Research Article

Jordan (α, β) -Derivations on Operator Algebras

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Received 20 March 2017; Accepted 30 April 2017; Published 1 August 2017

Academic Editor: Sergei Silvestrov

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Let $\mathcal{A}$ be a CSL subalgebra of a von Neumann algebra acting on a Hilbert space H . It is shown that any Jordan (α, β) -derivation on $\mathcal{A}$ is an (α, β) -derivation, where α, β are any automorphisms on $\mathcal{A}$. Moreover, the n th power (α, β) -maps on $\mathcal{A}$ are investigated.

1. Introduction and Preliminaries

Throughout the paper, let H be a complex Hilbert space. Denote by $B(H)$ the algebra of all bounded linear operators on H and by I the identity operator on H . The terms “projection” and “subspace” will mean “orthogonal projection” and “norm closed linear manifold.” For convenience, a subspace will be identified with the projection onto it. If M_α is a collection of subspaces of H , then the smallest subspace containing each M_α will be denoted by $\bigvee M_\alpha$ and the largest subspace contained in each M_α will be denoted by $\bigwedge M_\alpha$. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of subspaces (or projections) that is closed under the usual lattice operations $\bigvee$ and $\bigwedge$, containing 0 and I . If $\mathcal{L}$ is a subspace lattice, denote by $\text{Alg } \mathcal{L}$ the algebra of all bounded operators in $B(H)$ which leave every subspace in $\mathcal{L}$ invariant; that is, $\text{Alg } \mathcal{L} = \{A : A \in B(H) \text{ and } AP = PAP \text{ for all } P \in \mathcal{L}\}$. Dually, for a subalgebra $\mathcal{A}$ of $B(H)$, denote by $\text{Lat } \mathcal{A}$ the lattice of all closed subspaces left invariant under every operator in $\mathcal{A}$; that is, $\text{Lat } \mathcal{A} = \{P : P \text{ is a subspace of } H \text{ and } AP = PAP \text{ for all } A \in \mathcal{A}\}$. A subspace lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = \text{Lat Alg } \mathcal{L}$; an algebra $\mathcal{A}$ is reflexive if $\mathcal{A} = \text{Alg Lat } \mathcal{A}$. If all projections in a subspace lattice $\mathcal{L}$ commute pairwise, then the subspace lattice $\mathcal{L}$ is called a commutative subspace lattice, or a CSL for short, and $\text{Alg } \mathcal{L}$ is called a CSL algebra. If $\mathcal{L}$ is a CSL, whose projections are contained in a von Neumann algebra $\mathcal{N}$ acting on a Hilbert space H , then $\mathcal{A} =$

$\mathcal{N} \cap \text{Alg } \mathcal{L}$ is called a CSL subalgebra of the von Neumann algebra $\mathcal{N}$.

Let $\mathcal{A}$ be an algebra. A linear map $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is called a derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$. Let us review some generalizations of the notion of derivations. Suppose that α and β are two automorphisms on $\mathcal{A}$. Recall that a linear map $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is an (α, β) -derivation if $\delta(AB) = \delta(A)\beta(B) + \alpha(A)\delta(B)$ for all $A, B \in \mathcal{A}$; and $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is a Jordan (α, β) -derivation if $\delta(A^2) = \delta(A)\beta(A) + \alpha(A)\delta(A)$ for all $A \in \mathcal{A}$. For the identity isomorphism $\text{id}_{\mathcal{A}}$ and an automorphism σ on $\mathcal{A}$, we refer to a $(\sigma, \text{id}_{\mathcal{A}})$ -derivation as a σ -derivation and a Jordan $(\sigma, \text{id}_{\mathcal{A}})$ -derivation as a Jordan σ -derivation. Note that an $(\text{id}_{\mathcal{A}}, \text{id}_{\mathcal{A}})$ -derivation is just a derivation. A linear map δ on $\mathcal{A}$ is called a generalized (α, β) -derivation if $\delta(AB) = \delta(A)\beta(B) + \alpha(A)d(B)$ for all $A, B \in \mathcal{A}$, where d is an (α, β) -derivation on $\mathcal{A}$. The concept of generalized (α, β) -derivations covers both the concepts of (α, β) -derivations and of generalized derivations. The notions of (α, β) -derivation as a generalization of the concept of derivation have been studied in ([1–5] and references therein).

It is straightforward to show that if δ is an (α, β) -derivation of $\mathcal{A}$ and $n \geq 2$ is an integer, then $\delta(A^n) = \sum_{j=0}^{n-1} \alpha(A^j)\delta(A)\beta(A^{n-j-1})$ for any A in $\mathcal{A}$, where $\alpha(A^0)B = B = \beta\beta(A^0)$ for any element B in $\mathcal{A}$. This equation is called the n th power (α, β) -property. When $n = 2$, the n th power (α, β) -property makes δ a Jordan (α, β) -derivation. The natural

question is that if δ is a linear map on $\mathcal{A}$ and satisfies the n th power (α, β) -property, must δ be an (α, β) -derivation?

The purpose of the present article is to answer the above question if $\mathcal{A}$ is a CSL subalgebra of a von Neumann algebra acting on a Hilbert space. It can be proved that δ is a Jordan (α, β) -derivation if δ is a linear map on $\mathcal{A}$ and satisfies the n th power (α, β) -property. So we need to prove that a Jordan (α, β) -derivation on $\mathcal{A}$ is an (α, β) -derivation.

The characterization of Jordan (α, β) -derivations on algebras or rings is a subject in various areas. Clearly, any (α, β) -derivation is a Jordan (α, β) -derivation. Ashraf et al. [6] and Lanski [7] provided us with beautiful counterexamples to illustrate that the converse statements are not true in general. However, a classical result of Herstein ([8], Theorem 3.3) shows that a Jordan derivation of a 2-torsion free prime ring is a derivation. This result was extended to 2-torsion free semiprime rings [9]. Brešar and Vukman proved [10] that a Jordan (α, β) -derivation on a 2-torsion free prime ring is an (α, β) -derivation. Then it was proved [7] that a Jordan (α, β) -derivation on a 2-torsion free semiprime ring is an (α, β) -derivation. It was proved [11] that a Jordan derivation on a CSL algebra is a derivation. And it was extended [12] that a Jordan derivation on a CSL subalgebra of von Neumann algebra is a derivation.

Motivated by these results, we are concerned with a Jordan (α, β) -derivation and a Jordan σ -derivation on $\mathcal{A}$, a CSL subalgebra of a von Neumann algebra, which is not a semiprime ring.

The main results are as follows.

Theorem 1. *Let $\mathcal{N}$ be a von Neumann algebra acting on a Hilbert space H , let $\mathcal{L}$ be a CSL, whose projections are contained in $\mathcal{N}$, and let $\mathcal{A} = \mathcal{N} \cap \text{Alg } \mathcal{L}$ be the CSL subalgebra of the von Neumann algebra $\mathcal{N}$. Suppose that σ is any automorphism on $\mathcal{A}$; then a Jordan σ -derivation on $\mathcal{A}$ is a σ -derivation.*

Theorem 2. *Let $\mathcal{N}$ be a von Neumann algebra acting on a Hilbert space H , let $\mathcal{L}$ be a CSL, whose projections are contained in $\mathcal{N}$, and let $\mathcal{A} = \mathcal{N} \cap \text{Alg } \mathcal{L}$ be the CSL subalgebra of the von Neumann algebra $\mathcal{N}$. Suppose that α, β are any automorphisms on $\mathcal{A}$; then a Jordan (α, β) -derivation on $\mathcal{A}$ is an (α, β) -derivation.*

Then we extend the Jordan (α, β) -derivation case to the n th power (α, β) -property case for arbitrary $n \geq 2$, and we extend the derivation case to the generalized derivation case.

Corollary 3. *Let $n > 1$ be an integer, let $\mathcal{A} = \mathcal{N} \cap \text{Alg } \mathcal{L}$ be a CSL subalgebra of a von Neumann algebra $\mathcal{N}$ acting on a Hilbert space H , and let α, β be any automorphisms on $\mathcal{A}$. If δ is a linear map on $\mathcal{A}$ such that*

$$\delta(A^n) = \delta(A) \beta(A^{n-1}) + \sum_{j=1}^{n-1} \alpha(A^j) \delta(A) \beta(A^{n-j-1}) \quad (1)$$

for any $A \in \mathcal{A}$, where $A^0 = I$, then δ is an (α, β) -derivation.

Corollary 4. *Let $n > 1$ be an integer, let $\mathcal{A} = \mathcal{N} \cap \text{Alg } \mathcal{L}$ be a CSL subalgebra of a von Neumann algebra $\mathcal{N}$ acting on a Hilbert space H , and let α, β be any automorphisms on $\mathcal{A}$. If δ and d are two linear maps on $\mathcal{A}$ such that*

$$\delta(A^n) = \delta(A) \beta(A^{n-1}) + \sum_{j=1}^{n-1} \alpha(A^j) d(A) \beta(A^{n-j-1}) \quad (2)$$

for any $A \in \mathcal{A}$, where $A^0 = I$, then d is an (α, β) -derivation and δ is a generalized (α, β) -derivation on $\mathcal{A}$ with respect to d .

2. Proof of Theorems 1 and 2

Throughout the paper, we suppose that $\mathcal{N}$ is a von Neumann algebra acting on a Hilbert space H , $\mathcal{L}$ is a CSL whose projections are contained in $\mathcal{N}$, and $\mathcal{A} = \mathcal{N} \cap \text{Alg } \mathcal{L}$ is the CSL subalgebra of the von Neumann algebra $\mathcal{N}$.

Lemma 5 (see [12]). *Suppose that $\mathcal{A} = \mathcal{N} \cap \text{Alg } \mathcal{L}$ is a CSL subalgebra of the von Neumann algebra $\mathcal{N}$. Denote $Q_1(H)$, or Q_1 simply, by the orthogonal projection onto the linear span of the set $\{PAP^\perp x : P \in \mathcal{L}, A \in \mathcal{A}, x \in H\}$ and $Q_2(H)$, or Q_2 simply, by the orthogonal projection onto the linear span of the set $\{P^\perp A^* P x : P \in \mathcal{L}, A \in \mathcal{A}, x \in H\}$, and $Q = Q_1(H) \vee Q_2(H)$. Then, one has the following:*

- (i) $Q, Q_1 \in \text{Lat } \mathcal{A}$. And $Q_1, Q_2, Q \in \mathcal{L}' \cap \mathcal{N} \subseteq \mathcal{A}$, where $\mathcal{L}'$ is the commutant of $\mathcal{L}$. Q_1 commutes with Q_2 . Furthermore, $Q^\perp A Q = Q A Q^\perp = 0$ for any $A \in \mathcal{A}$, so that $\mathcal{A} = Q \mathcal{A} Q \oplus Q^\perp \mathcal{A} Q^\perp$.
- (ii) If $Q \neq I$, then $Q^\perp \mathcal{A} Q^\perp$ is a von Neumann algebra on $Q^\perp H$.

Proposition 6. *Let $\mathcal{A} = \mathcal{N} \cap \text{Alg } \mathcal{L}$ be a CSL subalgebra of a von Neumann algebra $\mathcal{N}$. Suppose that σ is an automorphism on $\mathcal{A}$ with the property that, for any projection $P \in \mathcal{A}$, there are $A_P, B_P \in \mathcal{A}$, depending on P , such that $\sigma(P) = A_P P B_P$. In such case, a Jordan σ -derivation on $\mathcal{A}$ is a σ -derivation.*

Proof. Let $\mathcal{A}$ be a CSL subalgebra of a von Neumann algebra $\mathcal{N}$. Suppose that δ is a Jordan σ -derivation on $\mathcal{A}$. For any A, B in $\mathcal{A}$,

$$\delta(A^2) = \delta(A) A + \sigma(A) \delta(A). \quad (3)$$

Since $AB + BA = (A + B)^2 - A^2 - B^2$,

$$\begin{aligned} \delta(AB + BA) &= \delta(A) B + \sigma(A) \delta(B) + \delta(B) A \\ &\quad + \sigma(B) \delta(A). \end{aligned} \quad (4)$$

Replacing B by $AB + BA$ in (4), we have that

$$\delta(ABA) = \delta(A) BA + \sigma(A) \delta(B) A + \sigma(AB) \delta(A). \quad (5)$$

In the sequel of proof, we choose an arbitrary nontrivial projection P in the set $(\mathcal{L} \cup \{Q, Q_1\})$. Let $P^\perp = I - P$. Then $P^\perp A P = 0$ and $A = PAP + PAP^\perp + P^\perp A P^\perp$ for any $A \in \mathcal{A}$.

From $\delta(P) = \delta(P)P + \sigma(P)\delta(P)$, we have that $\sigma(P)\delta(P)P = \sigma(P^\perp)\delta(P)P^\perp = 0$. By the condition of σ ,

$$\begin{aligned}\delta(P) &= \sigma(P)\delta(P)P^\perp + \sigma(P^\perp)\delta(P)P \\ &= \sigma(P)\delta(P)P^\perp.\end{aligned}\quad (6)$$

For any $A \in \mathcal{A}$, by (4), we have that

$$\begin{aligned}\delta(PAP^\perp) &= \delta(PPAP^\perp + PAP^\perp P) \\ &= \delta(P)PAP^\perp + \sigma(P)\delta(PAP^\perp) \\ &\quad + \delta(PAP^\perp)P + \sigma(PAP^\perp)\delta(P).\end{aligned}\quad (7)$$

Since $\delta(P)PAP^\perp = \sigma(P)\delta(P)P^\perp PAP^\perp = 0$ and $\sigma(PAP^\perp)\delta(P) = 0$,

$$\delta(PAP^\perp) = \sigma(P)\delta(PAP^\perp) + \delta(PAP^\perp)P. \quad (8)$$

It follows that $\sigma(P)\delta(PAP^\perp)P = \sigma(P^\perp)\delta(PAP^\perp)P^\perp = 0$. Since $\sigma(P^\perp)\delta(PAP^\perp)P = 0$ by the condition of σ ,

$$\delta(PAP^\perp) = \sigma(P)\delta(PAP^\perp)P^\perp. \quad (9)$$

By the fact that

$$\begin{aligned}2\delta(PAP) &= \delta(P(PAP) + (PAP)P) \\ &= \delta(P)PAP + \sigma(P)\delta(PAP) + \delta(PAP)P \\ &\quad + \sigma(PAP)\delta(P),\end{aligned}\quad (10)$$

we get that

$$\sigma(P^\perp)\delta(PAP) = 0. \quad (11)$$

Similarly,

$$\begin{aligned}2\delta(P^\perp AP^\perp) &= \delta(P^\perp(P^\perp AP^\perp) + (P^\perp AP^\perp)P^\perp) \\ &= \delta(P^\perp)P^\perp AP^\perp + \sigma(P^\perp)\delta(P^\perp AP^\perp) \\ &\quad + \delta(P^\perp AP^\perp)P^\perp \\ &\quad + \sigma(P^\perp AP^\perp)\delta(P^\perp).\end{aligned}\quad (12)$$

It follows that

$$\delta(P^\perp AP^\perp)P = 0. \quad (13)$$

By (3), $\delta(I) = 0$ and $\delta(P^\perp) = -\delta(P) = -\sigma(P)\delta(P)P^\perp$. Let $A, B \in \mathcal{A}$. Then by (11) and (13), we have that

$$\begin{aligned}\delta(APBP^\perp) &= \delta((PAP)(PBP^\perp) + (PBP^\perp)(PAP)) \\ &= \delta(PAP)PBP^\perp + \sigma(PAP)\delta(PBP^\perp) \\ &\quad + \delta(PBP^\perp)PAP + \sigma(PBP^\perp)\delta(PAP) \\ &= \delta(PAP)PBP^\perp + \sigma(PAP)\delta(PBP^\perp) \\ &= \delta(A)PBP^\perp + \sigma(A)\delta(PBP^\perp).\end{aligned}\quad (14)$$

Let $A, B, T \in \mathcal{A}$, $P \in \mathcal{L}$. Then we have that

$$\begin{aligned}\delta(ABPTP^\perp) &= \delta(AB)PTP^\perp + \sigma(AB)\delta(PTP^\perp), \\ \delta(ABPTP^\perp) &= \delta(APBPTP^\perp) \\ &= \delta(A)PBPTP^\perp + \sigma(A)\delta(PBPTP^\perp) \\ &= \delta(A)BPTP^\perp + \sigma(A)\delta(BPTP^\perp) \\ &= \delta(A)BPTP^\perp \\ &\quad + \sigma(A)[\delta(B)PTP^\perp + \sigma(B)\delta(PTP^\perp)].\end{aligned}\quad (15)$$

It follows that $(\delta(AB) - (\delta(A)B + \sigma(A)\delta(B)))PTP^\perp = 0$. The arbitrariness of T and P gives that

$$(\delta(AB) - (\delta(A)B + \sigma(A)\delta(B)))Q_1 = 0. \quad (16)$$

Similarly, for any $A, B \in \mathcal{A}$, we have that

$$\begin{aligned}\delta(PAP^\perp B) &= \delta((PAP^\perp)(P^\perp BP^\perp) + (P^\perp BP^\perp)(PAP^\perp)) \\ &= \delta(PAP^\perp)P^\perp BP^\perp + \sigma(PAP^\perp)\delta(P^\perp BP^\perp) \\ &\quad + \delta(P^\perp BP^\perp)PAP^\perp + \sigma(P^\perp BP^\perp)\delta(PAP^\perp) \\ &= \delta(PAP^\perp)P^\perp BP^\perp + \sigma(PAP^\perp)\delta(P^\perp BP^\perp) \\ &= \delta(PAP^\perp)B + \sigma(PAP^\perp)\delta(B).\end{aligned}\quad (17)$$

Let $A, B, T \in \mathcal{A}$, $P \in \mathcal{L}$. Then we have that

$$\begin{aligned}\delta(PTP^\perp AB) &= \delta(PTP^\perp)AB + \sigma(PTP^\perp)\delta(AB), \\ \delta(PTP^\perp AB) &= \delta(PTP^\perp AP^\perp B) \\ &= \delta(PTP^\perp AP^\perp)B \\ &\quad + \sigma(PTP^\perp AP^\perp)\delta(B) \\ &= \delta(PTP^\perp A)B + \sigma(PTP^\perp A)\delta(B) \\ &= [\delta(PTP^\perp)A + \sigma(PTP^\perp)\delta(A)]B \\ &\quad + \sigma(PTP^\perp)\sigma(A)\delta(B).\end{aligned}\quad (18)$$

It follows that

$$\sigma(PTP^\perp)(\delta(AB) - (\delta(A)B + \sigma(A)\delta(B))) = 0, \quad (19)$$

$$PTP^\perp \cdot \sigma^{-1}(\delta(AB) - (\delta(A)B + \sigma(A)\delta(B))) = 0.$$

So

$$\begin{aligned}(\sigma^{-1}(\delta(AB) - (\delta(A)B + \sigma(A)\delta(B))))^* \cdot P^\perp T^* P \\ = 0\end{aligned}\quad (20)$$

by involution. The arbitrariness of T and P gives that

$$(\sigma^{-1}(\delta(AB) - (\delta(A)B + \sigma(A)\delta(B))))^* \cdot Q_2 = 0, \quad (21)$$

$$\sigma(Q_2)(\delta(AB) - (\delta(A)B + \sigma(A)\delta(B))) = 0. \quad (22)$$

We now define a new linear map $\Delta : \mathcal{A} \rightarrow \mathcal{A}$ by

$$\Delta(A) = \delta(A) - (\sigma(A)\delta(Q_1) - \delta(Q_1)A) \quad (23)$$

$(A \in \mathcal{A}).$

Then Δ is also a Jordan σ -derivation. Since $\delta(Q_1) = \sigma(Q_1)\delta(Q_1)Q_1^\perp$ by (6), it follows that $\Delta(Q_1) = 0$ and $\Delta(Q_1^\perp) = \Delta(I - Q_1) = 0$. Note that $TQ_1 = Q_1TQ_1$ and $Q_1^\perp T = Q_1^\perp TQ_1^\perp$ for each $T \in \mathcal{A}$. Then

$$\begin{aligned} \Delta(TQ_1) &= \Delta(Q_1TQ_1) \\ &= \Delta(Q_1)TQ_1 + \sigma(Q_1)\Delta(T)Q_1 \\ &\quad + \sigma(Q_1T)\Delta(Q_1) = \sigma(Q_1)\Delta(T)Q_1 \quad (24) \\ &= \sigma(I)\Delta(T)Q_1 - \sigma(Q_1^\perp)\Delta(T)Q_1 \\ &= \sigma(I)\Delta(T)Q_1 = \Delta(T)Q_1, \end{aligned}$$

$$\begin{aligned} \Delta(Q_1^\perp T) &= \Delta(Q_1^\perp TQ_1^\perp) \\ &= \Delta(Q_1^\perp)TQ_1^\perp + \sigma(Q_1^\perp)\Delta(T)Q_1^\perp \\ &\quad + \sigma(Q_1^\perp T)\Delta(Q_1^\perp) = \sigma(Q_1^\perp)\Delta(T)Q_1^\perp \quad (25) \\ &= \sigma(Q_1^\perp)\Delta(T) - \sigma(Q_1^\perp)\Delta(T)Q_1 \\ &= \sigma(Q_1^\perp)\Delta(T). \end{aligned}$$

Claim. If $Q_1 \vee Q_2 = I$, then δ is a σ -derivation.

For this, it suffices obviously to prove that Δ is a σ -derivation. In fact, let $A, B \in \mathcal{A}$. Then equalities (16) and (22) hold if δ is replaced by Δ . By (24), we have that

$$(\Delta(ABQ_1) - (\Delta(A)BQ_1 + \sigma(A)\Delta(BQ_1)))Q_1^\perp = 0. \quad (26)$$

By (16) and (26), we have that

$$\Delta(ABQ_1) = \Delta(A)BQ_1 + \sigma(A)\Delta(BQ_1). \quad (27)$$

By (22),

$$\begin{aligned} &\sigma(Q_2)(\Delta(Q_1^\perp AB) - (\Delta(Q_1^\perp A)B + \sigma(Q_1^\perp A)\Delta(B))) \\ &= 0. \end{aligned} \quad (28)$$

Since $Q_1 \vee Q_2 = I$, $\sigma(Q_1^\perp) = \sigma(Q_1^\perp Q_2)$ and

$$\begin{aligned} &\sigma(Q_1^\perp)(\Delta(Q_1^\perp AB) - (\Delta(Q_1^\perp A)B + \sigma(Q_1^\perp A)\Delta(B))) \\ &= 0. \end{aligned} \quad (29)$$

By (25), we get that

$$\begin{aligned} &\sigma(Q_1)(\Delta(Q_1^\perp AB) - (\Delta(Q_1^\perp A)B + \sigma(Q_1^\perp A)\Delta(B))) \\ &= 0. \end{aligned} \quad (30)$$

It follows that

$$\Delta(Q_1^\perp AB) = \Delta(Q_1^\perp A)B + \sigma(Q_1^\perp A)\Delta(B). \quad (31)$$

Equation (14) gives that

$$\Delta(AQ_1BQ_1^\perp) = \Delta(A)Q_1BQ_1^\perp + \sigma(A)\Delta(Q_1BQ_1^\perp). \quad (32)$$

By (17), we have that

$$\Delta(Q_1AQ_1^\perp B) = \Delta(Q_1AQ_1^\perp)B + \sigma(Q_1AQ_1^\perp)\Delta(B). \quad (33)$$

By (24), (25), (26), (27), (31), (32), and (33),

$$\begin{aligned} \Delta(AB) &= \Delta(ABQ_1 + AQ_1BQ_1^\perp + Q_1AQ_1^\perp B + Q_1^\perp AB) \\ &= \Delta(A)BQ_1 + \sigma(A)\Delta(BQ_1) \\ &\quad + \Delta(A)Q_1BQ_1^\perp + \sigma(A)\Delta(Q_1BQ_1^\perp) \\ &\quad + \Delta(Q_1AQ_1^\perp)B + \sigma(Q_1AQ_1^\perp)\Delta(B) \\ &\quad + \Delta(Q_1^\perp A)B + \sigma(Q_1^\perp A)\Delta(B) \\ &= \Delta(A)(BQ_1 + Q_1BQ_1^\perp) \\ &\quad + \sigma(A)(\Delta(BQ_1) + \Delta(Q_1BQ_1^\perp)) \\ &\quad + (\Delta(Q_1AQ_1^\perp) + \Delta(Q_1^\perp A))B \\ &\quad + (\sigma(Q_1AQ_1^\perp) + \sigma(Q_1^\perp A))\Delta(B) \quad (34) \\ &= \Delta(A)(Q_1BQ_1 + Q_1BQ_1^\perp) \\ &\quad + \sigma(A)(\Delta(Q_1BQ_1) + \Delta(Q_1BQ_1^\perp)) \\ &\quad + (\Delta(Q_1AQ_1^\perp) + \Delta(Q_1^\perp AQ_1^\perp))B \\ &\quad + (\sigma(Q_1AQ_1^\perp) + \sigma(Q_1^\perp AQ_1^\perp))\Delta(B) \\ &= \Delta(A)(Q_1B) + \sigma(A)\Delta(Q_1B) + \Delta(AQ_1^\perp)B \\ &\quad + \sigma(AQ_1^\perp)\Delta(B) \\ &= \Delta(AQ_1)B + \sigma(A)\Delta(Q_1B) + \Delta(AQ_1^\perp)B \\ &\quad + \sigma(A)\Delta(Q_1^\perp B) = \Delta(A)B + \sigma(A)\Delta(B). \end{aligned}$$

It follows that Δ is a σ -derivation, and so is δ . The proof of *Claim* is complete.

For the general case, let $Q = Q_1(\mathcal{L}) \vee Q_2(\mathcal{L})$. By Lemma 5, the algebra $\mathcal{A}$ can be written as the direct sum $\mathcal{A} = Q\mathcal{A}Q \oplus Q^\perp\mathcal{A}Q^\perp$. We now define a new linear map $\Delta_1 : \mathcal{A} \rightarrow \mathcal{A}$ by

$$\Delta_1(A) = \delta(A) - (\sigma(A)\delta(Q) - \delta(Q)A) \quad (A \in \mathcal{A}). \quad (35)$$

Then Δ_1 is a Jordan σ -derivation and $\Delta_1(Q) = \Delta_1(Q^\perp) = 0$. For any $A \in \mathcal{A}$, by (5) and the assumption of σ ,

$$\begin{aligned} \Delta_1(QAQ) &= \sigma(Q)\Delta_1(A)Q = A_QQB_Q\Delta_1(A)Q \\ &= Q(A_QQB_Q\Delta_1(A))Q \in Q\mathcal{A}Q, \\ \Delta_1(Q^\perp AQ^\perp) &= \Delta_1(Q^\perp(Q^\perp AQ^\perp)Q^\perp) \\ &= \sigma(Q^\perp)\Delta_1(Q^\perp AQ^\perp)Q^\perp. \end{aligned} \quad (36)$$

Then $\Delta_1(Q^\perp A Q^\perp)$ is of form $\Delta_1(Q^\perp A Q^\perp) = Q A_1 Q^\perp + Q^\perp A_2 Q^\perp$. By Lemma 5, $Q A_1 Q^\perp = 0$. It follows that

$$\Delta_1(Q^\perp A Q^\perp) = Q^\perp A_2 Q^\perp \in Q^\perp \mathcal{A} Q^\perp \quad (37)$$

for any $A \in \mathcal{A}$. Then Δ_1 maps $Q\mathcal{A}Q$, $Q^\perp \mathcal{A} Q^\perp$ onto themselves, respectively. So Δ_1 can be written as the direct sum $\Delta_{11} \oplus \Delta_{12}$, where Δ_{11}, Δ_{12} are the restrictions of Δ_1 on $Q\mathcal{A}Q$, $Q^\perp \mathcal{A} Q^\perp$, respectively. And Δ_{11}, Δ_{12} are both Jordan σ -derivations. Since

$$\begin{aligned} Q\mathcal{A}Q &= \{T \in Q\mathcal{N}Q : (Q - QP)TQP = 0 \text{ for any } P \in \mathcal{L}\} \quad (38) \\ &= Q\mathcal{N}Q \cap \text{Alg}(Q\mathcal{L}), \end{aligned}$$

we have that $Q\mathcal{A}Q$ is a CSL subalgebra of the von Neumann algebra $Q\mathcal{N}Q$. For any $P \in \mathcal{L}$, $A \in \mathcal{A}$, and $x \in H$, we have that $QAQ^\perp = 0$ and

$$\begin{aligned} PAP^\perp x &= QPAP^\perp x = PQAP^\perp x = PQA(Q - QP)x \\ &= QPA(Q - QP)x. \end{aligned} \quad (39)$$

Since

$$\begin{aligned} Q_1(\mathcal{L}) &= \overline{\text{span}}\{PAP^\perp x : P \in \mathcal{L}, A \in \mathcal{A}, x \in H\}, \\ Q_1(Q\mathcal{L}) &= \overline{\text{span}}\{QPA(Q - QP)x : P \in \mathcal{L}, A \in \mathcal{A}, x \in H\}, \end{aligned} \quad (40)$$

we have that $Q_1(\mathcal{L}) = Q_1(Q\mathcal{L})$ and similarly $Q_2(\mathcal{L}) = Q_2(Q\mathcal{L})$. It follows that $Q_1(Q\mathcal{L}) \vee Q_2(Q\mathcal{L}) = Q$ is the identity element of $Q\mathcal{A}Q$. By the Claim, we have that Δ_{11} is a σ -derivation. Notice that $Q^\perp \mathcal{A} Q^\perp$ is von Neumann algebra and von Neumann algebra is a semiprime ring. By Theorem 2 of [7], we know that Δ_{12} is also a σ -derivation. Consequently, Δ_1 is a σ -derivation and so is δ , as required. $\square$

Lemma 7. Suppose that $\mathcal{A} = \mathcal{N} \cap \text{Alg } \mathcal{L}$ is CSL subalgebra of the von Neumann algebra $\mathcal{N}$. Denote $Q_1(H)$, or Q_1 simply, by the orthogonal projection onto the linear span of the set $\{PAP^\perp x : P \in \mathcal{L}, A \in \mathcal{A}, x \in H\}$; $Q_2(H)$, or Q_2 simply, by the orthogonal projection onto the linear span of the set $\{P^\perp A^* P x : P \in \mathcal{L}, A \in \mathcal{A}, x \in H\}$, and $Q = Q_1(H) \vee Q_2(H)$. Then $Q, Q_1 \in \mathcal{L}''$, the double commutant of $\mathcal{L}$.

Proof. Let $B \in \mathcal{L}'$.

Since $\mathcal{L}' = \text{Alg } \mathcal{L} \cap (\text{Alg } \mathcal{L})^*$, we have that $BQ_1 = Q_1 BQ_1$ and $B^* Q_1 = Q_1 B^* Q_1$. It follows that $BQ_1 = Q_1 B$. Hence $Q_1 \in \mathcal{L}''$.

Similarly, $B^* P^\perp A^* P x = P^\perp B^* P^\perp A^* P x + P B^* P^\perp A^* P x = P^\perp B^* P^\perp A^* P x$ for any $P \in \mathcal{L}, A \in \mathcal{A}, x \in H$. It follows that $B^* Q_2 = Q_2 B^* Q_2$ and $Q_2 B = Q_2 BQ_2$ by involution. Noting that $BQ_2 = Q_2 BQ_2$. So $BQ_2 = Q_2 B$. Hence $Q_2 \in \mathcal{L}''$.

It follows that $Q = Q_1 + Q_2 - Q_1 Q_2 \in \mathcal{L}''$. $\square$

Lemma 8 (Theorem 2.1, [13]). Let $\mathcal{L}_1, \mathcal{L}_2$ be commutative subspace lattices on Hilbert spaces H_1 and H_2 , respectively, and $\phi : \text{Alg } \mathcal{L}_1 \rightarrow \text{Alg } \mathcal{L}_2$ be an isomorphism. Let $\mathcal{M}$ be a

masa (maximal abelian self-adjoint subalgebra) contained in $\text{Alg } \mathcal{L}_1$. Then there is a bounded invertible operator $Y : H_1 \rightarrow H_2$ and an automorphism $\rho : \text{Alg } \mathcal{L}_1 \rightarrow \text{Alg } \mathcal{L}_1$ such that

- (i) $\rho(M) = M$ for all $M \in \mathcal{M}$.
- (ii) $\phi(A) = Y\rho(A)Y^{-1}$ for any $A \in \text{Alg } \mathcal{L}_1$.

Remark 9 (Remark 1, [13]). Lemma 8 holds for an isomorphism $\phi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are assumed to contain masas.

Proof of Theorem 1. Suppose that δ is a Jordan σ -derivation on $\mathcal{A} = \mathcal{N} \cap \text{Alg } \mathcal{L}$. We first note that any CSL subalgebra of a von Neumann algebra contains a masa $\mathcal{M}$ which contains $\mathcal{L}''$, the double commutant of $\mathcal{L}$.

In fact, for a CSL $\mathcal{L}$, $\mathcal{L}' = \text{Alg } \mathcal{L} \cap (\text{Alg } \mathcal{L})^*$, which gives that $\mathcal{L}'$ is a von Neumann algebra. It follows that $\mathcal{L}' \cap \mathcal{L}''$ is an abelian von Neumann algebra. Since $\mathcal{L}'' \subseteq \mathcal{L}'$, $\mathcal{L}'' = \mathcal{L}' \cap \mathcal{L}''$ is an abelian von Neumann algebra.

As in the proof of Proposition 6, we can choose an arbitrary nontrivial projection P in $(\mathcal{L} \cup \{Q, Q_1\})$. Lemma 7 gives that $Q, Q_1 \in \mathcal{L}'' \subseteq \mathcal{M}$, the masa (maximal abelian self-adjoint subalgebra). Lemma 8 with Remark 9 following it gives that the automorphism σ and the projection P meet the property of Proposition 6. Using the proof of Proposition 6, we can similarly prove that a Jordan σ -derivation on $\mathcal{A}$ is a σ -derivation. $\square$

Proof of Theorem 2. Suppose that δ is a Jordan (α, β) -derivation on $\mathcal{A} = \mathcal{N} \cap \text{Alg } \mathcal{L}$. For any $A \in \mathcal{A}$,

$$\begin{aligned} \delta(A^2) &= \alpha(A)\delta(A) + \delta(A)\beta(A), \\ (\beta^{-1} \circ \delta)(A^2) &= (\beta^{-1} \circ \alpha)(A)(\beta^{-1} \circ \delta)(A) \\ &\quad + (\beta^{-1} \circ \delta)(A)(\beta^{-1} \circ \beta)(A). \end{aligned} \quad (41)$$

Let $\sigma = \beta^{-1} \circ \alpha$ and $\Delta = \beta^{-1} \circ \delta$; then the above equality gives that Δ is a Jordan σ -derivation. It follows that Δ is a σ -derivation by Theorem 1. So

$$\begin{aligned} (\beta^{-1} \circ \delta)(AB) &= (\beta^{-1} \circ \alpha)(A)(\beta^{-1} \circ \delta)(B) \\ &\quad + (\beta^{-1} \circ \delta)(A)(\beta^{-1} \circ \beta)(B), \quad (42) \\ \delta(AB) &= \alpha(A)\delta(B) + \delta(A)\beta(B) \end{aligned}$$

for any $A, B \in \mathcal{A}$, which gives that δ is an (α, β) -derivation on $\mathcal{A}$. $\square$

Example 10. Let $\mathcal{A}$ be CSL subalgebra of von Neumann algebra. If there is an invertible operator $Y \in \mathcal{A}$ such that Y is not the identity operator I on H . Consider the mapping σ on $\mathcal{A}$ by $\sigma(A) = Y^{-1}AY$. Then σ is an automorphism on $\mathcal{A}$ and σ meets the property in Proposition 6. Let $\delta = \sigma - \text{id}_{\mathcal{A}}$, where $\text{id}_{\mathcal{A}}$ is the identity function on $\mathcal{A}$. It is easy to see that δ is a σ -derivation but not a derivation on $\mathcal{A}$.

For example, let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis for $\mathbb{C}^4$. We use the notation [124] for the span of the first, second,

and fourth of these vectors and so on. $\mathcal{A}_4$ is the collection of 4×4 matrices of the form

$$\begin{pmatrix} a & 0 & b & c \\ 0 & d & e & f \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & h \end{pmatrix}, \quad (43)$$

where $a, b, \dots, h$ are complex numbers. The algebra is reflexive and its lattice consists of the subspaces $\{0, [1], [2], [12], [123], [124], [1234]\}$. $\mathcal{A}_4$ is a CSL subalgebra of a von Neumann algebra. Let $Y = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then Y is invertible and $Y^{-1} = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Consider the mapping σ on $\mathcal{A}_4$ by $\sigma(A) = Y^{-1}AY$; that is,

$$\begin{aligned} \sigma : \begin{pmatrix} a & 0 & b & c \\ 0 & d & e & f \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & h \end{pmatrix} \\ \mapsto \begin{pmatrix} a & 0 & a+b-g & a+c-h \\ 0 & d & d+e-g & d+f-h \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & h \end{pmatrix}. \end{aligned} \quad (44)$$

It is obvious that σ is an automorphism on $\mathcal{A}_4$ and σ meets the property in Proposition 6. Define a mapping δ on $\mathcal{A}_4$ by

$$\begin{aligned} \delta : \begin{pmatrix} a & 0 & b & c \\ 0 & d & e & f \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & h \end{pmatrix} \\ \mapsto \begin{pmatrix} 0 & 0 & a-g+b & a-h+c \\ 0 & 0 & d-g+e & d-h+f \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (45)$$

It is easy to see that δ is a σ -derivation but not a derivation on $\mathcal{A}_4$.

It shows that Proposition 6 and Theorems 1 and 2 in the present paper generalize Theorem 1.2 of [12].

3. Generalized n th Power (α, β) -Maps on CSL Subalgebras of Von Neumann Algebras

The following proposition is essentially due to [7]. For the sake of completeness, we outline the proof.

Proposition 11. Let $n > 1$ be an integer, let $\mathcal{A}$ be a unital operator algebra on a Hilbert space H with unit I , and let α, β be automorphisms on $\mathcal{A}$. If δ is a linear map on $\mathcal{A}$ such that

$$\begin{aligned} \delta(A^n) &= \delta(A) \beta(A^{n-1}) \\ &+ \sum_{j=1}^{n-1} \alpha(A^j) \delta(A) \beta(A^{n-j-1}) \end{aligned} \quad (46)$$

for any $A \in \mathcal{A}$, where $A^0 = I$, then δ is a Jordan (α, β) -derivation.

Proof. When $n = 2$, our assumption is $\delta(A^2) = \delta(A)\beta(A) + \alpha(A)\delta(A)$, and we reduce the general situation to this case. Substituting I for A in our assumed relation (46) gives that $\delta(I) = 0$. Next, replace A with $A + zI$ for any integer z . The result of this substitution, using the additivity of δ and $\delta(I) = 0$, is

$$\begin{aligned} \delta((A + zI)^n) \\ = \delta(A + zI) \beta((A + zI)^{n-1}) \\ + \sum_{j=1}^{n-1} \alpha((A + zI)^j) \delta(A + zI) \beta((A + zI)^{n-j-1}). \end{aligned} \quad (47)$$

Expanding (47), now using the additivity of α and β , we collect terms containing z^k to write $\sum_{k=0}^n z^k P_k(A) = 0$. Note that $P_0(A)$ is the relation assumed in the proposition, so it is zero, which leads to

$$\sum_{k=1}^n z^k P_k(A) = 0. \quad (48)$$

We replace z by $1, 2, \dots, n$ in turn in equality (48). Expressing the resulting system of n homogeneous equations, we see that the coefficient matrix of the system is a Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^n \\ \dots & \dots & \dots & \dots \\ n & n^2 & \dots & n^n \end{pmatrix}. \quad (49)$$

Since the determinant of the matrix is different from zero, it follows that the system has only trivial solution, so each $P_k(A) = 0$. In particular, $P_{n-2}(A) = 0$. We get that

$$\begin{aligned} \binom{n}{n-2} \delta(A^2) &= \binom{n-1}{n-2} \delta(A) \beta(A) + \alpha(A) \delta(A) \\ &+ \binom{n-1}{n-2} \delta(A) \beta(A) \\ &+ \binom{2}{1} \alpha(A) \delta(A) \end{aligned}$$

$$\begin{aligned}
& + \binom{n-2}{n-3} \delta(A) \beta(A) \\
& + \binom{3}{2} \alpha(A) \delta(A) \\
& + \binom{n-3}{n-4} \delta(A) \beta(A) + \cdots \\
& + \binom{n-1}{n-2} \alpha(A) \delta(A) \\
& + \delta(A) \beta(A) \\
& + \binom{n-1}{n-2} \alpha(A) \delta(A).
\end{aligned} \tag{50}$$

It follows that $\delta(A^2) = \delta(A)\beta(A) + \alpha(A)\delta(A)$. The proof is complete. $\square$

The proof of Corollary 3 is as follows.

Proof. It follows from Proposition 11 and Theorem 2. $\square$

The proof of Corollary 4 is as follows.

Proof. It is similar to the proof of Theorem 3 in [7]. $\square$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

Part of the work was done during the first author's visit to Queen's University, Canada. The first author gratefully acknowledges the kind hospitality from the institute, especially Professor Jamie Mingo. The authors would like to thank Professor Jamie Mingo for his many very helpful comments and suggestions that helped to improve the presentation of this paper in the preparation of this paper. This work was supported by the National Nature Science Foundation of China (nos. 11401273 and 11371279), the Natural Science Research Foundation of Jiangxi Province Education Department (no. GJJ160915), and the Special Fund Project of the Young and Middle-Aged Teachers' Development Plan of the Ordinary Undergraduate Colleges and Universities in Jiangxi Province.

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