the class $\{D \cap A : A \in \mathscr{B}\}$.

The next result is proved on page 177 in [9].

Lemma 2. *Let G denote the unit circle in the plane endowed with its natural topology. For every infinite discrete space Γ with $\mathfrak{m} = \operatorname{card}(\Gamma)$, the spaces $\mathscr{M}(\beta\Gamma)$ and $\mathscr{M}(G^{2^{\mathfrak{m}}})$ are order-isometric.*

3. The Results

It is well known that two Polish spaces are Borel isomorphic if and only if they have the same cardinality ([11, p. 451] or [12, p. 38]). It follows that *two compact metrizable spaces Ω_1, Ω_2 are Baire isomorphic if and only if*

$$\operatorname{card}(\Omega_1) = \operatorname{card}(\Omega_2), \tag{1}$$

whence the product of $\mathfrak{m} \geq \aleph_0$ compact metrizable spaces is Baire isomorphic both to the Cantor cube $\mathbf{2}^{\mathfrak{m}}$ and to $I^{\mathfrak{m}}$. Hence, by [13, Theorem H, p. 229] and [13, Theorem D, p. 239], we obtain the following.

Proposition A. *Let Ω be a compact metrizable space.*

(a) *If Ω is uncountable, then $\mathscr{M}(\Omega)$ is order-isometric both to $\mathscr{M}(I)$ and to $\mathscr{M}(\Delta)$.*

(b) *If $(\Omega_t)_{t\in\theta}$ is a family of compact metrizable spaces with $\operatorname{card}\theta = \mathfrak{m} \geq \aleph_0$, then*

$$\mathscr{M}\left(\prod_{t\in\theta}\Omega_t\right) \cong \mathscr{M}(I^{\mathfrak{m}}) \cong \mathscr{M}(\mathbf{2}^{\mathfrak{m}}). \tag{2}$$

In particular, each of the spaces $\mathscr{M}(\Omega^{\mathfrak{m}})$, $\mathscr{M}(\Omega \times I^{\mathfrak{m}})$, $\mathscr{M}(\Omega \times \mathbf{2}^{\mathfrak{m}})$, and $\mathscr{M}(I^{\mathfrak{m}})$ is order-isometric to $\mathscr{M}(\mathbf{2}^{\mathfrak{m}})$.

It is also known that if Ω is a compact metrizable and *uncountable* space, then, by Milutin's result [5, Theorem 21.5.10], the spaces of continuous functions $C(\Omega)$ and $C(I)$ are linearly homeomorphic yet non-order-isomorphic [5, Theorem 7.8.1], in general. Thus Proposition A may be considered as a dual version of Milutin's theorem, essentially stronger than the classical one.

Let us notice that condition (1) applies also for spaces of Radon measures built on scattered spaces. The classical result [5, Corollary 19.7.7] says that *if Ω is a scattered compact space* (i.e., every nonempty closed subset of Ω has an isolated point), then $\mathscr{M}(\Omega)$ consists of atomic measures only; that is, *$\mathscr{M}(\Omega)$ is order-isometric to $\ell_1(\Omega)$*. The examples of scattered spaces are furnished, for example, by order intervals [5, pp. 151–156], Mrówka spaces [14, 15] (cf. [16, Section 3]), and Stone spaces of superatomic Boolean algebras [17, Theorem, p. 1146], [18]. Hence we obtain the following complement to part (a) of Proposition A.

Proposition B. *Let Ω_1, Ω_2 be two infinite compact scattered spaces. Then condition (1) implies that $\mathscr{M}(\Omega_1) \cong \mathscr{M}(\Omega_2)$.*

In this paper, we shall prove and apply the following theorem which, by the above results, is essential in the class of compact Hausdorff spaces nonhomeomorphic either to products of metrizable spaces or to scattered spaces.

Theorem 3. *Let Ω_1, Ω_2 be two compact Hausdorff spaces. If*

(i) *Ω_1 and Ω_2 are each homeomorphic to a closed subset of the other space, or*

(ii) *Ω_1 and Ω_2 are each continuously mapped onto the other space and they are extremally disconnected;*

then $\mathscr{M}(\Omega_1) \cong \mathscr{M}(\Omega_2)$.

Proof.

Part (i). Let Ω_1 be a closed subspace of Ω_2. We shall show that $\mathscr{M}(\Omega_1)$ is order-isometric to a projection band of $\mathscr{M}(\Omega_2)$. The formula

$$(P\mu)(B) := \mu(B \cap \Omega_1), \quad B \in \mathscr{B}(\Omega_2) \tag{3}$$

defines an order projection P from $\mathscr{M}(\Omega_2)$ onto the projection band $\mathscr{M}_{\Omega_1}(\Omega_2)$ of all Radon measures concentrated on Ω_1. Since $\mathscr{B}(\Omega_1) = \Omega_1 \cap \mathscr{B}(\Omega_2)$, the mapping $\mathscr{M}_{\Omega_1}(\Omega_2) \ni \nu \to \nu_{|\Omega_1}$ is an order-isometry from $\mathscr{M}_{\Omega_1}(\Omega_2)$ onto $\mathscr{M}(\Omega_2)$. Similarly, $\mathscr{M}(\Omega_2)$ is order-isometric to a projection band in $\mathscr{M}(\Omega_1)$. By Lemma 1, we obtain that $\mathscr{M}(\Omega_1) \cong \mathscr{M}(\Omega_2)$.

Part (ii). Observe that if ϕ maps Ω_1 onto Ω_2, then there is a closed subset V of Ω_1 such that the restriction $\phi_{|V}$ is a homeomorphism from V onto Ω_2 (see, e.g., [5, Proposition 7.1.13 and Theorem 24.2.10]). Now we apply part (i). □

Corollary 4. *Let Ω be a compact space, and let Γ be an infinite discrete space with $\mathfrak{m} = \operatorname{card}\Gamma$. Then $\mathscr{M}(\beta\Gamma) \cong \mathscr{M}(\Omega^{\mathfrak{m}}) \cong \mathscr{M}(I^{2^{\mathfrak{m}}})$.*

Proof. This is an immediate consequence of Lemma 2 and part (b) of Proposition A. □

Corollary 5. *Let Ω be a metrizable compact space, and let K denote an infinite closed subspace of $\beta\mathbf{N}$. Then $\mathscr{M}(K) \cong \mathscr{M}(\beta\mathbf{N}) \cong \mathscr{M}(\Omega^{2^{\aleph_0}}) \cong \mathscr{M}(I^{2^{\aleph_0}})$.*

Proof. Since K contains a homeomorphic copy of $\beta\mathbf{N}$ [19, p. 71], from part (i) of the Theorem we obtain that $\mathscr{M}(K) \cong \mathscr{M}(\beta\mathbf{N})$. The remaining order isometries follow from Corollary 4. □

The corollary below is a generalization of part (b) of Proposition A.

Corollary 6. *Let Ω be a compact space of weight $\mathfrak{m} \geq \aleph_0$. If Ω contains a homeomorphic copy of I^m, or Ω is 0-dimensional and contains a homeomorphic copy of $\mathbf{2}^{\mathfrak{m}}$, then $\mathscr{M}(\Omega) \cong \mathscr{M}(\mathbf{2}^{\mathfrak{m}})$.*

In particular, if K is a compact Hausdorff space with weight $\leq \mathfrak{m}$, then $\mathscr{M}(K \times I^{\mathfrak{m}}) \cong \mathscr{M}(K \times \mathbf{2}^{\mathfrak{m}}) \cong \mathscr{M}(\mathbf{2}^{\mathfrak{m}})$.

Proof. The first part follows from the universality of the Tychonoff and Cantor cubes, Theorem 3(i) and Proposition A(b). □

If Ω is compact and of weight $\leq \mathfrak{m}$, the product $\Omega \times I^{\mathfrak{m}}$ is of weight $\mathfrak{m}$; thus $\mathscr{M}(\Omega \times I^{\mathfrak{m}}) \cong \mathscr{M}(\mathbf{2}^{\mathfrak{m}})$ by the first part. Moreover, because $I^{\mathfrak{m}}$ and $\mathbf{2}^{\mathfrak{m}}$ are Baire isomorphic, the products $\Omega \times I^{\mathfrak{m}}$ and $\Omega \times \mathbf{2}^{\mathfrak{m}}$ are Baire isomorphic, too. Hence $\mathscr{M}(\Omega \times I^{\mathfrak{m}}) \cong \mathscr{M}(\Omega \times \mathbf{2}^{\mathfrak{m}})$.

4. Examples

The examples presented in this section are motivated by the result stated in [9, Exercise 5]: *If X and Y are L_1-predual spaces such that each of them is linearly isomorphically embedded into the other space, then their dual spaces, X^* and Y^*, are linearly isomorphic*. We shall show by examples the possibility of X^* and Y^* being order-isometric with no linear isomorphic embedding of either of these spaces into the other one. To this end, we shall apply the following strengthening of Pełczyński's observation [5, pp. 366-367] that if Γ is an uncountable set then there is no isometric embedding of $C(\beta\Gamma)$ into $C(I^{\mathfrak{m}})$ for every infinite cardinal $\mathfrak{m}$.

Lemma 7. *Let Ω denote either one of the spaces: $N^* := \beta\mathbf{N} \setminus \mathbf{N}$ or $\beta\Gamma$, where Γ is an uncountable set. Then there is no linear isomorphic embedding of $C(\Omega)$ into $C(I^{\mathfrak{m}})$, for every infinite cardinal number $\mathfrak{m}$.*

Proof. We follow an idea of the proof of the above-mentioned Pełczyński's result. By the remark of Pełczyński [5, p. 367], the space $C(I^{\mathfrak{m}})$ has an equivalent strictly convex norm. By Partington's results [20, 21], in each of the either cases, $C(\Omega)$ contains an isometric copy of ℓ_∞. Hence, $C(\Omega)$ cannot be embedded into $C(I^{\mathfrak{m}})$. □

Example 8. Let Γ be a discrete space with $\operatorname{card}\Gamma = \mathfrak{m} > \aleph_0$. From Corollary 4 we know that the spaces $\mathscr{M}(\beta\Gamma)$ and $\mathscr{M}(I^{2^{\mathfrak{m}}})$ are order-isometric.

On the other hand, by Lemma 7, there is no linear isomorphic embedding from $C(\beta\Gamma)$ into $C(I^{2^{\mathfrak{m}}})$.

In the next example we show that a similar property holds for the pair $\mathscr{M}(\mathbf{N}^*)$ and $\mathscr{M}(I^{2^{\aleph_0}})$.

Example 9. By Corollary 5, the spaces $\mathscr{M}(\mathbf{N}^*)$ and $\mathscr{M}(I^{2^{\aleph_0}})$ are order-isometric, yet, by Lemma 7, $C(\mathbf{N}^*)$ does not embed linearly isomorphically into $C(I^{2^{\aleph_0}})$.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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