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Research Article

New Result on the Critical Exponent for Solution of an Ordinary Fractional Differential Problem

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We give some background of fractional Cauchy problems and correct an existing result of Cauchy problem for the fractional differential inequality, and we also give an example to illustrate our statement.

1. Introduction

It is well known that many engineering and industrial systems involve the convection-diffusion process such as in seepage flows [1], oil flow through porous media in underground reservoirs [2, 3], and flow of chemical and metal solutions through porous media in heap leaching [4, 5]. Traditionally, for the transport of solute in nonheterogeneous porous media, based on the conservation principle and Ficks law, we can use the integer-order differential equation to describe the whole convection-diffusion process. However, it has been established that the convection diffusion equation based on Ficks law fails to model the anomalous feature of diffusion in porous media, and thus fractional order convection-diffusion equations have the potential to accurately model the convection-diffusion process [6–10]. On the other hand, fractional order integral and derivative operators are nonlocal which can describe the behaviour of many complex processes and systems with long-memory in time. This implies that fractional order models possess much more advantage than integer order models, especially in viscoelasticity, electrochemistry, and porous media [11–15], and many new applications for fractional models in various fields have also been reported recently [16–34].

Fractional Cauchy problems are important in physics to model anomalous diffusion [35], and one has already shown that some transfer processes in a medium and a universal response of electromagnetic, acoustic, and mechanical influence can be described by the fractional Cauchy problem

$$\begin{aligned} &\left(D_{+}^{\alpha}x\right)(t) = f(t, x(t)), \quad t > 0,\ 0 < \alpha < 1, \\ &\left(D_{+}^{\alpha-1}x\right)(0+) = b, \end{aligned} \tag{1}$$

which has attracted a large number of attention from many researchers and some results on existence of solution have been established in weighted spaces of continuous functions [36, 37]. However, in the case of lack of regularity, there are very few results for nonexistence of ordinary fractional differential equations and inequalities [26]. Recently, in the absence of regularity, Laskri and Tatar [26] considered the following Cauchy problem for the fractional differential inequality with a polynomial nonlinearity and variable coefficient:

$$\begin{aligned} &D_{+}^{\alpha}u(t) \geq t^{\beta}|u(t)|^{m}, \quad t > 0,\ 0 < \alpha < 1, \\ &D_{+}^{\alpha-1}u(0+) = b \in \mathbb{R}. \end{aligned} \tag{2}$$

By using the test function method and the integration by parts formula, they established the following nonexistence result with critical exponent separating existence from nonexistence of solutions.

Theorem 1. *Assume that $\beta > -\alpha$ and $1 < m \le (\beta+1)/(1-\alpha)$. Then, problem (2) does not admit global nontrivial solutions when $b \ge 0$.*

However, by checking this result carefully one can find that this theorem shows some flaws. In fact, under the above assumptions, problem (2) admits global nontrivial solutions when $b \ge 0$, as shown in the following example.

Example. Consider the following Cauchy problem for fractional differential inequality:

$$\begin{aligned} D_+^{1/2} u(t) &\ge t^{-1/6} |u(t)|^{3/2}, \quad t > 0, \\ D_+^{-1/2} u(0+) &= I_+^{1/2} u(0+) = 0. \end{aligned} \tag{3}$$

Conclusion. The fractional differential inequality (3) has at least a global nontrivial solution.

Proof. Let $\alpha = 1/2$, $\beta = -1/6$, $m = 3/2$, and $b = 0$; then we have $\beta > -\alpha$ and $1 < m = 3/2 \le (\beta+1)/(1-\alpha) = 5/3$. And thus the Cauchy problem (3) satisfies all of the conditions of Theorem 1.

But we can easily find that the following function is a global nontrivial solution of the fractional differential inequality (3):

$$u(t) = \begin{cases} \left[\Gamma\left(\frac{3}{2}\right)\right]^2 t^{1/2}, & 0 < t \le 1, \\ \left[\Gamma\left(\frac{3}{2}\right)\right]^2 t^{-4/5}, & t > 1. \end{cases} \tag{4}$$

In fact, by using the general relationship of Riemann-Liouville fractional derivative (see Miller and Ross [27] (p36, 6.6)), we have

$$D_+^{\alpha} t^p = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} t^{p-\alpha}, \quad \alpha > 0, \ p > -1. \tag{5}$$

Thus we divide the proof into two cases.

Case 1. If $0 < t \le 1$, then

$$\begin{aligned} D_+^{1/2} u(t) &= D_+^{1/2}\left(\left[\Gamma\left(\frac{3}{2}\right)\right]^2 t^{1/2}\right) = \left[\Gamma\left(\frac{3}{2}\right)\right]^3, \\ t^{\beta} |u(t)|^m &= t^{-1/6}\left[\left[\Gamma\left(\frac{3}{2}\right)\right]^2 t^{1/2}\right]^{3/2} \\ &= \left[\Gamma\left(\frac{3}{2}\right)\right]^3 t^{7/12}. \end{aligned} \tag{6}$$

It follows from $0 < t \le 1$ that

$$\begin{aligned} D_+^{1/2} u(t) &\ge t^{-1/6} |u(t)|^{3/2}, \\ D_+^{-1/2} u(0+) &= I_+^{1/2} u(0+) \\ &= \lim_{t\to 0+} \frac{[\Gamma(3/2)]^2}{\Gamma(1/2)} \int_0^t (t-s)^{-1/2} s^{1/2}\, ds \\ &= \lim_{t\to 0+} \frac{[\Gamma(3/2)]^3}{\Gamma(2)} t = 0. \end{aligned} \tag{7}$$

Case 2. If $t > 1$, we have

$$\begin{aligned} D_+^{1/2} u(t) &= D_+^{1/2}\left(\left[\Gamma\left(\frac{3}{2}\right)\right]^2 t^{-4/5}\right) \\ &= \left[\Gamma\left(\frac{3}{2}\right)\right]^3 t^{-13/10}, \\ t^{\beta} |u(t)|^m &= t^{-1/6}\left[\left[\Gamma\left(\frac{3}{2}\right)\right]^2 t^{-4/5}\right]^{3/2} \\ &= \left[\Gamma\left(\frac{3}{2}\right)\right]^3 t^{-41/30}. \end{aligned} \tag{8}$$

It follows from $t > 1$ that

$$D_+^{1/2} u(t) \ge t^{-1/6} |u(t)|^{3/2}. \tag{9}$$

So function (4) is indeed a global nontrivial solution of the Cauchy problem for the fractional differential inequality (3), and the conclusion is proved. □

2. Correction of Theorem 1

In this section, we firstly point out the flaws in the proof of Theorem 1 in [26] and then give the correct result and the corresponding proof on nonexistence of solution for Cauchy problem (3).

In the proof of Theorem 1 in [26], the authors get the following formulas by defining a test function φ and using the properties of Riemann-Liouville fractional integral:

$$\begin{aligned} &-\left(I_+^{1-\alpha} u\right)(0^+) - \frac{1}{\Gamma(1-\alpha)} \\ &\quad \cdot \int_0^T \varphi'(t) \int_0^t (t-s)^{-\alpha} u(s)\, ds\, dt \\ &\quad \ge \int_0^T t^{\beta} |u(t)|^m \varphi(t)\, dt, \\ &\frac{1}{\Gamma(1-\alpha)} \int_0^T \varphi'(t) \int_0^t (t-s)^{-\alpha} u(s)\, ds\, dt \le \frac{1}{m} \\ &\quad \cdot \int_0^T t^{\beta} |u(t)|^m \varphi(t)\, dt + \frac{\left(\max\left\{1, 2^{\beta/m}\right\}\right)^{m'}}{m'} \\ &\quad \cdot T^{-\beta m'/m} \int_{T/2}^T \left(I_{T_-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}}\right)(t)\, dt, \end{aligned} \tag{10}$$

which corresponds to (11) and (14) in paper [26], respectively. From (10), they obtained

$$\begin{aligned} &-\left(I_+^{1-\alpha} u\right)(0^+) \\ &\quad + \frac{\left(\max\left\{1, 2^{\beta/m}\right\}\right)^{m'}}{m'} T^{-\beta m'/m} \int_{T/2}^T \left(I_{T_-}^{1-\alpha} \frac{|\varphi'|}{\varphi^{1/m}}\right)(t)\, dt \\ &\quad \ge \left(1 - \frac{1}{m}\right) \int_0^T t^{\beta} |u(t)|^m \varphi(t)\, dt. \end{aligned} \tag{11}$$

However, it follows from (10) that the correct form should be

$$\begin{aligned}
&-\left(I_{+}^{1-\alpha}u\right)(0^{+}) \\
&-\frac{2}{\Gamma(1-\alpha)}\int_{0}^{T}\varphi'(t)\int_{0}^{t}(t-s)^{-\alpha}u(s)\,ds\,dt \\
&+\frac{\left(\max\left\{1,2^{\beta/m}\right\}\right)^{m'}}{m'}T^{-\beta m'/m}\int_{T/2}^{T}\left(I_{T_{-}}^{1-\alpha}\frac{|\varphi'|}{\varphi^{1/m}}\right)(t)\,dt \\
&\geq\left(1-\frac{1}{m}\right)\int_{0}^{T}t^{\beta}|u(t)|^{m}\varphi(t)\,dt,
\end{aligned} \tag{12}$$

instead of (11), which leads to a wrong result.

In fact, under the assumption of Theorem 1, problem (2) has infinitely many global nontrivial positive solutions; for example, according to the proof of the above example and noticing that $m > 1$, for any $\mu \in [1,\infty)$, the following functions

$$u(t)=\begin{cases}[\Gamma(\alpha)]^{1/(m-1)}t^{\alpha}, & 0<t\leq 1,\\ [\Gamma(\alpha)]^{1/(m-1)}t^{-\mu-(\alpha+\beta)/(m-1)}, & t>1\end{cases} \tag{13}$$

are global nontrivial positive solutions of problem (2).

Thus we give the correct statement as follows.

Theorem 2. *Assume that $\beta > -\alpha$ and $1 < m \leq (\beta+1)/(1-\alpha)$. Then, when $b \geq 0$, Cauchy problem (2) has infinitely many global nontrivial positive solutions. However it does not admit global nontrivial negative solutions.*

Proof. Take a test function $\varphi(t) \in C^{1}([0,\infty),[0,+\infty))$, where φ is nonincreasing such that

$$\varphi(t)=\begin{cases}1, & t\in\left[0,\frac{T}{2}\right],\\ 0, & t\in[T,\infty).\end{cases} \tag{14}$$

Suppose, on the contrary, that a global nontrivial negative solutions $u(t) \leq 0$ exist for all time $t > 0$. Following the proof of [26], we only correct formula (11). By (10), one has

$$\begin{aligned}
&-\left(I_{+}^{1-\alpha}u\right)(0^{+}) \\
&-\frac{2}{\Gamma(1-\alpha)}\int_{0}^{T}\varphi'(t)\int_{0}^{t}(t-s)^{-\alpha}u(s)\,ds\,dt \\
&+\frac{\left(\max\left\{1,2^{\beta/m}\right\}\right)^{m'}}{m'}T^{-\beta m'/m}\int_{T/2}^{T}\left(I_{T_{-}}^{1-\alpha}\frac{|\varphi'|}{\varphi^{1/m}}\right)(t)\,dt \\
&\geq\left(1-\frac{1}{m}\right)\int_{0}^{T}t^{\beta}|u(t)|^{m}\varphi(t)\,dt.
\end{aligned} \tag{15}$$

Noticing that the test function $\varphi(t) \in C^{1}([0,\infty))$ satisfies $\varphi(t) \geq 0$ and φ is nonincreasing for some $T > 0$ and the solution $u(t) \leq 0$ we have

$$\frac{1}{\Gamma(1-\alpha)}\int_{0}^{T}\varphi'(t)\int_{0}^{t}(t-s)^{-\alpha}u(s)\,ds\,dt\geq 0. \tag{16}$$

Consequently, (15) and (16) guarantee that (11) holds. The rest of proof is similar to those of [26]. □

Remark 3. Although we correct the result of [26], the accurate critical exponent for solution is still unknown.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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