Weak Estimates of Singular Integrals with Variable Kernel and Fractional Differentiation on Morrey-Herz Spaces

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Let \( T \) be the singular integral operator with variable kernel defined by

\[
Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x,x - y)}{|x - y|^n} f(y) dy,
\]

where \( \Omega(x, z) \) satisfies the following conditions:

\[
\Omega(x, \lambda z) = \Omega(x, z), \quad \text{for any } x, z \in \mathbb{R}^n, \lambda > 0,
\]

\[
\int_{S^{n-1}} \Omega(x, z') d\sigma(z') = 0, \quad \text{for any } x \in \mathbb{R}^n.
\]

As we all know, the singular integrals with variable kernel played an important role in the theory of nondivergent elliptic equations with discontinuous coefficients (see [1, 2]). Some properties for various of the singular integrals with variable kernel have been obtained by authors; for example, see [3–6] and their references. In the Mihlin conditions, Calderón and Zygmund proved the boundedness of \( T \) on the \( L^2(S^{n-1}) \) (see [7]).

Let \( 0 \leq \gamma \leq 1 \). For tempered distributions \( f \in \mathcal{S}'(\mathbb{R}^n) (n = 1, 2, \ldots) \), the fractional differentiation operators \( D^\gamma \) defined by \( D^\gamma f(x) = (|\xi|^\gamma \widehat{f}(\xi))^\gamma(x) \). Let \( I_{\gamma} \) be the Riesz potential operator of order \( \gamma \) defined on the space of tempered distributions modulo polynomials by setting \( I_{\gamma} f(\xi) = |\xi|^{-\gamma} \widehat{f}(\xi) \). It is easy to see that a locally integrable function \( b \in L_1(\text{BMO}(\mathbb{R}^n)) \) if and only if \( D^\gamma b \in \text{BMO}(\mathbb{R}^n) \). Strichartz (see [8]) showed that \( I_{\gamma}(\text{BMO}(\mathbb{R}^n)) \) is a space of functions modulo constants which is properly contained in \( \text{Lip}_\gamma(\mathbb{R}^n) \), where \( \gamma \in (0, 1) \).

Denote \( \mathcal{H}_m \) to be the space of spherical harmonic homogeneous polynomials of degree \( m \). Let \( \dim \mathcal{H}_m = d_m \) and \( \{Y_{m,j}\}_{j=1}^{d_m} \) be an orthonormal system of \( \mathcal{H}_m \). It is well known that \( \{Y_{m,j}\}_{j=1}^{d_m}, m = 0, 1, \ldots \) is a complete orthonormal system in \( L^2(S^{n-1}) \) (see [9]). Let us expand the function \( \Omega(x, z') \) in spherical harmonics

\[
\Omega(x, z') = \sum_{m=0}^{d_m} \sum_{j=1}^{d_m} b_{m,j}(x) Y_{m,j}(z'),
\]

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The main purpose of this paper is to generalize the above

on the weak Morrey-Herz spaces. The answer is affirmative.

In 2016, Tao and Yang obtained the boundedness of those

operatorsontheweightedMorrey-Herzspaces(see[6]).Anat-

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respectively. In [7], Calderón and Zygmund found that

these operators are closely related to the second order linear

equations with variables coefficients and established

the following boundedness of the operators

Let

and

and

are

with homogeneousMorrey-Herzspaces and weakMorrey-Herzspaces.

Then one has

one has

Theorem 2. Let

Suppose that

and

satisfy (2) and (3). If

satisfies (9) and

satisfies

one has

Furthermore, we also consider the cases

As we all know, D is the square root of Laplacian operator

and

is the identity operator

In this case, we obtain the following results.

Theorem 3. Let

Suppose that

satisfies (2) and (3), and

Then one has

one has

Theorem 4. Let

Suppose that

satisfies (2), (3), and

Then one has

one has

max

Theorem A (see [7]). Let

be measurable set

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such that

If any index

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respectively. In [7], Calderón and Zygmund found that

these operators are closely related to the second order linear

elliptic equations with variables coefficients and established

the following boundedness of the operators

Let

and

are

with homogeneousMorrey-Herzspaces and weakMorrey-Herzspaces.

Then one has

one has

Theorem 5. Let

Suppose that

and

satisfy (2) and (3). If

satisfies (10) and

satisfies (12), then one has

2. Preliminaries and Main Lemmas

In this section, we shall recall the definitions of the homo-
geousMorrey-Herzspaces and weak Morrey-Herzspaces.

Our main results are stated as follows.

\[ a_{m,j}(x) = \int_{\mathbb{R}^n} \Omega(x, z') \frac{Y_{m,j}(z')}{|z'|} \, d\sigma(z'). \]  

\[ T_{m,j}f(x) = \left( \frac{Y_{m,j}}{|z'|} \ast f \right)(x). \]  

\[ Tf(x) = \sum_{m \geq 1} \sum_{j = 1}^{d_m} \left( -1 \right)^m T_{m,j}(\overline{a}_{m,j}(x)f)(x). \]  

\[ T_{\ast}f(x) = \sum_{m = 1}^{\infty} \sum_{j = 1}^{d_m} \left( -1 \right)^m T_{m,j}(\overline{a}_{m,j}(x)f)(x). \]  

\[ T_{\ast}\overline{T}_{\ast} = \overline{T}_{\ast}T_{\ast} \]  

\[ \Omega(\alpha, \lambda) \]  

\[ C_{\beta}^{C_{\infty}}, \beta > 1 \]  

\[ (1) \| (TD - DT_{\ast}^\ast) f \|_{L^p} \leq \| f \|_{L^p}; \]  

\[ (2) \| (T_{\ast}^\ast - \overline{T}_{\ast}^\ast) f \|_{L^{\infty}} \leq \| f \|_{L^p}; \]  

\[ (3) \| (T_{\ast} + \overline{T}_{\ast} - 1) f \|_{L^p} \leq \| f \|_{L^p}. \]  

In 2015, Chen and Zhu proved that Theorem A was also true on Weighted Lebesgue space and Morrey space (see [10]). In 2016, Tao and Yang obtained the boundedness of those operators on the weighted Morrey-Herz spaces (see [6]). A natural question is whether these operators also have boundedness on the weak Morrey-Herz spaces. The answer is affirmative. The main purpose of this paper is to generalize the above results to the cases of weak Morrey-Herz spaces \( WMK^{\alpha,\lambda}_{p,\gamma}(\mathbb{R}^n) \) (see Definition 7 in the next section).

\[ \max \| \frac{\partial^i}{\partial y^j} \Omega(\alpha, \lambda) \|_{L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1})} < \infty. \]  

\[ \| (T_{\ast} + \overline{T}_{\ast} - 1) f \|_{WMK^{\alpha,\lambda}_{p,\gamma}(\mathbb{R}^n)} \leq \| f \|_{WMK^{\alpha,\lambda}_{p,\gamma}(\mathbb{R}^n)}. \]  

2. Preliminaries and Main Lemmas

In this section, we shall recall the definitions of the homoge-
geousMorrey-Herzspaces and weak Morrey-Herzspaces.
Furthermore, the weak estimates of $T_{m,j}$ defined by (6) and a class of Calderón-Zygmund operators will be established on Morrey-Herz spaces.

The well-known Morrey spaces, introduced originally by Morrey [11] in relation to the study of partial differential equations, were widely investigated during last decades, including the study of classical operators of harmonic analysis in various generalizations of these spaces. Morrey-type spaces appeared to be quite useful in the study of the local behavior of the solutions of partial differential equations, a priori estimates, and other topics. They are also widely used in applications to regularity properties of solutions to PDE including the study of Navier-Stokes equations (see [12] and references therein). The ideas of Morrey [11] were further developed by Campanato [13]. A more systematic study of these (and even more general) spaces, we refer the readers to see [12, 14–21]. In 1964, Beurling [22] first introduced some fundamental forms of Herz spaces to study convolution algebras. Later Herz [23] gave versions of the spaces defined below in a slightly different setting. Since then, the theory of Herz spaces has been significantly developed, and these spaces have turned out to be quite useful in harmonic analysis. For instance, they were used by Baernstein and Sawyer [24] to characterize the multipliers on the classical Hardy spaces and used by Lu and Yang [25] in the study of partial differential equations. More results and further details can be found in [26–28]. On the basis of above available results, the theory of the homogeneous Morrey-Herz spaces goes back to Lu-Xu [29] who considered the boundedness of a class of sublinear operators; also see [6, 30, 31] for more further results. Next we give the following notation. For each $k \in \mathbb{Z}$, we denote $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $A_k = B_k - B_{k-1}$, $\chi_k = \chi_{A_k}(x)$.

**Definition 6** (see [29]). Let $\alpha \in \mathbb{R}^n$, $0 < p \leq \infty$, $0 < q < \infty$, and $\lambda \geq 0$. The homogeneous Morrey-Herz spaces $\mathcal{M}\mathcal{K}_{p,\lambda}^\alpha(\mathbb{R}^n)$ are defined by

$$M\mathcal{K}_{p,\lambda}^\alpha(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n \setminus 0), \|f\|_{M\mathcal{K}_{p,\lambda}^\alpha(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{M\mathcal{K}_{p,\lambda}^\alpha(\mathbb{R}^n)} = \sup_{k \in \mathbb{Z}} 2^{-\lambda k} \left( \sum_{k=-\infty}^{\infty} 2^{k\lambda} \|f|_{B_k}\|_{L^p(\mathbb{R}^n)}^p \right)^{1/p},$$

and the usual modifications should be made when $p = \infty$.

In what follows, for any $k \in \mathbb{Z}$ and $\gamma > 0$, let $m_k(y, f) = \{|x \in A_k : |f(x)| > \gamma\}$.

**Definition 7** (see [29]). Let $\alpha \in \mathbb{R}^n$, $0 < p \leq \infty$, $0 < q < \infty$, and $\lambda \geq 0$. A measurable function $f$ is said to belong to the homogeneous weak Morrey-Herz spaces $W\mathcal{M}\mathcal{K}_{p,\lambda}^\alpha(\mathbb{R}^n)$, if

$$W\mathcal{M}\mathcal{K}_{p,\lambda}^\alpha(\mathbb{R}^n) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n \setminus 0), \|f\|_{W\mathcal{M}\mathcal{K}_{p,\lambda}^\alpha(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{W\mathcal{M}\mathcal{K}_{p,\lambda}^\alpha(\mathbb{R}^n)} = \sup_{\gamma > 0} \sup_{k \in \mathbb{Z}} 2^{-\lambda k} \left( \sum_{k=-\infty}^{\infty} 2^{k\lambda} m_k(y, f)^{p/q} \right)^{1/p} < \infty,$$

where the usual modifications are made when $p = \infty$.

**Lemma 8** (see [32]). $(X, \mathcal{A}, \mu)$ is said to be a homogeneous space. Let $0 \leq \lambda \leq \infty$, $0 \leq l < 1$, $1 < q_1 < 1/l$, $l/q_2 = 1/q_1 - l$, $\lambda - 1/q_2 < \alpha < \lambda + 1 - 1/q_1$, and $0 < p_1 \leq p_2 < \infty$, if a sublinear operator $T_l$ meets the following requirements:

$$|T_l(f(x))| \leq C \int \frac{|f(y)|}{\mu(B(x, d(x, y)))} \, dy, \quad x \notin \text{supp} \, f.$$
Let $2Q$ be a cube with sides twice as long, and let $Q$ and $R$ be cubes with sides twice as long. For $\lambda > 0$, we have

\[
\left\lfloor \lambda \right\rfloor \leq \frac{2^n}{\lambda} \int_{\mathbb{R}^n} |T_{m,j}b(x)|^2 \, dx
\]

Let $2Q_l$ be the cube with the same center as $Q_l$ and whose sides are twice as long, and let $(Q_l^*)^* = \bigcup_{l=1}^{\infty} 2Q_l$. Then we have $|Q_l^*| \leq 2^n|Q_l^3|$, and together with characteristic of left in (2), we can obtain

\[
|Q_{l+1}| = \sum_{j=1}^{\infty} |Q_j| \leq \sum_{j=1}^{\infty} \int_{Q_j} |f(x)| \, dx \leq \frac{1}{\lambda} \|f\|_1.
\]

Thus we have

\[
\left\lfloor \lambda \right\rfloor \leq \frac{2^n}{\lambda} \|f\|_1 + \frac{2}{\lambda} \int_{\mathbb{R}^n \setminus Q_{l+1}} |T_{m,j}b(x)| \, dx.
\]

Notice that $|T_{m,j}b(x)| \leq \sum_{l=1}^{\infty} |T_{m,l}b_l(x)|$ almost everywhere. Hence, to complete the proof of the weak (1, 1) inequality it will suffice to show that

\[
\sum_{l=1}^{\infty} \int_{\mathbb{R}^n \setminus Q_{l+1}} |T_{m,j}b_l(x)| \, dx \leq \|f\|_1.
\]

For any $b_l$, $x \notin 2Q_l$, the formula

\[
T_{m,j}b_l(x) = \int_{Q_l} b_l(y) Y_{m,j}(x-y) \frac{1}{|x-y|^n} \, dy
\]

is still valid. Denote the center of $Q_j$ by $z_{Q_j}$, and then we have

\[
\int_{\mathbb{R}^n \setminus Q_j} |T_{m,j}b_l(x)| \, dx = \int_{\mathbb{R}^n \setminus Q_j} \left| Y_{m,j}(x-y) b_l(y) \right| \frac{1}{|x-y|^n} \, dy \, dx
\]

\[
\leq \int_{\mathbb{R}^n \setminus Q_j} b_l(y) Y_{m,j}(x-y) \frac{1}{|x-y|^n} \, dy \, dx
\]

without loss of generality, for $p = 2$, we have

\[
\left\lfloor \lambda \right\rfloor \leq \frac{2^n}{\lambda} \int_{\mathbb{R}^n} |T_{m,j}g(x)|^2 \, dx
\]

\[
\leq \frac{2^n}{\lambda^2} \left( \sum_{l=1}^{\infty} |Q_l| \right) \int_{\mathbb{R}^n} |g(x)|^2 \, dx
\]

\[
\leq \frac{2^n}{\lambda^2} \left( \sum_{l=1}^{\infty} \int_{Q_l} |f(x)| \, dx \right) \int_{\mathbb{R}^n} |g(x)|^2 \, dx
\]

\[
= \frac{2^n}{\lambda^2} \sum_{l=1}^{\infty} |Q_l| \int_{\mathbb{R}^n \setminus Q_{l+1}} |T_{m,j}b_l(x)| \, dx.
\]

Notice that

\[
\int_{\mathbb{R}^{n+2}} \frac{|Q_j|}{|x-z_Q|^n} \, dx = \sum_{k=1}^{\infty} \int_{\mathbb{R}^{n+2}} |Q_j| \frac{|Q_j|}{|x-z_Q|^n} \, dx
\]

\[
\leq \sum_{k=1}^{\infty} \frac{|2^{n+2}Q_{j+1}|}{|x-z_Q|^n} \frac{|Q_j|}{|x-z_Q|^n} \leq \sum_{k=1}^{\infty} 2^{-kn} < \infty.
\]

Then for any $T_{m,j}$, it is clear that

\[
\left\lfloor \lambda \right\rfloor \leq \frac{2^n}{\lambda^2} \sum_{l=1}^{\infty} |Q_l| \int_{\mathbb{R}^n \setminus Q_{l+1}} |T_{m,j}b_l(x)| \, dx.
\]

Summing up the estimates above for $T_{m,j}g(x)$ and $T_{m,j}b(x)$, we finish the proof of Lemma 9.

\[\square\]

**Lemma 10.** Let $0 \leq \lambda \leq \infty$, $\lambda - 1 < \alpha < \lambda$, and $0 < p < \infty$. Let $T$ be a generalized Calderón-Zygmund operator, and then $T$ is bounded from $MK^\alpha_{p,1}(\mathbb{R}^n)$ to WM$K^\alpha_{p,1}(\mathbb{R}^n)$; namely,

\[
\|Tf(x)\|_{WMK^\alpha_{p,1}(\mathbb{R}^n)} \leq \|f(x)\|_{MK^\alpha_{p,1}(\mathbb{R}^n)}.
\]

**Proof.** It is well known that $T$ is weak (1) (e.g., see [33]). Noticing that $T$ satisfying (18) with $l = 0$, then we can obtain Lemma 10 by using Lemma 8. \[\square\]
Lemma 12. Let $0 \leq \lambda \leq \infty$, $\lambda - 1 < \alpha < \lambda$, and $0 < p < \infty$; $T_{m,j}$ defined by (6) is bounded from $MK_k^{a,\lambda}_{p,1}(\mathbb{R}^n)$ to $WMK_k^{a,\lambda}_{p,1}(\mathbb{R}^n)$, and

$$
\left\| T_{m,j}f(x) \right\|_{WMK_k^{a,\lambda}_{p,1}(\mathbb{R}^n)} \leq m^{n/2} \left\| f(x) \right\|_{MK_k^{a,\lambda}_{p,1}(\mathbb{R}^n)}.
$$

Proof. By applying the fact that $T_{m,j}f(x) = Y_{m,j} \cdot f(x)$ and $|Y_{m,j}| \leq m^{(n-2)/2}$ (7), we can easily obtain

$$
\left\| T_{m,j}f(x) \right\| \leq m^{n/2} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^n} dy.
$$

Noticing that the interior integral above is meeting the condition in (18) for $l = 0$ and $T_{m,j}$ is weak (1) on the basis of Lemma 9, it is not difficult to deduce that $T_{m,j}$ is bounded from $MK_k^{a,\lambda}_{p,1}(\mathbb{R}^n)$ to $WMK_k^{a,\lambda}_{p,1}(\mathbb{R}^n)$. Hence

$$
\left\| T_{m,j}f(x) \right\|_{WMK_k^{a,\lambda}_{p,1}(\mathbb{R}^n)} \leq m^{n/2} \left\| f(x) \right\|_{MK_k^{a,\lambda}_{p,1}(\mathbb{R}^n)}.
$$

This completes the proof of Lemma 12. □

Lemma 13. Let $t(x)$ be a homogeneous of degree $-n - 1$ and locally integrable in $|x| > 0$. Let $b \in Lip(\mathbb{R}^n)$ and

$$
Kf(x) = \lim_{\varepsilon \to 0} \int_{|x-y|<\varepsilon} t(x) (b(x) - b(y)) f(y) dy.
$$

If $t(x) \in C(\mathbb{S}^{n-1})$ and $t(x) \, d\sigma(x) = 0$, then, for $0 \leq \lambda \leq \infty$, $\lambda - 1 < \alpha < \lambda$, and $0 < p < \infty$, one has

$$
\left\| Kf \right\|_{WMK_k^{a,\lambda}_{p,1}(\mathbb{R}^n)} \leq \left\| \frac{\partial f}{\partial x_j} \right\|_{WMK_k^{a,\lambda}_{p,1}(\mathbb{R}^n)}.
$$

Proof. With an argument similar to that used in the proof of Lemma 5.2 in [10], it is not difficult to obtain Lemma 13. Thus, we omit the details here. □

3. Proofs of Theorems

Proof of Theorem 1. Let

$$
\Omega(x, y) = \sum_{m=1}^{d_m} \sum_{j=1}^{a_{m,j}} Y_{m,j}(y).
$$

From [3], for any $x$, we can write the coefficients $a_{m,j}$ as

$$
a_{m,j}(x) = (-1)^m m^{-n} (m + n - 2)^{-n} \int_{S^{n-1}} L^{n} (\Omega(x, y')) Y_{m,j}(y') d\sigma(y'),
$$

where $K(x) \in C^2(S^{n-1})$, $\int_{S^{n-1}} K(x) d\sigma(x) = 0$, and $K(\lambda x) = \lambda^{-n} K(x)$, for $x \in \mathbb{R}^n \setminus \{0\}$, $\lambda > 0$, and then one has that, for $0 \leq \lambda \leq \infty$, $\lambda - 1 < \alpha < \lambda$, $0 < p < \infty$, and $f \in C_0^\infty$, the operator

$$
\left\| [b, T] \frac{\partial f}{\partial x_j} \right\|_{WMK_k^{a,\lambda}_{p,1}(\mathbb{R}^n)} \leq \max_{|\beta| \leq 2} \left\| \partial^\beta K \right\|_{L^p(S^{n-1})} \left\| \nabla b \right\|_{L^{\infty}} \left\| f \right\|_{MK_k^{a,\lambda}_{p,1}(\mathbb{R}^n)}.
$$

Proof. With an argument similar to that used in the proof of Lemma 5.2 in [10], it is not difficult to obtain Lemma 13. Thus, we omit the details here. □
Moreover, by the fact that $[b, D^r]$ is a generalized Calderón-Zygmund operator (see [35]), which is defined by
\begin{equation}
[b, D^r] f(x) = C(y) \int_{\mathbb{R}^n} \frac{(b(x) - b(y))}{|x-y|^{n+r}} f(y) \, dy.
\end{equation}
(47)

Thus we can get that $[b, D^r] f(x)$ is bounded from $MK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)$ to $WMK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)$ by applying Lemma 10; namely,
\begin{equation}
\| [b, D^r] f \|_{WMK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)} \leq \| D^r b \|_{BMO} \| f \|_{MK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}.
\end{equation}
(48)

Then by $d_m = m^{n-2}$ (see [36]), (46), (48), and Lemma 11, we have
\begin{equation}
\| (TD^r - D^r T)f \|_{WMK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}
\leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \| [a_{m,j}, D^r] T_{m,j} f \|_{WMK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}
\leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \| D^r a_{m,j} \|_{BMO} \| T_{m,j} f \|_{WMK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}
\leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \| D^r a_{m,j} \|_{L^\infty} \| f \|_{WMK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}
\leq m^{n-2} m^{n/2} \| f \|_{MK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)} \leq \| f \|_{MK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}.
\end{equation}
(49)

Now let us turn to estimate (2). By applying the definition of $T^4$ and $T^*$ we can deduce that
\begin{equation}
(T^4 - T^*) D^r f = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} (-1)^m \left[ a_{m,j}, T_{m,j} \right] D^r f.
\end{equation}
(50)

In order to estimate $WMK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)$ norm of $(T^* - T^4)D^r$, we first consider $[b, T_{m,j}] D^r$ for any fixed $b \in L_1$ (BMO). Noting that $b(x) - b(y) = (b(x) - b(z)) - (b(y) - b(z))$, for any $x, y, z \in \mathbb{R}^n$, then we have
\begin{equation}
[b, T_{m,j}] D^r f = [b, D^r T_{m,j}] f - T_{m,j} [b, D^r] f.
\end{equation}
(51)

Thus, by (48) and Lemma 11, we get
\begin{equation}
\| T_{m,j} [b, D^r] f \|_{WMK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)} \leq m^{n/2} \| D^r b \|_{BMO} \| f \|_{MK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}.
\end{equation}
(52)

Further, we estimate the $WMK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)$ norm of $[b, D^r T_{m,j}]$. From the fact that $[b, D^r T_{m,j}] f$ is a generalized Calderón-Zygmund operator with kernel (see [10])
\begin{equation}
|k_{m,j}(x, y)| \leq m^{n/2 - 1 + \gamma} \| D^r b \|_{BMO} \frac{1}{|x-y|^r},
\end{equation}
(53)
then we get by Lemma 10
\begin{equation}
\| [b, D^r T_{m,j}] f \|_{WMK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)} \leq m^{n/2 + \gamma} \| D^r b \|_{BMO} \| f \|_{MK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}.
\end{equation}
(54)

Then, combining (52) with (54), we have
\begin{equation}
\| [b, D^r T_{m,j}] f \|_{WMK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)} \leq \| [b, D^r T_{m,j}] f \|_{WMK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}
+ \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \| D^r a_{m,j} \|_{L^\infty} \| f \|_{MK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}
\leq m^{n/2} \| D^r b \|_{BMO} \| f \|_{MK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}
+ m^{n/2} \| D^r b \|_{BMO} \| f \|_{MK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}
\leq m^{n/2} \| D^r b \|_{BMO} \| f \|_{MK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}.
\end{equation}
(55)

By estimates (46), (50), and (55), we get
\begin{equation}
\| (T^4 - T^*) D^r f \|_{WMK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}
\leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \| [a_{m,j}, T_{m,j}] D^r f \|_{WMK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}
\leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \| D^r a_{m,j} \|_{L^\infty} \| f \|_{MK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}
\leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \| D^r a_{m,j} \|_{L^\infty} \| f \|_{MK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}
\leq m^{n/2} \| D^r b \|_{BMO} \| f \|_{MK^{\alpha, \lambda}_{p,1}(\mathbb{R}^n)}.
\end{equation}
(56)

Thus we finish the proof of Theorem 1.

\textit{Proof of Theorem 2.} Let
\begin{equation}
T_1 f(x) = \int_{\mathbb{R}^n} \Omega_1 \frac{(x, x - y)}{|x-y|^n} f(y) \, dy,
\end{equation}
(57)
\begin{equation}
T_2 f(x) = \int_{\mathbb{R}^n} \Omega_2 \frac{(x, x - y)}{|x-y|^n} f(y) \, dy.
\end{equation}
(58)

Write
\begin{equation}
\Omega_1 (x, y) = \sum_{m=1}^{d_m} a_{m,j} (x) Y_{m,j} (y),
\end{equation}
(58)
where

\[ a_{m,j}(x) = \int_{\mathbb{R}^n} \Omega_1(x, z') Y_{m,j}(z') \mathrm{d}z', \]

\[ b_{\lambda,\mu}(x) = \int_{\mathbb{R}^n} \Omega_2(x, z') Y_{\lambda,\mu}(z') \mathrm{d}z'. \]

(59)

For any \( x \in \mathbb{R}^n \), with a similar argument used in the proof of Theorem 1 in terms of (9) and (10), we can obtain that

\[ \left\| a_{m,j} \right\|_{L^\infty} \leq m^{-2n}, \]

\[ \left\| D^\gamma b_{\lambda,\mu} \right\|_{L^\infty} \leq m^{-2n}. \]

(60)

Let \( T_{m,j}f(x) = Y_{m,j}/| \cdot |^\nu * f(x) \) and \( T_{\lambda,\mu}f(x) = Y_{\lambda,\mu}/| \cdot |^\nu * f(x) \). Since \( \Omega_1(x, y) \) and \( \Omega_2(x, y) \) satisfy (3), then we get

\[ T_1 f(x) = \sum_{m \geq 1} \sum_{j=1}^{d_m} a_{m,j}(x) T_{m,j} f(x), \]

\[ T_2 f(x) = \sum_{\lambda \geq 1} \sum_{\mu=1}^{d_\lambda} b_{\lambda,\mu}(x) T_{\lambda,\mu} f(x). \]

(61)

Write ((10))

\[ (T_1 \ast T_2) f(x) = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{d_\lambda} a_{m,j}(x) b_{\lambda,\mu}(x) \left( T_{m,j} T_{\lambda,\mu} f \right)(x), \]

\[ (T_1 T_2) f(x) = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{d_\lambda} a_{m,j} T_{m,j} \left( b_{\lambda,\mu} T_{\lambda,\mu} f \right)(x). \]

(62)

Then

\[ (T_1 \ast T_2 - T_1 T_2) D^\gamma f = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{d_\lambda} a_{m,j} \]

\[ \left( b_{\lambda,\mu}(x) T_{m,j} - T_{m,j} b_{\lambda,\mu}(x) \right) T_{\lambda,\mu} D^\gamma f \]

\[ = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{d_\lambda} a_{m,j} \left( b_{\lambda,\mu}(x) T_{m,j} - T_{m,j} b_{\lambda,\mu}(x) \right) \]

\[ \left. \cdot D^\gamma T_{\lambda,\mu} f \right\} = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{d_\lambda} a_{m,j} \left[ b_{\lambda,\mu}(x) T_{m,j} \right] D^\gamma T_{\lambda,\mu} f. \]

(63)

Therefore, together with (55), (60), and Lemma II, we obtain

\[ \left\| (T_1 \ast T_2 - T_1 T_2) D^\gamma f \right\|_{WMK^{\alpha,\lambda}_{p,1}(\mathbb{R}^n)} \]

\[ \leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{d_\lambda} \left\| a_{m,j} \right\|_{L^\infty} \]

\[ \cdot \left\| \left( b_{\lambda,\mu} T_{m,j} \right) D^\gamma T_{\lambda,\mu} f \right\|_{WMK^{\alpha,\lambda}_{p,1}(\mathbb{R}^n)} \]

\[ \leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \sum_{\lambda=1}^{d_\lambda} \left\| a_{m,j} \right\|_{L^\infty} \left\| D^\gamma b_{\lambda,\mu} \right\|_{BMO} \]

\[ \cdot m^{n/2+\nu} \left\| T_{\lambda,\mu} f \right\|_{WMK^{\alpha,\lambda}_{p,1}(\mathbb{R}^n)} \]

\[ \leq \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \left\| a_{m,j} \right\|_{L^\infty} \left\| D^\gamma b_{\lambda,\mu} \right\|_{BMO} \]

\[ \cdot m^{n/2+\lambda^{n/2+\nu}} \left\| f \right\|_{WMK^{\alpha,\lambda}_{p,1}(\mathbb{R}^n)} \]

\[ \leq \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \left\| a_{m,j} \right\|_{L^\infty} \left\| D^\gamma b_{\lambda,\mu} \right\|_{BMO} \]

\[ \cdot m^{n/2+\lambda^{n/2+\nu}} \lambda^{n/2} \left\| f \right\|_{WMK^{\alpha,\lambda}_{p,1}(\mathbb{R}^n)} \]

\[ \leq \sum_{\lambda=1}^{\infty} \sum_{\mu=1}^{d_\lambda} \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} \left\| a_{m,j} \right\|_{L^\infty} \left\| D^\gamma b_{\lambda,\mu} \right\|_{BMO} \]

\[ \cdot m^{n/2+\lambda^{n/2+\nu}} \lambda^{n/2} \left\| f \right\|_{WMK^{\alpha,\lambda}_{p,1}(\mathbb{R}^n)} \]

\[ \leq \left\| f \right\|_{WMK^{\alpha,\lambda}_{p,1}(\mathbb{R}^n)} \cdot \]

This finishes the proof of Theorem 2.

\[ \square \]

**Proof of Theorem 3.** We can estimate term (1) exactly as we did for the corresponding boundedness in Theorem 1 in the above arguments. Thus, we have only to prove (2) and (3) of Theorem 3. In order to do this, we use the same notations as in the proof of Theorem 2. By using the fact that \( \Omega_1(x, y) \) and \( \Omega_2(x, y) \) satisfy (10), therefore, we have

\[ \left\| a_{m,j} \right\|_{L^\infty} \leq m^{-2n}, \]

\[ \left\| b_{\lambda,\mu} \right\|_{L^\infty} \leq \lambda^{-2n}. \]

(65)

Firstly, let us prove (2). As in the proof of Theorem 1, we can get

\[ (T_1^* - T_1^*) \mathcal{F} f = \sum_{m=1}^{\infty} \sum_{j=1}^{d_m} (-1)^m \left[ a_{m,j}, T_{m,j} \right] \mathcal{F} f. \]

(66)

As \( [b, T_{m,j}] \) is a special Calderón-Zygmund operator, it is bounded from Morrey-Herz spaces \( MK^{\alpha,\lambda}_{p,1}(\mathbb{R}^n) \) to weak Morrey-Herz spaces \( WMK^{\alpha,\lambda}_{p,1}(\mathbb{R}^n) \) by applying Lemma 10. Thus we have

\[ \left\| [b, T_{m,j}] \mathcal{F} f \right\|_{WMK^{\alpha,\lambda}_{p,1}(\mathbb{R}^n)} \leq m^{n/2} \left\| b \right\|_{L^\infty} \left\| \mathcal{F} f \right\|_{MK^{\alpha,\lambda}_{p,1}(\mathbb{R}^n)}. \]

(67)

Then by (65), we get

\[ \left\| (T_1^* - T_1^*) \mathcal{F} f \right\|_{WMK^{\alpha,\lambda}_{p,1}(\mathbb{R}^n)} \]

\[ \leq \sum_{m=1}^{\infty} m^{n/2} m^{-3n/2} \left\| \mathcal{F} f \right\|_{MK^{\alpha,\lambda}_{p,1}(\mathbb{R}^n)} \leq \left\| f \right\|_{MK^{\alpha,\lambda}_{p,1}(\mathbb{R}^n)}. \]

(68)

Thus conclusion (2) is proved.
We now estimate (3). Write

\[ (T_1 \ast T_2 - T_1 T_2) f = \sum_{m=1}^{\infty} \sum_{j=1}^{d_n} \sum_{\lambda=1}^{d_1} b_{k,m,j} T_m f. \]  
(69)

Therefore, by (65), (67), and Lemma II, we get

\[ \| (T_1 \ast T_2 - T_1 T_2) f \|_{WMK_{p,1}^n(R^n)} \leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_n} \sum_{\lambda=1}^{d_1} \| b_{k,m,j} \|_{L^\infty} \| b_{k,m,j} \|_{L^\infty}. \]

From this and (12), we get for \( k = 1, \ldots, n, \)

\[ \left\| \frac{\partial a_{m,j}}{\partial x_k} \right\|_{L^\infty} \leq m^{-2n}. \]  
(74)

By using the fact that \( \| \mathcal{R}_{\lambda} g \|_{WMK_{p,1}^n(R^n)} \leq \| g \|_{MK_{p,1}^n(R^n)}, d_m = m^{-2}, \) and Lemma II, then we have

\[ \| I_1 \|_{WMK_{p,1}^n(R^n)} \]

\[ \leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_n} \sum_{\lambda=1}^{d_1} \| \mathcal{R}_{\lambda} \left( \frac{\partial a_{m,j}}{\partial x_k} T_m f \right) \|_{WMK_{p,1}^n(R^n)} \]

\[ \leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_n} \sum_{\lambda=1}^{d_1} \| \mathcal{R}_{\lambda} \left( a_{m,j} T_m f \right) \|_{WMK_{p,1}^n(R^n)} \]

\[ \leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_n} \sum_{\lambda=1}^{d_1} \| a_{m,j} \|_{L^\infty} \| T_m f \|_{WMK_{p,1}^n(R^n)} \]

By Lemma 13 and (74), a trivial computation shows that, for \( I_2, \)

\[ \| I_2 \|_{WMK_{p,1}^n(R^n)} \]

\[ \leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_n} \sum_{\lambda=1}^{d_1} \| a_{m,j} \|_{L^\infty} \| T_m f \|_{WMK_{p,1}^n(R^n)} \]

\[ \leq \sum_{m=1}^{\infty} \sum_{j=1}^{d_n} \sum_{\lambda=1}^{d_1} \| a_{m,j} \|_{L^\infty} \| T_m f \|_{WMK_{p,1}^n(R^n)} \]

Combining the estimates above, we arrive at the desired boundedness

\[ \|(TD - DT) f \|_{WMK_{p,1}^n(R^n)} \leq \| f \|_{MK_{p,1}^n(R^n)}. \]  
(77)

We posterior prove conclusion (2). Write \( D = \sum_{k=1}^{n} \mathcal{R}_k (\partial / \partial x_k); \) we have

\[ (T^4 - T^*) Df (x) = \sum_{m=1}^{\infty} \sum_{j=1}^{d_n} (-1)^m \left[ \bar{a}_{m,j} T_m f \right] Df (x) \]

\[ = \sum_{m=1}^{\infty} \sum_{j=1}^{d_n} \sum_{k=1}^{\infty} \sum_{\lambda=1}^{d_1} \left[ \bar{a}_{m,j} T_m f \right] \frac{\partial}{\partial x_k} \mathcal{R}_{\lambda} (\mathcal{R}_{\lambda} f) (x). \]  
(78)

We now turn to estimate the \( WMK_{p,1}^n(R^n) \) norm of \( \bar{a}_{m,j} T_m (\partial / \partial x_k) (\mathcal{R}_{\lambda} f). \) Applying (74), Lemma 13, and the fact that for any multi-index \( \beta \) and \( x \in R^n \setminus \{0\}, m = 1, 2, \ldots \) (see [7]),

\[ \left| \bar{a}^{\beta} (|x|^m) Y_{m,j} \right| \leq C(n) |x|^{m-1} m^{1/2}. \]  
(79)
Hence, we get
\[
\left\| \nabla a_{m,j} \right\|_{L^\infty} \leq \max_{|\beta| \leq 2} \left\| \partial^\beta Y_{m,j} \right\|_{L^{\infty}(\mathbb{R}^n)} \left\| R_k f \right\|_{WM^{s,\alpha}_p(\mathbb{R}^n)}
\]
\[
\leq m^{-2n} m^{n/2+1} \left\| f \right\|_{WM^{s,\alpha}_p(\mathbb{R}^n)}
\]
\[
\leq m^{-3n/2+1} \left\| f \right\|_{WM^{s,\alpha}_p(\mathbb{R}^n)}.
\]  

Combining the estimates of \((78)\) with \((80)\), we have
\[
\left\| \partial_x \right\|_{L^\infty} \leq m^{-2n}, \quad \left\| \nabla b_{\lambda,\mu} \right\|_{L^\infty} \leq \lambda^{-2n}.
\]

Write \(D = \sum_{k=1}^{n} (\partial / \partial x_k) R_k\), and it then follows that
\[
\left\| (T_1 \circ T_2 - T_1 T_2) Df \right\|_{WM^{s,\alpha}_p(\mathbb{R}^n)}
\]
\[
\leq \sum_{m=1}^{\infty} \sum_{k=1}^{n} \sum_{j=1}^{d_k} \sum_{\lambda=1}^{d_\lambda} \left\| a_{m,j} \right\|_{L^\infty} \cdot \left\| [b_{\lambda,\mu}, T_{m,j}] \left( \sum_{k=1}^{n} \frac{\partial}{\partial x_k} R_k T_{\lambda,\mu} f \right) \right\|_{WM^{s,\alpha}_p(\mathbb{R}^n)}
\]  

The above estimates, via Lemma 13, lead to that
\[
\left\| (T_1 \circ T_2 - T_1 T_2) Df \right\|_{WM^{s,\alpha}_p(\mathbb{R}^n)}
\]
\[
\leq \sum_{m=1}^{\infty} \sum_{k=1}^{n} \sum_{j=1}^{d_k} \sum_{\lambda=1}^{d_\lambda} \left\| a_{m,j} \right\|_{L^\infty} \left\| \nabla b_{\lambda,\mu} \right\|_{L^\infty} \cdot \max_{|\beta| \leq 2} \left\| \partial^\beta Y_{m,j} \right\|_{L^{\infty}(\mathbb{R}^n)} \left\| T_{\lambda,\mu} R_k f \right\|_{WM^{s,\alpha}_p(\mathbb{R}^n)}.
\]  

We thus obtain from \((79)\), \((83)\), and Lemma II that
\[
\left\| (T_1 \circ T_2 - T_1 T_2) Df \right\|_{WM^{s,\alpha}_p(\mathbb{R}^n)}
\]
\[
\leq \sum_{m=1}^{\infty} m^{n/2-1} m^{-2n} m^{n/2+1} \sum_{\lambda=1}^{\infty} \lambda^{n/2-1} \lambda^{-2n} \lambda^{n/2}
\]
\[
\leq \left\| f \right\|_{WM^{s,\alpha}_p(\mathbb{R}^n)}.
\]

Consequently, the proof of Theorem 5 is finished.


