Research Article

On Topological Properties of Metrics Defined via Generalized “Linking Construction”

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We analyze topological properties of metric spaces obtained by using Szász's construction, which we used to call generalized “linking construction.” In particular, we provide necessary and sufficient conditions for completeness of metric spaces obtained in this way. Moreover, we examine the relation between Szász's construction and the “linking construction.” A particular attention is drawn to the “floor” metric, the analysis of which provides some interesting observations.

1. Introduction

In the paper [1] the authors described a construction of a large class of hyperconvex metric spaces (for an introduction to the theory of hyperconvex metric spaces we refer the reader to [2] or [3]). This construction is based on using Chebyshev subsets of a normed space endowed with a hyperconvex metric. In particular, it contains two classical hyperconvex metric spaces, $\mathbb{R}^2$ endowed with the “river” metric or with the radial metric, as special cases (cf. [4]). Let us also add that the theory of hyperconvex metric spaces is closely connected with the theory of $\mathbb{R}$ -trees (for more information about $\mathbb{R}$ -trees the reader can see, e.g., [5–7]).

On the other hand, in paper [8] the authors showed a general “linking construction” yielding hyperconvex spaces. That construction also encompasses the “river” metric and the radial metric. Let us add that a similar concept appears in [9], where it was used to study existence of certain mappings between Banach spaces. A slight generalization of the “linking construction” along with several examples can be found in the paper [10].

An interesting generalization of the two constructions mentioned above was proposed by Szász in the paper [11]. In that construction, instead of a metric projection and a collinearity relation, Szász used an arbitrary function and a suitable relation defined on a set under consideration.

The main goal of this note is to examine some topological properties of metric spaces obtained by using Szász's construction. The main emphasis will be put on completeness of metric spaces produced by Szász's construction. In particular, in Section 4 we provide necessary and sufficient conditions for completeness of those metric spaces while in Section 5 we deal with their boundedness.

In Section 6, completing this note, we concentrate on one particular case of Szász's construction, namely, on the so-called “floor” metric, defined on the set of real numbers $\mathbb{R}$. We provide some interesting observations concerning that metric. It turns out, for example, that balls in that metric space need not be connected nor compact. We will give formulae which allow calculating the Kuratowski measure of noncompactness of bounded subsets included in that metric space. Let us emphasize that in general it is not easy to find concrete formulae which would allow calculating the Kuratowski measure of noncompactness. Such formulae for bounded subsets of $\mathbb{R}^2$ endowed with the “river” metric or with the radial metric can be found in paper [12].

2. Preliminaries

In this section, we recall the definition of a metric introduced by Szász, next we mention some of its basic properties, and we give some examples.

Definition 1 (see [11]). Let (X, d) be a metric space, $T : X \rightarrow X$ be a function, g be a metric on $T(X)$, and “ $\sim$ ” be an equivalence relation on X such that $x \sim y$ implies $T(x) = T(y)$. Define the function $\varphi : X \times X \rightarrow [0, +\infty)$ by the formula

$$\varphi(x, y) = \begin{cases} d(x, y), & \text{if } x \sim y, \\ d(x, T(x)) + g(T(x), T(y)) + d(T(y), y), & \text{otherwise.} \end{cases} \quad (1)$$

It turns out that the above defined function φ is a metric on X .

Proposition 2 (see [11]). *Under the assumptions and notation of Definition 1, φ is a metric on X .*

Remark 3. In the sequel we will use the symbol $d|_{T(X)}$ to denote the metric on $T(X)$ inherited from d .

The following obvious observation will be very useful in the sequel.

Proposition 4. *Under the assumptions and notation of Definition 1 and assuming that $g = d|_{T(X)}$, the metric φ is stronger than d .*

Proof. It is enough to observe that $d(x, y) \leq \varphi(x, y)$ for any $x, y \in X$. $\square$

In the next examples we mention some particular cases covered by Definition 1.

Example 1.

- (a) Let (X, d) be a metric space, $T : X \rightarrow X$ be a function, “ $\sim$ ” be the equality relation, and $g = d|_{T(X)}$.
- (b) Let (X, d) be a metric space, $T : X \rightarrow X$ be a function, “ $\sim$ ” be the relation defined by $x \sim y \Leftrightarrow T(x) = T(y)$, and $g = d|_{T(X)}$.
- (c) Let (X, d) be a metric space, $T : X \rightarrow X$ be a function, “ $\sim$ ” be the equality relation, and g be a metric on X .

$$\varphi_a(x, y) = \begin{cases} d_c(x, y), & \text{if } x, y \in X_c, \\ d_i(x, y), & \text{if } x, y \in X_i, \\ d_c(x, x_i) + d_i(x_i, y), & \text{if } x \in X_c, y \in X_i, \\ d_i(x, x_i) + d_c(x_i, y), & \text{if } x \in X_i, y \in X_c, \\ d_i(x, x_i) + d_c(x_i, x_j) + d_j(x_j, y), & \text{if } x \in X_i, y \in X_j, i \neq j. \end{cases} \quad (3)$$

(Let us note that the five cases in the above formula are not pairwise disjoint, but if more than one applies, the result is in fact the same in either case.)

Remark 8. The above metric can be also realized using Szász’s construction in the following way.

Example 2. Let $C_{[a,b]}^\infty$ be the space of functions $f : [a, b] \rightarrow \mathbb{R}$ differentiable infinitely many times and d be the “maximum” metric: $d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$. Let $T(f) = f'$ for all $f \in C_{[a,b]}^\infty$. Of course, $T : C_{[a,b]}^\infty \rightarrow C_{[a,b]}^\infty$, so we may deduce that the function $\varphi : C_{[a,b]}^\infty \times C_{[a,b]}^\infty \rightarrow \mathbb{R}_+$ defined by the formula

$$\varphi = \begin{cases} 0, & \text{if } f = g, \\ d(f, f') + d(f', g') + d(g', f), & \text{otherwise,} \end{cases} \quad (2)$$

is a metric in $C_{[a,b]}^\infty$.

It turns out that there is a strong connection between limit points of (X, φ) and fixed points of the mapping T .

Proposition 5. *Under the assumptions and notation of Definition 1 and assuming that “ $\sim$ ” is the equality relation, each limit point of (X, φ) is a fixed point of T .*

Proof. Let (x_n) be a sequence of points of X converging to some $x \in X$ such that each x_n is distinct from x . This means that $d(x, T(x)) \leq d(x, T(x)) + g(T(x), T(x_n)) + d(T(x_n), x_n) = \varphi(x, x_n)$; since $\lim_{n \rightarrow \infty} \varphi(x, x_n) = 0$, it must be the case that $d(x, T(x)) = 0$. $\square$

Remark 6. Let us consider a particular case of Example 1(b). Namely, let $X = \mathbb{R}^2$ and let φ be the “river” metric. Then $T((x, y)) = (x, 0)$ for $(x, y) \in \mathbb{R}^2$ and obviously not all limit points of (X, φ) are fixed points of T .

3. Relation to the “Linking Construction”

In this short section we are going to compare Szász’s construction with the so-called linking construction considered in paper [8].

Definition 7. Let (X_c, d_c) be a metric space and $\{(X_i, d_i)\}_{i \in I}$ be a family of metric spaces. Let us assume that $X_c \cap X_i = \{x_i\}$ for each $i \in I$ and that $X_i \cap X_j \subset \{x_i\}$ for any $i, j \in I$ (i.e., each pair of X_i ’s is either disjoint or intersects at x_i). Let us define $X = X_c \cup \bigcup_{i \in I} X_i$ and let $\varphi_a : X \times X \rightarrow [0, +\infty)$ be defined by the formula

Let $T : X \rightarrow X$ be defined by the formula

$$T(x) = \begin{cases} x_i, & \text{if } x \in X_i \text{ for some } i \in I, \\ x, & \text{otherwise.} \end{cases} \quad (4)$$

For any $x, y \in X$, we declare that $x \sim y$ if $x = y$ or $x, y \in X_i \setminus \{x_i\}$ for some $i \in I$. Finally, we define $g = d|_{T(X)} = \varphi_a$.

In order to check that φ and φ_a coincide, one has to examine several cases. As an example, let us consider one of them, namely, $x \in X_c$ and $y \in X_i$. Then, either $x \sim y$ (which can only happen if $x = y = x_i$, and in that case $\varphi(x, y) = \varphi_a(x, y) = 0$), or $x \not\sim y$. Then we obtain $\varphi_a(x, y) = d_c(x, x_i) + d_i(x_i, y) = \varphi_a(x, x_i) + \varphi_a(x_i, y)$. On the other hand, $\varphi(x, y) = d(x, T(x)) + g(T(x), T(y)) + d(T(y), y) = d(x, x) + g(x, x_i) + d(x_i, y) = \varphi_a(x, x_i) + \varphi_a(x_i, y)$.

4. Completeness

In this section we are going to prove a sufficient and necessary condition for the metric φ to be complete (with the natural assumption that $g = d|_{T(X)}$).

Throughout this section we will frequently make use of the fact that for any points $x, y \in X$, the obvious inequalities $d(x, y) \leq \varphi(x, y) \leq d(x, T(x)) + d(T(x), T(y)) + d(T(y), y)$ hold. In particular, if some sequence is φ -Cauchy (resp., φ -convergent), that is, it is a Cauchy (respectively, convergent) sequence with respect to the metric φ , then it is d -Cauchy (resp., d -convergent to the same limit).

Also, we will sometimes say shortly that some points of X are “equivalent” to mean that they are equivalent with respect to the relation “ $\sim$ ”.

Finally, we will often assume that all but finitely many of the terms (i.e., *almost all* terms) of some sequence (x_n) are pairwise equivalent with respect to the relation “ $\sim$ ”; in such cases, in order to shorten the considerations, we will silently drop enough of the beginning terms so that the rest of them are equivalent.

Let us now present a way to express the fact that a sequence is φ -Cauchy in terms of the metric d and the relation “ $\sim$ ”.

Proposition 9. *With the assumptions and notation of Definition 1, with the additional assumption that $g = d|_{T(X)}$, a sequence (x_n) of points in X is φ -Cauchy if and only if (x_n) is d -Cauchy, and*

- (i) *almost all x_n 's are equivalent with respect to the relation “ $\sim$,” or*
- (ii) $d(x_n, T(x_n)) \rightarrow 0$.

Proof. Let the sequence (x_n) be φ -Cauchy. It is then d -Cauchy. If almost all of its terms are equivalent, the proof is complete. Assume the contrary. Fix $\epsilon > 0$ and choose N large enough so that $\varphi(x_m, x_n) < \epsilon$ if $m, n \geq N$. For any $n \geq N$ there exists some $m \geq N$ such that $x_m \not\sim x_n$. We have then

$$\begin{aligned} d(x_n, T(x_n)) &\leq d(x_n, T(x_m)) + d(T(x_m), T(x_n)) \\ &\quad + d(T(x_m), x_m) = \varphi(x_n, x_m) < \epsilon. \end{aligned} \quad (5)$$

Assume now that the sequence (x_n) is d -Cauchy. If almost all of its terms are equivalent, the metrics d and φ coincide for those terms and hence (x_n) is φ -Cauchy. Assume now that $d(x_n, T(x_n)) \rightarrow 0$. Fix $\epsilon > 0$ and let N be large enough so that

$d(x_n, T(x_n)) < (1/5)\epsilon$ for $n \geq N$ and $d(x_m, x_n) < (1/5)\epsilon$ for $m, n \geq N$. Then, if $m, n \geq N$, we obtain

$$\begin{aligned} \varphi(x_m, x_n) &\leq d(x_m, T(x_m)) + d(T(x_m), T(x_n)) \\ &\quad + d(T(x_n), x_n) \\ &\leq d(x_m, T(x_m)) + d(T(x_m), x_m) \\ &\quad + d(x_m, x_n) + d(x_n, T(x_n)) \\ &\quad + d(T(x_n), x_n) \leq \epsilon. \end{aligned} \quad (6)$$

□

Theorem 10. *With the assumptions and notation of Definition 1, if we assume additionally that $g = d|_{T(X)}$, the metric φ is complete if and only if the following two conditions are true for any d -Cauchy sequence (x_n) such that $(T(x_n))$ is also d -Cauchy.*

- (1) *If $d(x_n, T(x_n)) \not\rightarrow 0$ and almost all x_n 's are equivalent, then there exists some $x \in X$ equivalent to the x_n 's such that $d(x_n, x) \rightarrow 0$.*
- (2) *If $d(x_n, T(x_n)) \rightarrow 0$, then $d(x_n, x) \rightarrow 0$ for some $x \in X$ such that the alternative, $x = T(x)$ or almost all x_n 's are equivalent to x , is true.*

Proof.

Sufficiency. Let (x_n) be φ -Cauchy and hence d -Cauchy. Let us consider two cases.

(I) *Almost All x_n 's Are Equivalent.* This means that the sequence $(T(x_n))$ is constant and hence d -Cauchy. If $d(x_n, T(x_n)) \not\rightarrow 0$, then $d(x_n, x) \rightarrow 0$ for some x equivalent to the x_n 's. This means that d and φ coincide for x_n 's and x and hence $\varphi(x_n, x) \rightarrow 0$. If on the other hand $d(x_n, T(x_n)) \rightarrow 0$, then $d(x_n, x) \rightarrow 0$ for some $x \in X$. Moreover, since $(T(x_n))$ is constant, denoting $y = T(x_n)$ we obtain $d(x_n, y) \rightarrow 0$ and hence $y = x$. If x is equivalent to x_n 's, d and φ coincide again on x_n 's and x and hence $\varphi(x_n, x) \rightarrow 0$. Otherwise we have $x = T(x)$ and we obtain

$$\begin{aligned} \varphi(x_n, x) &\leq d(x_n, T(x_n)) + d(T(x_n), T(x)) \\ &\quad + d(T(x), x). \end{aligned} \quad (7)$$

The first term on the right-hand side above tends to zero and the latter two terms are equal to zero, and hence $\varphi(x_n, x) \rightarrow 0$.

(II) *It Is Not the Case That Almost All x_n 's Are Equivalent.* Proposition 9 implies that $d(x_n, T(x_n)) \rightarrow 0$. Since (x_n) is d -Cauchy and $d(x_n, T(x_n)) \rightarrow 0$, the sequence $(T(x_n))$ is also d -Cauchy. This means that $d(x_n, x) \rightarrow 0$ for some $x = T(x)$. We have the estimate

$$\begin{aligned} \varphi(x_n, x) &\leq d(x_n, T(x_n)) + d(T(x_n), T(x)) \\ &\quad + d(T(x), x) \\ &\leq d(x_n, T(x_n)) + d(T(x_n), x_n) + d(x_n, x) \\ &\quad + d(x, T(x)) + d(T(x), x) \rightarrow 0 \end{aligned} \quad (8)$$

and the proof is complete.

Necessity. Assume that the metric φ is complete. Let (x_n) be a d -Cauchy sequence such that $(T(x_n))$ is also d -Cauchy.

If $d(x_n, T(x_n)) \rightarrow 0$ and almost all x_n 's are equivalent, this means that d and φ coincide for x_n 's and hence the sequence (x_n) is φ -Cauchy—hence φ -convergent and hence d -convergent to some $x \in X$. If x were not equivalent to x_n 's, we would have

$$\begin{aligned} \varphi(x_n, x) &= d(x_n, T(x_n)) + d(T(x_n), T(x)) \\ &\quad + d(T(x), x); \end{aligned} \quad (9)$$

as $\varphi(x_n, x) \rightarrow 0$, it would be the case that also $d(x_n, T(x_n)) \rightarrow 0$, which is a contradiction.

Assume now that $d(x_n, T(x_n)) \rightarrow 0$. From Proposition 9 we know that the sequence (x_n) is φ -Cauchy, hence φ -convergent and hence d -convergent to some $x \in X$. For the sake of contradiction, assume that $x \neq T(x)$ and that we can find a pair of nonequivalent terms arbitrarily far in the sequence (x_n) . We may thus pass to a subsequence of x_n 's not equivalent to x . Since

$$\begin{aligned} \varphi(x_n, x) &= d(x_n, T(x_n)) + d(T(x_n), T(x)) \\ &\quad + d(T(x), x) \end{aligned} \quad (10)$$

and $x \neq T(x)$, we obtain a contradiction with the φ -convergence of the sequence (x_n) . $\square$

As a corollary, we are going to show that some very natural properties of d and T imply completeness of φ .

Corollary 11. *With the assumptions and notation of Definition 1, if we assume additionally that $g = d|_{T(X)}$, the mapping T is continuous (with respect to the metric d), the space (X, d) is complete, and the equivalence classes of the relation “ $\sim$ ” are closed with respect to the metric d ; then the space (X, φ) is complete.*

Proof. Assume that the sequences (x_n) and $(T(x_n))$ are d -Cauchy. If $d(x_n, T(x_n)) \rightarrow 0$ and almost all x_n 's are equivalent, then (x_n) is d -convergent to some $x \in X$ due to d -completeness of X and x is equivalent to x_n 's due to the equivalence classes of the relation “ $\sim$ ” being closed with respect to the metric d .

Let now $d(x_n, T(x_n)) \rightarrow 0$. Again, (x_n) is d -convergent to some $x \in X$. By continuity of T , we have $T(x_n) \rightarrow T(x)$. Finally, $d(x, T(x)) \leq d(x, x_n) + d(x_n, T(x_n)) + d(T(x_n), T(x)) \rightarrow 0$ and hence $x = T(x)$. $\square$

The following three examples show that while the above theorem gives only a sufficient and not necessary conditions for completeness of φ , releasing either one of these three conditions may result in an incomplete space.

Example 3. If T is discontinuous, the metric φ may be incomplete. To see this, let us define $X := \mathbb{R}$, let d be the Euclidean metric and $g = d|_{T(X)}$, let the relation “ $\sim$ ” be the equality relation, and let

$$T(x) := \begin{cases} x, & \text{if } x \neq 0, \\ -1, & \text{if } x = 0. \end{cases} \quad (11)$$

We have $\varphi(x, y) = |x - y|$, if x, y are simultaneously equal or not equal to zero, and $\varphi(x, 0) = 1 + |x + 1|$, if $x \neq 0$. Let us consider the sequence $(1/n)_{n \in \mathbb{N}}$. Clearly, it is φ -Cauchy, but not φ -convergent.

Let us notice that, putting $x_n = 1/n$, we obtain a Cauchy sequence such that $(T(x_n))$ is also a Cauchy sequence and $d(x_n, T(x_n)) = 0$, but while $d(x_n, 0) \rightarrow 0$, neither the equality $0 = T(0)$ nor the equivalence of almost all x_n 's holds. In other words, the second assumption of Theorem 10 is not satisfied.

Example 4. If (X, d) is not complete, the metric φ may be incomplete. To see this, let $X := (0, 1)$, let d be the Euclidean metric and $g = d|_{T(X)}$, T be the identity, and let “ $\sim$ ” be the equality relation. Then φ is the Euclidean metric on $(0, 1)$. Again, considering the sequence $(1/n)$ shows that the assumptions of Theorem 10 are not satisfied.

Example 5. If the equivalence classes of the relation “ $\sim$ ” are not closed with respect to the metric d , the metric φ may be incomplete. To see this, let us define $X := \mathbb{R}$, let d be the Euclidean metric and $g = d|_{T(X)}$, let T be a constant mapping (say, $T(x) := 0$ for any $x \in \mathbb{R}$), and let $x \sim y$ if and only if, $\lfloor x \rfloor = \lfloor y \rfloor$. Let us consider the sequence (x_n) defined by the formula $x_n = 1 - 1/n$. It is obvious that it is φ -Cauchy. If it were φ -convergent to some $x \in \mathbb{R}$, Proposition 4 would imply its d -convergence to x ; hence $x = 1$. But $\varphi(1 - 1/n, 1) = (2n - 1)/n \xrightarrow{n \rightarrow \infty} 2 \neq 0$.

Once again, let us notice that both the sequences (x_n) and $(T(x_n))$ are d -Cauchy, $d(x_n, T(x_n)) = |1 - 1/n| \rightarrow 0$, and all x_n 's are equivalent. However, the sequence (x_n) is not d -convergent to a point in the same equivalence class.

Now, we are going to illustrate Theorem 10 and Corollary 11 with the following two examples.

Example 6. Let us consider the “river” metric. Here, $X = \mathbb{R}^2$, d is the Euclidean metric, $T(x, y) = (x, 0)$, the relation “ $\sim$ ” is defined by the formula $(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow x_1 = x_2$, and $g = d|_{T(X)}$. It is easy to see that all the conditions of Corollary 11 are satisfied and hence the “river” metric is complete.

Example 7. Let us now consider the radial metric. Put $X = \mathbb{R}^2$, let d be the Euclidean metric, $T(x, y) = (0, 0)$ for all $x, y \in \mathbb{R}$ (hence g is trivial), and define the relation “ $\sim$ ” by the formula

$$\begin{aligned} (x_1, y_1) \sim (x_2, y_2) &\Leftrightarrow \\ \{x_1 = y_1 = x_2 = y_2 = 0, \text{ or } (x_1, y_1), (x_2, y_2) & \\ \neq (0, 0) \text{ and } \exists \lambda > 0 : (x_2, y_2) = \lambda(x_1, y_1)\} & \end{aligned} \quad (12)$$

Let us notice that while d is complete and T is d -continuous, the equivalence classes of the relation “ $\sim$ ” are not closed with respect to the metric d . Despite that, the radial metric is complete. Indeed, let us take any d -Cauchy sequence (x_n) . Obviously, $(T(x_n))$ is a constant sequence and hence d -Cauchy. If $d(x_n, 0) \rightarrow 0$ and almost all x_n 's lie on the same ray, this means that (x_n) is d -convergent to some nonzero point on the same ray and (1) is satisfied. If on the other hand $d(x_n, 0) \rightarrow 0$, then (2) holds, because $(0, 0) = T(0, 0)$.

5. Boundedness

In this very short section we are going to discuss the problem of boundedness of the metric defined in Definition 1.

Proposition 12. *Under the assumptions and notation of Definition 1, if the metrics d and g are bounded, then so is φ . On the other hand, if φ is bounded, then so is g . Moreover, if $g = d|_{T(X)}$ and φ is bounded, then so is d .*

Proof. Let us first assume that d and g are bounded by some constant M . Then, for any $x, y \in X$, we obtain $\varphi(x, y) \leq d(x, T(x)) + g(T(x), T(y)) + d(T(y), y) \leq 3M$.

Assume now that φ is bounded. Let $x' = T(x)$ and $y' = T(y)$ be two distinct points in $T(X)$. Then $g(x', y') \leq d(x, x') + g(x', y') + d(y', y) = \varphi(x, y)$, so g is bounded. Furthermore, if $g = d|_{T(X)}$, then for any $x, y \in X$ we either have $x \sim y$ (and in this case, $d(x, y) \leq d(x, T(x)) + d(T(x), T(y)) + d(T(y), y) = \varphi(x, y)$) or $x \sim y$ (and in this case $d(x, y) = \varphi(x, y)$). $\square$

Remark 13. Let us notice that, in the last part of the above proposition, we assume that d restricted to $T(X)$ is equal to g and hence bounded, and from that we infer the boundedness of d on the whole X .

Remark 14. If $g \neq d|_{T(X)}$, then the boundedness of φ does not have to imply the boundedness of d . Indeed, consider $X = \mathbb{R}$, let $T : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $T(x) = \lfloor x \rfloor$, let “ $\sim$ ” be the equality relation, and let d be the Euclidean metric and g the discrete metric. Then φ is bounded (in fact, (X, φ) has the diameter of 3), but (X, d) is not.

6. The “Floor” Metric

For now, we will concentrate on one particular case of Definition 1, given in the following example.

Example 8. Let $X := \mathbb{R}$, $d(x, y) = g(x, y) = |x - y|$, “ $\sim$ ” be the equality relation, and $T(x) = \lfloor x \rfloor$. Applying Definition 1 we obtain a certain metric on $\mathbb{R}$, which in what follows will be denoted by φ .

Remark 15. By setting $T(x) = \lceil x \rceil$, that is, letting $T(x)$ be the smallest integer not less than x , we obtain a metric with similar properties—in fact, it is isometric to the one given here. A family of metrics with similar properties can be generated by functions of the form $T(x) = \lfloor ax + b \rfloor$ or $T(x) = \lceil ax + b \rceil$.

We are now going to establish some properties of the metric defined in Example 8.

Proposition 16. *The metric defined in Example 8 is neither homogeneous, nor translation invariant, nor metrically convex.*

Proof. To see that φ is not homogeneous, let us observe that $\varphi(1/2, 1) = 3/2$, but $\varphi(1, 2) = 1$. To see that it is not translation invariant, let us observe that $\varphi(0, 1/2) = 1/2$, but $\varphi(1/2, 1) = 3/2$. To see that it is not metrically convex, let us

observe first that $\varphi(0, 1) = 1$. Now if $z \in \mathbb{R} \setminus \{0, 1\}$, we have $\varphi(0, z) + \varphi(z, 1) > 1$. Indeed, if $z > 1$, then $\varphi(0, z) > 1$, and if $z < 1$ (and $z \neq 0$), then $\varphi(z, 1) > 1$. $\square$

Proposition 17. *The metric defined in Example 8 is complete.*

Let us notice that we can easily see that the assumptions of Theorem 10 are satisfied, although—since the mapping T is discontinuous at integer points—we cannot apply Corollary 11. We shall, however, show how to prove the completeness directly.

Proof. Let (x_n) be a φ -Cauchy sequence. If it has a constant subsequence, there is nothing to prove. If this is not the case, we may assume (passing to a subsequence if necessary) that all x_n 's are pairwise distinct. We may also assume that $x_n \in [k, k+1)$ for some $k \in \mathbb{Z}$. We will prove that $\lim_{n \rightarrow \infty} x_n = k$. Assume for the sake of contradiction that $\exists \varepsilon > 0 \forall N \in \mathbb{N} \exists n \geq N : x_n \in [k + \varepsilon, k + 1)$. That means that $\exists \varepsilon > 0 \forall N \in \mathbb{N} \exists m, n \geq N : x_m, x_n \in [k + \varepsilon, k + 1)$, which implies $\varphi(x_m, x_n) \geq 2\varepsilon$ —contradiction. $\square$

We will now describe balls in the metric space $(\mathbb{R}, \varphi)$. We will start with the case when the center is an integer.

Remark 18. If $a \geq b$, we define $[a, b) := \emptyset$ and similarly with other kinds of intervals.

Proposition 19. *Let $x \in \mathbb{Z}$ and $r > 0$. Then the ball centered at x with radius r with respect to the metric defined in Example 8 has the form*

$$B(x, r) = [x - \lfloor r \rfloor, x - \lfloor r \rfloor + r - \lfloor r \rfloor) \cup [x - \lfloor r \rfloor + 1, x + r). \quad (13)$$

Proof. Let us consider several cases.

- (i) $y < x - \lfloor r \rfloor$. Then $\lfloor y \rfloor \leq y < x - \lfloor r \rfloor$ and hence $r < \lfloor r \rfloor + 1 \leq x - \lfloor y \rfloor$, so $\varphi(y, x) = y - \lfloor y \rfloor + x - \lfloor y \rfloor > x - \lfloor y \rfloor > r$.
- (ii) $x - \lfloor r \rfloor \leq y < x - \lfloor r \rfloor + r - \lfloor r \rfloor$. Then $\lfloor y \rfloor = x - \lfloor r \rfloor$ and hence $y - \lfloor y \rfloor < r - \lfloor r \rfloor$, so $\varphi(y, x) = y - \lfloor y \rfloor + x - \lfloor y \rfloor < r$.
- (iii) $x - \lfloor r \rfloor + r - \lfloor r \rfloor \leq y < x - \lfloor r \rfloor + 1$. Then $\lfloor y \rfloor = x - \lfloor r \rfloor$, so $\varphi(x, y) = y - \lfloor y \rfloor + x - \lfloor y \rfloor = y - \lfloor y \rfloor + \lfloor r \rfloor \geq r - \lfloor r \rfloor + \lfloor r \rfloor = r$.
- (iv) $x - \lfloor r \rfloor + 1 \leq y < x$. Since $x - r \lfloor r \rfloor + 1 \leq y$, also $x - \lfloor r \rfloor + 1 \leq \lfloor y \rfloor$ and hence $x - \lfloor y \rfloor + 1 \leq \lfloor r \rfloor$. We obtain now $\varphi(y, x) = y - \lfloor y \rfloor + x - \lfloor y \rfloor < x - \lfloor y \rfloor + 1 \leq \lfloor r \rfloor \leq r$.
- (v) $y = x$; then obviously $\varphi(y, x) < r$.
- (vi) $x < y < x + r$. Then $x \leq \lfloor y \rfloor$ and $\varphi(y, x) = y - x < r$.
- (vii) $x + r \leq y$. Then $\varphi(y, x) = y - x \geq r$. $\square$

Let us now consider a ball with an arbitrary center.

Proposition 20. Let $x \in \mathbb{R}$ and $r > 0$. Then the ball centered at x with radius r with respect to the metric defined in Example 8 has the form $B(x, r) = B(\lfloor x \rfloor, r - (x - \lfloor x \rfloor)) \cup \{x\}$.

Proof. Let $y \in B(x, r)$. If $y = x$, there is nothing to prove. Let $y \neq x$. Then $y - \lfloor y \rfloor + \lfloor y \rfloor - \lfloor x \rfloor < r - (x - \lfloor x \rfloor)$. Since the left-hand side of this expression equals $\varphi(y, \lfloor x \rfloor)$, we are done.

Let now $y \in B(\lfloor x \rfloor, r - (x - \lfloor x \rfloor))$. Again we may assume $y \neq x$, and similar computation shows that $y \in B(x, r)$. $\square$

Remark 21. In a similar way we may prove that if $x \in \mathbb{R}$ and $r \geq 0$, then $\bar{B}(x, r) = \bar{B}(\lfloor x \rfloor, r - (x - \lfloor x \rfloor)) \cup \{x\}$ and that if $x \in \mathbb{Z}$ and $r \geq 0$, then $\bar{B}(x, r) = [x - \lfloor r \rfloor, x - \lfloor r \rfloor + r - \lfloor r \rfloor] \cup [x - \lfloor r \rfloor + 1, x + r]$.

Now we describe the form of open subsets of the metric space $(\mathbb{R}, \varphi)$.

Proposition 22 (form of open sets). *A subset of $\mathbb{R}$ is φ -open if and only if it is a disjoint union of maximal intervals (i.e., such that the union of any two or more of these intervals is not an interval itself) such that any of these intervals whose right endpoint is an integer is right-open.*

Proof. It is known that any subset of $\mathbb{R}$ can be represented by a disjoint union of maximal intervals (even in a unique way). It is thus enough to prove that a necessary and sufficient condition for an interval to be φ -open is that if its right endpoint is an integer, the interval in question is right-open.

Let us notice that an open ball with a noninteger center x and sufficiently small radius is a singleton $\{x\}$ and that an open ball with an integer center x and sufficiently small radius $\epsilon > 0$ is the interval $[x, x + \epsilon)$. Let now I be an interval with left endpoint a and right endpoint b . If $x \in (a, b)$, then clearly x lies in the interior of I . If $a \in I$, then a must be an interior point of I . If $b \in I$, then b is in the interior of I iff $b \notin \mathbb{Z}$, and the proof is complete. $\square$

Corollary 23. *A subset of $\mathbb{R}$ is closed with respect to the metric φ if and only if it is a disjoint union of maximal intervals (i.e., such that the union of any two or more of these intervals is not an interval itself) such that any of these intervals whose left endpoint is integer is left-closed.*

Remark 24. Balls in $(\mathbb{R}, \varphi)$ need not be connected nor compact. For instance, $B(1, 3/2) = [0, 1/2) \cup [1, 5/2)$ is the union of two open sets $[0, 1/2)$ and $[1, 5/2)$. On a similar note, $\bar{B}(1, 3/2) = [0, 1/2] \cup [1, 5/2]$, and these two intervals are closed with respect to the metric φ .

Consider now the ball $\bar{B}(0, 1/2) = [0, 1/2]$. Since it can be covered by an infinite family of disjoint open sets, namely, $\{[0, 1/4]\} \cup \{x\} \mid x \in [1/4, 1/2]\}$, it is not compact.

Now we are going to establish the formula which allows calculating the Kuratowski measure of noncompactness of bounded subsets of $\mathbb{R}$ endowed with the “floor” metric (for the definition and the basic properties of the Kuratowski measure of noncompactness we refer the reader to [13] or to [14]).

Proposition 25. Let $A \subset [k, k + 1)$ for some $k \in \mathbb{Z}$. The Kuratowski measure of noncompactness of A , considered as a subset of $(\mathbb{R}, \varphi)$, is given by the formula

$$\alpha(A) = 2 \max(\{x - k \mid x \text{ is the limit point of } A\} \cup \{0\}). \quad (14)$$

(Let us note that for the sake of this proposition and its corollary, when we talk of a limit point of some subset of $\mathbb{R}$, we mean the limit point with respect to the Euclidean topology and not the topology generated by the metric φ . The same holds for limits of sequences.)

Proof. First observe that A has a limit point if, and only if, it is infinite; if A is finite, obviously $\alpha(A) = 0$. From now on we will assume that A is infinite.

Let us define $a := \sup\{x - k \mid x \text{ is the limit point of } A\} \in [0, 1]$. Let us observe first that this supremum is actually attained, since a limit of a sequence of limit points of any given set is also its limit point.

Let us first prove that $\alpha(A) \leq 2a$. If $a = 1$, this claim is obvious, since $\alpha(A) \leq \text{diam}[k, k + 1) = 2$. Let us assume that $a \in [0, 1)$ and choose any $\epsilon > 0$ such that $a + (1/2)\epsilon < 1$. The family $\{[k, k + a + (1/2)\epsilon)\} \cup \{x\} \mid x \in A \cap [k + a + (1/2)\epsilon, k + 1)\}$ is a finite covering of A by sets of diameter no greater than $2a + \epsilon$.

Let us now prove that actually $\alpha(A) = 2a$; that is, there is no finite covering of A with sets of diameter strictly less than $2a$. Choose $d \in (0, 2a)$; of course, there are infinitely many points of A in the interval $I := [k + (1/2)d, k + 1)$. Choose two distinct points $x, y \in I$; we have $\varphi(x, y) = (x - k) + (y - k) > d$. Any finite covering of A would have to contain a set containing more than one point of I —but this means that the diameter of this set would be greater than d . $\square$

Corollary 26. Let A be a nonempty, bounded subset of $(\mathbb{R}, \varphi)$. The Kuratowski measure of noncompactness of A is given by the formula

$$\alpha(A) = 2 \cdot \max_{k \in \mathbb{Z}} \max(\{x - k \mid x \text{ is the limit point of } A \cap [k, k + 1)\} \cup \{0\}). \quad (15)$$

Proof. It is enough to observe that the outer maximum in the above formula is well-defined, since only finitely many of the terms involved may be positive and the rest are equal to zero. Then, it follows from the maximum property of the measure of noncompactness. $\square$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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