Research Article

Redefinition of τ -Distance in Metric Spaces

Tomonari Suzuki

Department of Basic Sciences, Kyushu Institute of Technology, Tobata, Kitakyushu 804-8550, Japan

Correspondence should be addressed to Tomonari Suzuki; suzuki-t@mns.kyutech.ac.jp

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The concept of τ -distance was introduced in 2001; on the other hand, that of τ -function was introduced by Lin and Du. Strongly inspired by τ -function, we introduce a new concept, which is a very slight generalization of τ -distance and is more natural than τ -distance. So we could say that we redefine τ -distance in some sense.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ the sets of all positive integers, all rational numbers, and all real numbers, respectively.

In 2001, the concept of τ -distance was introduced in order to generalize results in [1–8] and others.

Definition 1 (see [9]). Let (X, d) be a metric space. Then a function p from $X \times X$ into $[0, \infty)$ is called a τ -distance on X if there exists a function η from $X \times [0, \infty)$ into $[0, \infty)$ and the following is satisfied:

- (τ_d1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$.
- (τ_d2) $\eta(x, 0) = 0$ and $\eta(x, t) \geq t$ for any $x \in X$ and $t \in [0, \infty)$, and η is concave and continuous in its second variable.
- (τ_d3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply $p(w, x) \leq \liminf_n p(w, x_n)$ for any $w \in X$.
- (τ_d4) $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$.
- (τ_d5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

We note that the metric d is one of τ -distances on X . See [9–15] and references therein for many examples and theorems concerning τ -distance. For instance, using τ -distance, Suzuki [14] gave a simple proof of Zhong's theorem [7].

In 2006, Lin and Du [16] introduced the following very interesting concept of τ -function, which is similar to that of τ -distance. However, both are independent.

Definition 2 (Lin and Du [16]). Let (X, d) be a metric space. Then a function p from $X \times X$ into $[0, \infty)$ is called a τ -function if the following conditions hold:

- (τ_f1) For any $x, y, z \in X$, $p(x, z) \leq p(x, y) + p(y, z)$.
- (τ_f2) If $x \in X$ and $\{y_n\}$ in X with $\lim_n y_n = y$ and $p(x, y_n) \leq M$ for some $M = M(x) > 0$ then $p(x, y) \leq M$.
- (τ_f3) For any sequence $\{x_n\}$ in X with $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, and if there exists a sequence $\{y_n\}$ in X such that $\lim_n p(x_n, y_n) = 0$, then $\lim_n d(x_n, y_n) = 0$.
- (τ_f4) For $x, y, z \in X$, $p(x, y) = 0$ and $p(x, z) = 0$ imply $y = z$.

In this paper, strongly inspired by τ -function, we introduce a new concept, which is a slight generalization of τ -distance and which is more natural than τ -distance.

Before the author wrote the paper [9], he had considered more than twenty definitions. Definition 1 was the most natural definition between them. Therefore, he defined τ -distance as in Definition 1, though Definition 1 was not the weakest. However, strongly inspired by Definition 2, he finds a more natural definition of “ τ -distance”; see Definition 7. Now he thinks that we have to change the definition of “ τ -distance.”

2. New Definition

In this section, we introduce a new definition of the notion of τ -distance. Before introducing it, we state something on τ -distance, which are strongly connected with the new definition.

Definition 3 (see [9]). Let p be a τ -distance on a metric space (X, d) . Then a sequence $\{x_n\}$ in X is called p -Cauchy if there exist a function η from $X \times [0, \infty)$ into $[0, \infty)$ satisfying (τ_d2) – (τ_d5) and a sequence $\{z_n\}$ in X such that $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$.

Lemma 4 (see [9]). Let p be a τ -distance on a metric space (X, d) . If $\{x_n\}$ is a p -Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence in the usual sense.

Lemma 5 (see [9]). Let p be a τ -distance on a metric space (X, d) . If a sequence $\{x_n\}$ in X satisfies $\lim_n p(z, x_n) = 0$ for some $z \in X$, then $\{x_n\}$ is a p -Cauchy sequence. Moreover, if a sequence $\{y_n\}$ in X also satisfies $\lim_n p(z, y_n) = 0$, then $\lim_n d(x_n, y_n) = 0$. In particular for $x, y, z \in X$, $p(z, x) = 0$ and $p(z, y) = 0$ imply $x = y$.

Lemma 6 (see [9]). Let p be a τ -distance on a metric space (X, d) . If a sequence $\{x_n\}$ in X satisfies $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a p -Cauchy sequence. Moreover, if a sequence $\{y_n\}$ in X satisfies $\lim_n p(x_n, y_n) = 0$, then $\{y_n\}$ is also a p -Cauchy sequence and $\lim_n d(x_n, y_n) = 0$.

Now we give the new definition.

Definition 7. Let (X, d) be a metric space and let p be a function from $X \times X$ into $[0, \infty)$. Then p is called a τ' -distance on X if the following holds:

- $(\tau'1)$ $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$.
- $(\tau'2)$ If $\lim_n \sup\{p(z_n, z_m) : m > n\} = 0$ and $\lim_n p(z_n, x_n) = 0$, then $\lim_n d(z_n, x_n) = 0$. Moreover, if $\{x_n\}$ converges to some $x \in X$, then $p(w, x) \leq \liminf_n p(w, x_n)$ for any $w \in X$.
- $(\tau'3)$ If $\lim_n p(z, x_n) = 0$, then $\lim_n d(x_n, x_{n+1}) = 0$ holds. Moreover, if $\{x_n\}$ converges to some $x \in X$, then $p(w, x) \leq \liminf_n p(w, x_n)$ for any $w \in X$.

The concept of τ' -distance is slightly weaker than that of τ -distance. However, the author considers that the notions of both are the same. Indeed, we can prove τ' -distance versions of all the existence theorems in [9, 11–14] using the same proof.

Proposition 8. Let p be a τ -distance on a metric space (X, d) . Then p is a τ' -distance.

Proof. (τ_d1) and $(\tau'1)$ are the same. $(\tau'2)$ follows from Lemma 6 and (τ_d3) . $(\tau'3)$ follows from Lemmas 4 and 5 and (τ_d3) . $\square$

Remark 9. A τ' -distance need not be a τ -distance. However, the only example we have is quite complicated and we have chosen not to include it.

3. Condition (CL)

We have introduced the concept of p -Cauchy for τ -distance; see Definition 3. Instead of this, for τ' -distance, we introduce Condition (CL), which is named after Cauchy and lower semicontinuity.

Definition 10. Let p be a τ' -distance on a metric space (X, d) . Let $\{x_\alpha : \alpha \in D\}$ be a net in X . Then $\{x_\alpha\}$ is said to satisfy Condition (CL) if the following holds:

(CL1) $\{x_\alpha\}$ is a Cauchy net in the usual sense.

(CL2) Either of the following holds:

- (i) $\{x_\alpha\}$ does not converge.
- (ii) If $\{x_\alpha\}$ converges to x , then $p(w, x) \leq \liminf_\alpha p(w, x_\alpha)$ holds for any $w \in X$.

The following lemma is important in this section.

Lemma 11. Let (X, d) be a metric space and let $\{x_\alpha : \alpha \in D\}$ be a net in X . Assume that $\{x_\alpha\}$ is not Cauchy. Let $\{\beta_n\}$ be a sequence in D and let h be a mapping on D . Then there exist a positive number ε and a sequence $\{f(n)\}$ in D such that

$$\begin{aligned} \beta_n &\leq f(n), \\ h(f(n)) &\leq f(n+1), \\ d(x_{f(n)}, x_{f(n+1)}) &\geq \varepsilon \end{aligned} \quad (1)$$

for any $n \in \mathbb{N}$.

Proof. Since $\{x_\alpha\}$ is not Cauchy, there exist a positive number ε and mappings g_1 and g_2 on D such that $\alpha \leq g_1(\alpha)$, $\alpha \leq g_2(\alpha)$, and $d(x_{g_1(\alpha)}, x_{g_2(\alpha)}) \geq 2\varepsilon$ for any $\alpha \in D$. Put $f(1) = \beta_1$ and choose $\gamma_2 \in D$ with $\gamma_2 \geq \beta_2$ and $\gamma_2 \geq h(f(1))$. Then either

$$\begin{aligned} d(x_{f(1)}, x_{g_1(\gamma_2)}) &\geq \varepsilon \\ \text{or } d(x_{f(1)}, x_{g_2(\gamma_2)}) &\geq \varepsilon \end{aligned} \quad (2)$$

holds. So we can put $f(2) = g_1(\gamma_2)$ or $f(2) = g_2(\gamma_2)$ such that $d(x_{f(1)}, x_{f(2)}) \geq \varepsilon$. Continuing this argument, we can obtain the desired result. $\square$

Remark 12. $\{x_{f(n)}\}$ is not necessarily a subnet of $\{x_\alpha\}$.

We do not use the concept of net in Definition 7. However, we can prove something on net as follows.

Lemma 13. Let p be a τ' -distance on a metric space (X, d) . Let $\{x_\alpha : \alpha \in D\}$ be a net in X satisfying $\lim_\alpha p(z, x_\alpha) = 0$ for some $z \in X$. Then the following holds:

- (i) $\{x_\alpha\}$ satisfies Condition (CL).

- (ii) If a net $\{y_\alpha : \alpha \in D\}$ in X also satisfies $\lim_\alpha p(z, y_\alpha) = 0$, then $\lim_\alpha d(x_\alpha, y_\alpha) = 0$ holds.

Proof. We choose a sequence $\{\beta_n\}$ in D such that $p(z, x_\alpha) \leq 1/n$ for $\alpha \geq \beta_n$. Arguing by contradiction, we assume $\{x_\alpha\}$ is not Cauchy. Let h be an arbitrary mapping from D into itself. Then by Lemma 11, there exist a positive number ε and a sequence $\{f(n)\}$ in D satisfying

$$\begin{aligned} \beta_n &\leq f(n), \\ d(x_{f(n)}, x_{f(n+1)}) &\geq \varepsilon \end{aligned} \quad (3)$$

for any $n \in \mathbb{N}$. Since

$$\lim_{n \rightarrow \infty} p(z, x_{f(n)}) \leq \lim_{n \rightarrow \infty} \frac{1}{n} = 0, \quad (4)$$

from $(\tau'3)$, we have

$$0 < \varepsilon \leq \lim_{n \rightarrow \infty} d(x_{f(n)}, x_{f(n+1)}) = 0, \quad (5)$$

which is a contradiction. Hence, $\{x_\alpha\}$ is Cauchy. Next, we assume $\{x_\alpha\}$ converges to x . Fix $w \in X$ and let $\eta \in \mathbb{R}$ with $\eta > \liminf_\alpha p(w, x_\alpha)$. Then we can choose a sequence $\{\gamma_n\}$ in D such that

$$\begin{aligned} \gamma_n &\geq \beta_n, \\ d(x_{\gamma_n}, x) &< \frac{1}{n}, \\ p(w, x_{\gamma_n}) &< \eta \end{aligned} \quad (6)$$

for any $n \in \mathbb{N}$. Since $\{p(z, x_{\gamma_n})\}$ converges to 0, we have from $(\tau'3)$

$$p(w, x) \leq \liminf_{n \rightarrow \infty} p(w, x_{\gamma_n}) \leq \eta. \quad (7)$$

Since η is arbitrary, we obtain $p(w, x) \leq \liminf_\alpha p(w, x_\alpha)$. Therefore, we have shown (i). Let us prove (ii). Define a directed set E and an order $\leq$ in E

$$\begin{aligned} E &= D \times \{0, 1\}, \\ (\alpha, i) &\leq (\beta, j) \iff \alpha \leq \beta. \end{aligned} \quad (8)$$

Define a net $\{z_{\alpha, j} : (\alpha, j) \in E\}$ by

$$z_{(\alpha, j)} = \begin{cases} x_\alpha & \text{if } j = 0 \\ y_\alpha & \text{if } j = 1. \end{cases} \quad (9)$$

Then since $\lim_{(\alpha, j)} p(z, z_{(\alpha, j)}) = 0$, $\{z_{(\alpha, j)}\}$ is Cauchy by (i). This implies $\lim_\alpha d(x_\alpha, y_\alpha) = 0$. $\square$

As a corollary of Lemma 13, we obtain the following sequential version. Compare Lemma 14 with Lemma 5.

Lemma 14. Let p be a τ' -distance on a metric space (X, d) . If a sequence $\{x_n\}$ in X satisfies $\lim_n p(z, x_n) = 0$ for some $z \in X$, then $\{x_n\}$ satisfies Condition (CL). Moreover, if a sequence $\{y_n\}$ in X also satisfies $\lim_n p(z, y_n) = 0$, then $\lim_n d(x_n, y_n) = 0$ holds. In particular, for $x, y, z \in X$, $p(z, x) = 0$ and $p(z, y) = 0$ imply $x = y$.

Lemma 15. Let $(D, \leq)$ be a directed set and let h be a mapping from D into itself such that $\alpha \leq h(\alpha)$ for any $\alpha \in D$. Let $\{x_\alpha : \alpha \in D\}$ be a net in a set X . Then $\{x_{h(\alpha)} : \alpha \in D\}$ is a subnet of $\{x_\alpha\}$.

Proof. Obvious. $\square$

Lemma 16. Let p be a τ' -distance on a metric space (X, d) . Let D be a directed set such that for any $\alpha \in D$ there exists $\beta \in D$ with $\beta > \alpha$. Let $\{z_\alpha : \alpha \in D\}$ be a net in X satisfying $\lim_\alpha \sup\{p(z_\alpha, z_\beta) : \beta > \alpha\} = 0$. Then the following holds:

- (i) $\{z_\alpha\}$ satisfies Condition (CL).
- (ii) If a net $\{x_\alpha : \alpha \in D\}$ in X satisfies $\lim_\alpha p(z_\alpha, x_\alpha) = 0$, then $\{x_\alpha\}$ satisfies Condition (CL) and $\lim_\alpha d(z_\alpha, x_\alpha) = 0$ holds.

Proof. We choose a sequence $\{\beta_n\}$ in D such that

$$\begin{aligned} \sup_{\beta > \alpha} p(z_\alpha, z_\beta) &\leq \frac{1}{n}, \\ p(z_\alpha, x_\alpha) &\leq \frac{1}{n} \end{aligned} \quad (10)$$

for any $\alpha \geq \beta_n$. We consider the following two cases:

- (a) There exists $\zeta \in D$ such that $\zeta \geq \alpha$ for any $\alpha \in D$.
- (b) For any $\alpha \in D$, there exists $\beta \in D$ with $\alpha \not\geq \beta$.

In the first case, from the assumption, we can take $\omega \in D$ with $\omega > \zeta$. Then since $\zeta \geq \beta_n$ and $\omega \geq \beta_n$ for any $n \in \mathbb{N}$, we have

$$p(z_\zeta, z_\omega) = p(z_\omega, z_\zeta) = p(z_\zeta, x_\zeta) = p(z_\omega, x_\omega) = 0. \quad (11)$$

So

$$p(z_\zeta, x_\omega) \leq p(z_\zeta, z_\omega) + p(z_\omega, x_\omega) = 0. \quad (12)$$

Similarly, $p(z_\omega, x_\zeta) = 0$. By Lemma 14, we obtain $z_\omega = x_\omega = z_\zeta$ for $\omega \geq \zeta$. Therefore, $\lim_\alpha d(z_\alpha, x_\alpha) = 0$. Since $\{x_\alpha\}$ converges to z_ζ , thus, $\{x_\alpha\}$ is Cauchy. Also, $p(w, z_\zeta) = \lim_\alpha p(w, x_\alpha)$ holds. Therefore, we have shown that $\{x_\alpha\}$ satisfies Condition (CL) and $\lim_\alpha d(z_\alpha, x_\alpha) = 0$. As in the above proof, we can prove that $\{z_\alpha\}$ satisfies Condition (CL). In the second case, we can define a mapping h from D into itself satisfying

$$\begin{aligned} \alpha &< h(\alpha), \\ \alpha &\not\geq \gamma \end{aligned} \quad (13)$$

for any $\gamma \geq h(\alpha)$. In fact, for $\alpha \in D$, there exists $\beta \in D$ with $\alpha \not\geq \beta$. We can choose $h(\alpha) \in D$ satisfying $h(\alpha) \geq \alpha$ and $h(\alpha) \geq \beta$. If there exists $\gamma \in D$ such that $\gamma \geq h(\alpha)$ and $\alpha \geq \gamma$, then $\alpha \geq h(\alpha)$ and hence $\alpha \geq \beta$, which is a contradiction. Similarly $h(\alpha) = \alpha$ implies a contradiction. Therefore, we have defined h . We will show that $\{x_\alpha\}$ satisfies Condition (CL). Arguing by contradiction, we assume $\{x_\alpha\}$ is

not Cauchy. Then by Lemma 11, there exist a positive number ε and a sequence $\{f(n)\}$ in D satisfying

$$\begin{aligned} \beta_n &\leq f(n) < h(f(n)) \leq f(n+1), \\ d(x_{f(n)}, x_{f(n+1)}) &\geq \varepsilon. \end{aligned} \quad (14)$$

From the definition of h , we note $f(m) > f(n)$ for $m, n \in \mathbb{N}$ with $m > n$. Since

$$\begin{aligned} \lim_{n \rightarrow \infty} p(z_{f(n)}, x_{f(n)}) &\leq \lim_{n \rightarrow \infty} \frac{1}{n} = 0, \\ \lim_{n \rightarrow \infty} p(z_{f(n)}, z_{f(n+1)}) &= \lim_{n \rightarrow \infty} \sup_{m > n} p(z_{f(n)}, z_{f(m)}) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} = 0, \end{aligned} \quad (15)$$

from $(\tau'2)$, we have

$$\lim_{n \rightarrow \infty} d(z_{f(n)}, z_{f(n+1)}) = \lim_{n \rightarrow \infty} d(z_{f(n)}, x_{f(n)}) = 0. \quad (16)$$

So

$$\begin{aligned} 0 < \varepsilon &\leq \lim_{n \rightarrow \infty} d(x_{f(n)}, x_{f(n+1)}) \leq \lim_{n \rightarrow \infty} (d(x_{f(n)}, z_{f(n)}) \\ &\quad + d(z_{f(n)}, z_{f(n+1)}) + d(z_{f(n+1)}, x_{f(n+1)})) = 0, \end{aligned} \quad (17)$$

which is a contradiction. Hence, $\{x_\alpha\}$ is Cauchy. Next, we assume $\{x_\alpha\}$ converges to x . Fix $w \in X$ and let $\eta \in \mathbb{R}$ with $\eta > \liminf_\alpha p(w, x_\alpha)$. Then we can choose a sequence $\{\gamma_n\}$ in D such that

$$\begin{aligned} \beta_n &\leq \gamma_n < h(\gamma_n) \leq \gamma_{n+1}, \\ d(x_{\gamma_n}, x) &< \frac{1}{n}, \\ p(w, x_{\gamma_n}) &< \eta. \end{aligned} \quad (18)$$

Since $\lim_n \sup\{p(z_{\gamma_n}, z_{\gamma_m}) : m > n\} = \lim_n p(z_{\gamma_n}, x_{\gamma_n}) = 0$, we have from $(\tau'2)$ that

$$p(w, x) \leq \liminf_{n \rightarrow \infty} p(w, x_{\gamma_n}) \leq \eta. \quad (19)$$

Since η is arbitrary, we obtain $p(w, x) \leq \liminf_\alpha p(w, x_\alpha)$. Therefore, we have shown that $\{x_\alpha\}$ satisfies Condition (CL). Let us prove $\lim_\alpha d(z_\alpha, x_\alpha) = 0$. Arguing by contradiction, we assume $\limsup_\alpha d(z_\alpha, x_\alpha) > 0$. Fix $\varepsilon \in \mathbb{R}$ with $\limsup_\alpha d(z_\alpha, x_\alpha) > \varepsilon > 0$. Then we can choose a sequence $\{\delta_n\}$ in D such that

$$\begin{aligned} \beta_n &\leq \delta_n < h(\delta_n) \leq \delta_{n+1}, \\ d(z_{\delta_n}, x_{\delta_n}) &> \varepsilon. \end{aligned} \quad (20)$$

Since $\lim_n \sup\{p(z_{\delta_n}, z_{\delta_m}) : m > n\} = \lim_n p(z_{\delta_n}, x_{\delta_n}) = 0$, we obtain from $(\tau'2)$ that

$$0 < \varepsilon \leq \lim_{n \rightarrow \infty} d(z_{\delta_n}, x_{\delta_n}) = 0, \quad (21)$$

which is a contradiction. Therefore, we have shown $\lim_\alpha d(z_\alpha, x_\alpha) = 0$. We have proved (ii). We shall show that $\{z_\alpha\}$ satisfies Condition (CL). Fix $w \in X$ and let $\eta \in \mathbb{R}$ with $\eta > \liminf_\alpha p(w, z_\alpha)$. Then we can define a mapping g from D into itself such that

$$\alpha \leq h(\alpha) \leq g(\alpha), \quad (22)$$

$$p(w, z_{g(\alpha)}) < \eta.$$

By Lemma 15, we note that $\{z_{g(\alpha)}\}$ is a subnet of $\{z_\alpha\}$. We have

$$\lim_{\alpha \in D} p(z_\alpha, z_{g(\alpha)}) \leq \lim_{\alpha \in D} \sup_{\beta > \alpha} p(z_\alpha, z_\beta) = 0. \quad (23)$$

So from (ii), $\{z_{g(\alpha)}\}$ satisfies Condition (CL) and $\lim_\alpha d(z_\alpha, z_{g(\alpha)}) = 0$ holds. So, $\{z_\alpha\}$ is Cauchy. We assume that $\{z_\alpha\}$ converges to z . Since $\{z_{g(\alpha)}\}$ also converges to z , we have

$$p(w, z) \leq \liminf_{\alpha \in D} p(w, z_{g(\alpha)}) \leq \eta. \quad (24)$$

Since η is arbitrary, we obtain $p(w, z) \leq \liminf_\alpha p(w, z_\alpha)$. Therefore, we have shown that $\{z_\alpha\}$ satisfies Condition (CL). We have proved (i). $\square$

As a corollary of Lemma 16, we obtain the following sequential version. Compare Lemma 17 with Lemma 6.

Lemma 17. *Let p be a τ' -distance on a metric space (X, d) . If a sequence $\{z_n\}$ in X satisfies $\lim_n \sup\{p(z_n, z_m) : m > n\} = 0$, then $\{z_n\}$ satisfies Condition (CL). Moreover, if a sequence $\{x_n\}$ in X satisfies $\lim_n p(z_n, x_n) = 0$, then $\{x_n\}$ satisfies Condition (CL) and $\lim_n d(z_n, x_n) = 0$ holds.*

Lemma 18. *Let p be a τ' -distance on a metric space (X, d) and let $\{z_n\}$ and $\{x_n\}$ be sequences in X satisfying $\lim_n \sup\{p(z_n, z_m) : m > n\} = 0$. Assume that, for any subsequence $\{f(n)\}$ of the sequence $\{n\}$ in $\mathbb{N}$, there exist subsequences $\{f \circ g(n)\}$ of $\{f(n)\}$ and $\{h(n)\}$ of $\{n\}$ satisfying $\lim_n p(z_{h(n)}, x_{f \circ g(n)}) = 0$. Then $\{x_n\}$ satisfies Condition (CL) and $\lim_n d(z_n, x_n) = 0$ holds.*

Proof. We note that $\{z_n\}$ is Cauchy from Lemma 17. Let $\{f(n)\}$ be an arbitrary subsequence of $\{n\}$. Then from the assumption, there exist subsequences $\{f \circ g(n)\}$ of $\{f(n)\}$ and $\{h(n)\}$ of $\{n\}$ satisfying $\lim_n p(z_{h(n)}, x_{f \circ g(n)}) = 0$. We note $\lim_n \sup\{p(z_{h(n)}, z_{h(m)}) : m > n\} = 0$. From $(\tau'2)$, we have $\lim_n d(z_{h(n)}, x_{f \circ g(n)}) = 0$ and hence

$$\begin{aligned} \lim_{n \rightarrow \infty} d(z_{f \circ g(n)}, x_{f \circ g(n)}) \\ \leq \lim_{n \rightarrow \infty} (d(z_{f \circ g(n)}, z_{h(n)}) + d(z_{h(n)}, x_{f \circ g(n)})) = 0. \end{aligned} \quad (25)$$

Since $\{f(n)\}$ is arbitrary, we obtain $\lim_n d(z_n, x_n) = 0$. $\{z_n\}$ is Cauchy and so is $\{x_n\}$. Next, we assume that $\{x_n\}$ converges to $x \in X$, fix $w \in X$, and choose a subsequence $\{f'(n)\}$ of $\{n\}$ satisfying $\lim_n p(w, x_{f'(n)}) = \liminf_n p(w, x_n)$. Then from the assumption, there exist subsequences $\{f' \circ g'(n)\}$ of $\{f'(n)\}$

and $\{h'(n)\}$ of $\{n\}$ satisfying $\lim_n p(z_{h'(n)}, x_{f' \circ g'(n)}) = 0$. By $(\tau'2)$, we obtain

$$p(w, x) \leq \lim_{n \rightarrow \infty} p(w, x_{f' \circ g'(n)}) = \liminf_{n \rightarrow \infty} p(w, x_n). \quad (26)$$

Therefore, $\{x_n\}$ satisfies Condition (CL). $\square$

4. Examples

In this section, in order to show that the concept of τ' -distance is more natural than that of τ -distance, we give some examples. Compare them with Propositions 4 and 5 in [9] and Proposition 4.2 in [11]. We note that we have not yet proved τ -distance versions of Propositions 19–21 below.

The following is connected with the result in Zhong [7]. See also Turinici [17].

Proposition 19. *Let p be a τ' -distance on a metric space (X, d) . Let b be a nonincreasing function from $[0, \infty)$ into $(0, \infty)$ with $\int_0^\infty b(t)dt = \infty$; and let Γ be a function from X into $[0, \infty)$ with $\Gamma(y) \leq \Gamma(x) + p(x, y)$ for $x, y \in X$. Then a function q from $X \times X$ into $[0, \infty)$ defined by*

$$q(x, y) = \int_{\Gamma(x)}^{\Gamma(x)+p(x,y)} b(t) dt \quad (27)$$

for $x, y \in X$ is also a τ' -distance on X .

Proof. Using $(\tau'1)$, we have

$$\begin{aligned} q(x, z) &= \int_{\Gamma(x)}^{\Gamma(x)+p(x,z)} b(t) dt \\ &\leq \int_{\Gamma(x)}^{\Gamma(x)+p(x,y)+p(y,z)} b(t) dt \\ &= \int_{\Gamma(x)}^{\Gamma(x)+p(x,y)} b(t) dt \\ &\quad + \int_{\Gamma(x)+p(x,y)}^{\Gamma(x)+p(x,y)+p(y,z)} b(t) dt \\ &\leq \int_{\Gamma(x)}^{\Gamma(x)+p(x,y)} b(t) dt + \int_{\Gamma(y)}^{\Gamma(y)+p(y,z)} b(t) dt \\ &= q(x, y) + q(y, z) \end{aligned} \quad (28)$$

for any $x, y, z \in X$ and hence $(\tau'1)_q$ holds. Define a function g from X into $(0, \infty)$ by

$$\int_{\Gamma(x)}^{\Gamma(x)+g(x)} b(t) dt = 1 \quad (29)$$

for $x \in X$. From the assumption of $\int_0^\infty b(t)dt = \infty$, g is well defined. If $q(x, y) \leq 1$, then we have $p(x, y) \leq g(x)$ from the definition of g . So,

$$\begin{aligned} b(\Gamma(x) + g(x)) p(x, y) &\leq b(\Gamma(x) + p(x, y)) p(x, y) \\ &= \int_{\Gamma(x)}^{\Gamma(x)+p(x,y)} b(\Gamma(x) + p(x, y)) dt \\ &\leq \int_{\Gamma(x)}^{\Gamma(x)+p(x,y)} b(t) dt = q(x, y). \end{aligned} \quad (30)$$

Hence, we note that

$$\begin{aligned} q(x, y) &\leq 1 \\ \text{implies } p(x, y) &\leq \frac{q(x, y)}{b(\Gamma(x) + g(x))}. \end{aligned} \quad (31)$$

Using (31), we next show $(\tau'2)_q$. We assume $\lim_n \sup\{q(z_n, z_m) : m > n\} = 0$ and $\lim_n q(z_n, x_n) = 0$. Without loss of generality, we may assume $\sup\{q(z_n, z_m) : m > n\} \leq 1$ and $q(z_n, x_n) \leq 1$ for any $n \in \mathbb{N}$. Then for $n \in \mathbb{N}$ with $n \geq 2$, we have

$$p(z_1, z_n) \leq \frac{q(z_1, z_n)}{b(\Gamma(z_1) + g(z_1))} \leq \frac{1}{b(\Gamma(z_1) + g(z_1))} \quad (32)$$

and hence

$$\begin{aligned} \Gamma(z_n) &\leq \Gamma(z_1) + p(z_1, z_n) \\ &\leq \Gamma(z_1) + \frac{1}{b(\Gamma(z_1) + g(z_1))}. \end{aligned} \quad (33)$$

This implies that $\{\Gamma(z_n)\}$ is bounded. From the definition of g , $\{\Gamma(z_n) + g(z_n)\}$ is also bounded and so is $\{1/b(\Gamma(z_n) + g(z_n))\}$. We put

$$M := \sup \left\{ \frac{1}{b(\Gamma(z_n) + g(z_n))} : n \in \mathbb{N} \right\} < \infty. \quad (34)$$

Then we have by (31)

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sup_{m > n} p(z_n, z_m) &\leq M \limsup_{n \rightarrow \infty} \sup_{m > n} q(z_n, z_m) = 0, \\ \lim_{n \rightarrow \infty} p(z_n, x_n) &\leq M \lim_{n \rightarrow \infty} q(z_n, x_n) = 0. \end{aligned} \quad (35)$$

By $(\tau'2)$, we obtain $\lim_n d(z_n, x_n) = 0$. Moreover, if $\{x_n\}$ converges to some $x \in X$, then $p(w, x) \leq \liminf_n p(w, x_n)$ for any $w \in X$. This yields $q(w, x) \leq \liminf_n q(w, x_n)$ for any $w \in X$. Therefore, $(\tau'2)_q$ holds. We can prove $(\tau'3)_q$ as in the proof of $(\tau'2)_q$. $\square$

We call that a nonempty subset A of X is called p -bounded if $\sup\{p(a, b) : a, b \in A\} < \infty$. The following is connected with the result in Ume [18].

Proposition 20. Let p be a τ' -distance on a metric space (X, d) . Let T be a set-valued mapping on X such that Tx is a nonempty p -bounded subset of X for any $x \in X$. Assume that $\lim_n x_n = z$, $y_n \in Tx_n$, and $\lim_n y_n = z$ imply $Tz = \{z\}$. Then a function q from $X \times X$ into $[0, \infty)$ defined by

$$q(x, y) = \max \left\{ \sup_{a \in Tx} \inf_{b \in Ty} p(a, b), \sup_{a \in Tx} p(a, y) \right\} \quad (36)$$

for any $x, y \in X$ is also a τ' -distance.

Proof. It is obvious that $\{x\}$ is p -bounded for any $x \in X$. For nonempty p -bounded subsets A and B of X , we put

$$\delta(A, B) = \sup_{a \in A} \inf_{b \in B} p(a, b). \quad (37)$$

Then $\delta(A, B) < \infty$. Indeed, for $a_0 \in A$, we have

$$\begin{aligned} \delta(A, B) &\leq \sup_{a \in A} \inf_{b \in B} (p(a, a_0) + p(a_0, b)) \\ &= \sup_{a \in A} p(a, a_0) + \inf_{b \in B} p(a_0, b) \\ &\leq \sup_{a_1, a_2 \in A} p(a_1, a_2) + \inf_{b \in B} p(a_0, b) < \infty. \end{aligned} \quad (38)$$

Using δ , we can write

$$q(x, y) = \max \{ \delta(Tx, Ty), \delta(Tx, \{y\}) \}. \quad (39)$$

From $(\tau'1)$, we have

$$\delta(A, C) \leq \delta(A, B) + \delta(B, C) \quad (40)$$

for nonempty p -bounded subsets A , B , and C of X . So, for $x, y, z \in X$, we obtain

$$\begin{aligned} q(x, z) &= \max \{ \delta(Tx, Tz), \delta(Tx, \{z\}) \} \\ &\leq \max \{ \delta(Tx, Ty) + \delta(Ty, Tz), \delta(Tx, Ty) \\ &\quad + \delta(Ty, \{z\}) \} \leq q(x, y) + q(y, z). \end{aligned} \quad (41)$$

This is $(\tau'1)_q$. We next show $(\tau'2)_q$. We assume $\lim_n \sup \{q(z_n, z_m) : m > n\} = 0$ and $\lim_n q(z_n, x_n) = 0$. We choose a subsequence $\{h(n)\}$ of the sequence $\{n\}$ such that $\sup \{q(z_{h(n)}, z_m) : m > h(n)\} < 2^{-n}$ for any $n \in \mathbb{N}$. Since $\delta(Tz_{h(n)}, Tz_{h(n+1)}) \leq q(z_{h(n)}, z_{h(n+1)}) < 2^{-n}$, we can choose a sequence $\{a_n\}$ in X such that

$$\begin{aligned} a_n &\in Tz_{h(n)}, \\ p(a_n, a_{n+1}) &< 2^{-n} \end{aligned} \quad (42)$$

for $n \in \mathbb{N}$. We note

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{m > n} p(a_n, a_m) &\leq \lim_{n \rightarrow \infty} \sup_{m > n} \sum_{j=n}^{m-1} p(a_j, a_{j+1}) \\ &\leq \lim_{n \rightarrow \infty} \sup_{m > n} \sum_{j=n}^{m-1} 2^{-j} \leq \lim_{n \rightarrow \infty} 2^{1-n} = 0. \end{aligned} \quad (43)$$

Let $\{f(n)\}$ be an arbitrary subsequence of $\{n\}$ and choose a subsequence $\{f \circ g(n)\}$ of $\{f(n)\}$ such that $h(n) < f \circ g(n)$ for any $n \in \mathbb{N}$. Then we have

$$\begin{aligned} \lim_{n \rightarrow \infty} p(a_n, z_{f \circ g(n)}) &\leq \lim_{n \rightarrow \infty} \delta(Tz_{h(n)}, \{z_{f \circ g(n)}\}) \\ &\leq \lim_{n \rightarrow \infty} q(z_{h(n)}, z_{f \circ g(n)}) = 0 \end{aligned} \quad (44)$$

and hence $\lim_n d(a_n, z_n) = 0$ from Lemma 18. We also have

$$\begin{aligned} \lim_{n \rightarrow \infty} p(a_n, x_{f \circ g(n)}) &\leq \lim_{n \rightarrow \infty} \delta(Tz_{h(n)}, \{x_{f \circ g(n)}\}) \\ &\leq \lim_{n \rightarrow \infty} q(z_{h(n)}, x_{f \circ g(n)}) \\ &\leq \lim_{n \rightarrow \infty} (q(z_{h(n)}, z_{f \circ g(n)}) + q(z_{f \circ g(n)}, x_{f \circ g(n)})) = 0 \end{aligned} \quad (45)$$

and hence $\lim_n d(a_n, x_n) = 0$ from Lemma 18. Thus, $\lim_n d(z_n, x_n) = 0$ holds. If $\{x_n\}$ converges to some $x \in X$, then $\{z_{h(n)}\}$ and $\{a_n\}$ also converge to x . So from the assumption, $Tx = \{x\}$ holds. Also, $p(w, x) \leq \liminf_n p(w, x_n)$ holds for any $w \in X$ from Lemma 18. We obtain

$$\begin{aligned} q(w, x) &= \delta(Tw, \{x\}) = \sup_{b \in Tw} p(b, x) \\ &\leq \sup_{b \in Tw} \liminf_{n \rightarrow \infty} p(b, x_n) \leq \liminf_{n \rightarrow \infty} \sup_{b \in Tw} p(b, x_n) \\ &= \liminf_{n \rightarrow \infty} \delta(Tw, \{x_n\}) \leq \liminf_{n \rightarrow \infty} q(w, x_n). \end{aligned} \quad (46)$$

We have shown $(\tau'2)_q$. Let us prove $(\tau'3)_q$. We assume $\lim_n q(z, x_n) = 0$. Fix $a \in Tz$. Then since

$$\lim_{n \rightarrow \infty} p(a, x_n) \leq \lim_{n \rightarrow \infty} \delta(Tz, \{x_n\}) \leq \lim_{n \rightarrow \infty} q(z, x_n) = 0, \quad (47)$$

$\{x_n\}$ satisfies Condition (CL) by Lemma 14. We assume that $\{x_n\}$ converges to x . We choose a sequence $\{y_n\}$ in X such that

$$\begin{aligned} y_n &\in Tx_n, \\ p(a, y_n) &\leq \delta(\{a\}, Tx_n) + \frac{1}{n} \end{aligned} \quad (48)$$

for $n \in \mathbb{N}$. Since

$$\begin{aligned} \lim_{n \rightarrow \infty} p(a, y_n) &\leq \lim_{n \rightarrow \infty} \delta(\{a\}, Tx_n) \leq \lim_{n \rightarrow \infty} \delta(Tz, Tx_n) \\ &\leq \lim_{n \rightarrow \infty} q(z, x_n) = 0, \end{aligned} \quad (49)$$

we have $\lim_n d(x_n, y_n) = 0$ by Lemma 14. Hence, $\{y_n\}$ converges to x . From the assumption, $Tx = \{x\}$ holds. As in the proof of $(\tau'2)$, we can prove $q(w, x) \leq \lim_n q(w, x_n)$ for any $w \in X$. $\square$

We finally prove the following. We note that our proof is very easy while the proof of Proposition 4.2 in [11] is difficult.

Proposition 21. Let (X, d) be a metric space. Let $\{p_j : j \in J\}$ be a set of τ' -distances on X . If a function q defined by

$$q(x, y) = \sup_{j \in J} p_j(x, y) \quad (50)$$

is a real-valued function, then q is a τ' -distance on X .

Proof. We have

$$\begin{aligned} q(x, z) &= \sup_{j \in J} p_j(x, z) \leq \sup_{j \in J} (p_j(x, y) + p_j(y, z)) \\ &\leq \sup_{j \in J} p_j(x, y) + \sup_{j \in J} p_j(y, z) \\ &= q(x, y) + q(y, z) \end{aligned} \quad (51)$$

for any $x, y, z \in X$. We next show $(\tau'2)_q$. We assume $\lim_n \sup\{q(z_n, z_m) : m > n\} = 0$ and $\lim_n q(z_n, x_n) = 0$. Then since $p_j \leq q$ for any $j \in J$, $\lim_n \sup\{p_j(z_n, z_m) : m > n\} = 0$ and $\lim_n p_j(z_n, x_n) = 0$ hold. Thus, from $(\tau'2)_{p_j}$, $\lim_n d(z_n, x_n) = 0$ holds. If $\{x_n\}$ converges to some $x \in X$, then $p_j(w, x) \leq \liminf_n p_j(w, x_n)$ holds for any $w \in X$ and $j \in J$ by using $(\tau'2)_{p_j}$ again. Using it, we have

$$\begin{aligned} q(w, x) &= \sup_{j \in J} p_j(w, x) \leq \sup_{j \in J} \liminf_{n \rightarrow \infty} p_j(w, x_n) \\ &\leq \liminf_{n \rightarrow \infty} \sup_{j \in J} p_j(w, x_n) = \liminf_{n \rightarrow \infty} q(w, x_n) \end{aligned} \quad (52)$$

for any $w \in X$. We have shown $(\tau'2)_q$. We can similarly prove $(\tau'3)_q$. $\square$

5. Kirk and Saliga's Fixed Point Theorem

Kirk and Saliga generalized Caristi's fixed point theorem [2, 3]; see Theorem 2.3 in [19]. The author thinks that the proof in [19] is splendid. In this section, we generalize Kirk and Saliga's fixed point theorem.

Let α be an ordinal number. We denote by α^+ and α^- the successor and the predecessor of α , respectively. We recall that α is *isolated* if α^- exists. α is *limit* if $\alpha \neq 0$ and α^- does not exist. For a set A , we denote by $\#A$ cardinal number of A .

Theorem 22. *Let (X, d) be a complete metric space with a τ' -distance p . Let T be a mapping on X . Let f be a function from X into $\mathbb{R}$ bounded from below such that $f(x) \leq \liminf_{\delta} f(x_{\delta})$ whenever a net $\{x_{\delta} : \delta \in E\}$ satisfies the following:*

- (a) $\{x_{\delta}\}$ converges to x .
- (b) For any $\delta \in E$, there exists $\varepsilon \in E$ satisfying $\varepsilon > \delta$.
- (c) $\lim_{\delta} \sup\{p(x_{\delta}, x_{\varepsilon}) : \varepsilon > \delta\} = 0$.
- (d) $f(x_{\delta}) \geq f(x_{\varepsilon})$ for $\delta, \varepsilon \in E$ with $\delta \leq \varepsilon$.

Assume $f(Tx) \leq f(x)$ for any $x \in X$, and there exists $\kappa \in \mathbb{N}$ such that

$$f(T^{\kappa}x) + p(x, Tx) \leq f(x) \quad (53)$$

for any $x \in X$. Fix $u \in X$. Let γ be the first uncountable ordinal number and put $D = \{\alpha : \alpha < \gamma\}$. Define a net $\{u_{\alpha} : \alpha \in D\}$ by

$$u_{\alpha} = \begin{cases} u & \text{if } \alpha = 0 \\ Tu_{\alpha^-} & \text{if } \alpha \text{ is isolated} \\ \lim [u_{\beta} : \beta < \alpha] & \text{if } \alpha \text{ is limit.} \end{cases} \quad (54)$$

Then $\{u_{\alpha}\}$ is well defined and there exists $\alpha \in D$ satisfying $Tu_{\alpha} = u_{\alpha}$ and $p(u_{\alpha}, u_{\alpha}) = 0$.

Proof. Define a function F from X into $\mathbb{R}$ by

$$F(x) = \sum_{j=0}^{\kappa-1} f(T^j x), \quad (55)$$

where T^0 is the identity mapping on X . It is obvious that

$$F(Tx) \leq F(Tx) + p(x, Tx) \leq F(x) \quad (56)$$

holds for any $x \in X$. We shall show by transfinite induction (P_{α}) that

- (a) u_{α} is well defined,
- (b) $F(u_{\alpha}) \leq F(u_{\alpha}) + p(u_{\beta}, u_{\alpha}) \leq F(u_{\beta})$ for $\beta < \alpha$,

for $\alpha < \gamma$. It is obvious that P_0 is true. Fix α with $0 < \alpha < \gamma$, and assume that P_{β} is true for $\beta < \alpha$. In the case where α is isolated, it is obvious that $u_{\alpha} = Tu_{\alpha^-}$ is well defined. We have

$$F(u_{\alpha}) \leq F(u_{\alpha}) + p(u_{\alpha^-}, u_{\alpha}) \leq F(u_{\alpha^-}). \quad (57)$$

For $\beta < \alpha^-$, we have

$$\begin{aligned} F(u_{\alpha}) &\leq F(u_{\alpha}) + p(u_{\beta}, u_{\alpha}) \\ &\leq F(u_{\alpha}) + p(u_{\beta}, u_{\alpha^-}) + p(u_{\alpha^-}, u_{\alpha}) \\ &\leq F(u_{\alpha^-}) + p(u_{\beta}, u_{\alpha^-}) \leq F(u_{\beta}). \end{aligned} \quad (58)$$

Hence, P_{α} is true. In the other case, where α is limit, since $\{F(u_{\beta}) : \beta < \alpha\}$ is a nonincreasing and F is bounded from below, $\{F(u_{\beta}) : \beta < \alpha\}$ converges and hence it is Cauchy. We have

$$\begin{aligned} \limsup_{\beta < \alpha} \{p(u_{\beta}, u_{\delta}) : \beta < \delta < \alpha\} \\ \leq \limsup_{\beta < \alpha} \{F(u_{\beta}) - F(u_{\delta}) : \beta < \delta < \alpha\} = 0. \end{aligned} \quad (59)$$

By Lemma 16, $\{u_{\beta} : \beta < \alpha\}$ satisfies Condition (CL). Since X is complete, $\{u_{\beta} : \beta < \alpha\}$ converges. Hence, u_{α} is well defined. For $\beta < \alpha$, we have

$$\begin{aligned} F(u_{\alpha}) &\leq F(u_{\alpha}) + p(u_{\beta}, u_{\alpha}) \leq \kappa f(u_{\alpha}) + p(u_{\beta}, u_{\alpha}) \\ &\leq \lim_{\delta < \alpha} F(u_{\delta}) + \liminf_{\delta < \alpha} p(u_{\beta}, u_{\delta}) \\ &= \liminf_{\delta < \alpha} (F(u_{\delta}) + p(u_{\beta}, u_{\delta})) \leq F(u_{\beta}). \end{aligned} \quad (60)$$

Hence, P_{α} is true. Therefore, by transfinite induction, P_{α} is true for any $\alpha < \gamma$. Arguing by contradiction, we assume the following:

- (a) $F(u_{\alpha^{++}}) < F(u_{\alpha})$ for any $\alpha \in D$.

Then there exists $s, t \in \mathbb{Q}$ satisfying $F(u_{\alpha^{++}}) < s < t < F(u_{\alpha})$. So we have

$$\#\mathbb{Q} = \#\mathbb{N} < \#\gamma \leq \#\mathbb{Q}, \quad (61)$$

which is a contradiction. Therefore, there exists $\alpha \in D$ satisfying $F(u_{\alpha^{++}}) = F(u_{\alpha})$. We have

$$\begin{aligned} F(u_{\alpha}) &= F(u_{\alpha^{++}}) \leq F(u_{\alpha^{++}}) + p(u_{\alpha}, u_{\alpha^{++}}) \\ &\leq F(u_{\alpha}), \\ F(u_{\alpha}) &= F(u_{\alpha^{++}}) \leq F(u_{\alpha^+}) \leq F(u_{\alpha^+}) + p(u_{\alpha}, u_{\alpha^+}) \\ &\leq F(u_{\alpha}). \end{aligned} \quad (62)$$

Hence, $p(u_{\alpha}, u_{\alpha^{++}}) = 0$ and $p(u_{\alpha}, u_{\alpha^+}) = 0$ hold. By Lemma 14, $u_{\alpha^+} = u_{\alpha^{++}}$ holds. Therefore, $Tu_{\alpha^+} = u_{\alpha^+}$ holds. $\square$

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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