Research Article

On the Existence of Positive Solutions for a Fourth-Order Boundary Value Problem

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By using the method of order reduction and the fixed point index, the existence of positive solutions for a fourth-order boundary value problem is studied. We provide conditions under which the existence results hold. Such conditions are related to the first eigenvalue corresponding to the relevant linear differential equation with dependence on the derivatives of unknown function.

1. Introduction

In this paper, we consider the following fourth-order ordinary differential equation boundary value problem (BVP):

$$u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0, 1], \quad (1)$$

$$u(0) = u'(0) = u''(1) = u'''(1) = 0,$$

where $f: [0, 1] \times \mathbb{R}_+^3 \times \mathbb{R}_- \rightarrow \mathbb{R}_+$ is continuous.

BVP (1) is used to model such phenomena as the deformations of an elastic beam in equilibrium state, whose one end-point is fixed and the other is freed. In mechanics, BVP (1) is called cantilever beam equation. Owing to its importance in mechanics, the existence of solutions to this problem has been studied by many authors; see [1–7] and references therein.

Very recently, Li [3] used the theory of the fixed point index to discuss the existence of solutions of BVP (1) when $f(t, x_0, x_1, x_2, x_3)$ is superlinear or sublinear growth on x_0, x_1, x_2, x_3 . It should be remarked that two constants 1 and λ_1 , where λ_1 is the first eigenvalue of the linear eigenvalue problem corresponding to BVP (1), play an important role in the discussion. The first eigenvalue principles were also used in [8–24].

However, none of these known results use the first eigenvalues of the corresponding linear differential equation that contains the derivative of the unknown function. This

is because the presence of derivatives in linear eigenvalue problem will make the study extremely difficult. To overcome this difficulty, we employ the method of order reduction to develop spectral properties of associated linear differential equation that contains the derivative of the unknown function. Then, we use the fixed point index theory to investigate the existence results of positive solutions for BVP (1) under some conditions concerning the first eigenvalue corresponding to the relevant linear eigenvalue problem that contains the derivative of the unknown function. It should be noted that the method of order reduction was also used in [7, 25] to transform higher order boundary value problem to a lower order integrodifferential equation.

2. Main Results

Let $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_- = (-\infty, 0]$. We use the classical Banach space $C[0, 1]$, $C^1[0, 1]$ with the norm $\|u\|_C = \max_{t \in [0, 1]} |u(t)|$, $\|u\|_{C^1} = \max\{\|u\|_C, \|u'\|_C\}$, respectively. Let P denote the cone of all nonnegative functions in $C[0, 1]$.

We assume the following hypothesis holds throughout this paper.

$$(H_1) \quad f \in C([0, 1] \times \mathbb{R}_+^3 \times \mathbb{R}_-, \mathbb{R}_+).$$

Firstly, we consider the existence of solutions to BVP (1) when the nonlinearity f satisfies the sublinear growth

property. To this end, we use the method of order reduction to transform BVP (1). For $v \in C[0, 1]$, we set

$$\begin{aligned} (T_3 v)(t) &= \int_t^1 v(s) ds, \\ (T_2 v)(t) &= \int_0^1 G_2(t, s) v(s) ds, \\ (T_1 v)(t) &= \int_0^1 G_1(t, s) v(s) ds, \end{aligned} \quad (2)$$

where

$$\begin{aligned} G_2(t, s) &= \begin{cases} t, & 0 \leq t \leq s \leq 1, \\ s, & 0 \leq s \leq t \leq 1, \end{cases} \\ G_1(t, s) &= \begin{cases} \frac{t^2}{2}, & 0 \leq t \leq s \leq 1, \\ \frac{2ts - s^2}{2}, & 0 \leq s \leq t \leq 1. \end{cases} \end{aligned} \quad (3)$$

Then we have $(T_1 v)''' = (T_2 v)'' = (T_3 v)' = -v$. From the above expressions, we easily see that $T_i : C[0, 1] \rightarrow C[0, 1]$ ($i = 1, 2, 3$) are three completely continuous linear operators, and G_1 and G_2 have the following properties:

$$\begin{aligned} \frac{t^2}{2} s &\leq G_1(t, s) \leq s, \\ ts &\leq G_2(t, s) \leq s, \\ \forall t, s &\in [0, 1]. \end{aligned} \quad (4)$$

Hence we conclude that

$$\begin{aligned} T_1(P_1) &\subset P_1 \\ &= \left\{ u \in C[0, 1] : u(t) \geq \frac{t^2}{2} \|u\|_C, \quad t \in [0, 1] \right\}. \end{aligned} \quad (5)$$

Using the above transformations T_i ($i = 1, 2, 3$), BVP (1) can be converted into the equivalent nonlinear integral equation:

$$\begin{aligned} v(t) &= \int_t^1 f(s, (T_1 v)(s), (T_2 v)(s), (T_3 v)(s), -v(s)) ds, \\ v &\in C[0, 1]. \end{aligned} \quad (6)$$

Define an operator A by

$$\begin{aligned} (Av)(t) &= \int_t^1 f(s, (T_1 v)(s), (T_2 v)(s), (T_3 v)(s), -v(s)) ds, \\ v &\in C[0, 1]. \end{aligned} \quad (7)$$

Then the existence of a positive solution of BVP (1) is equivalent to the existence of a nontrivial fixed point of A on

P . Now condition (H_1) implies that $A : P \rightarrow P$ is a completely continuous operator.

For $\mathbf{a} = (a_1, b_1, c_1, d_1) \in \mathbb{R}_+^4$ with $a_1^2 + b_1^2 \neq 0$, we define linear operator $K_{\mathbf{a}}$ by

$$\begin{aligned} (K_{\mathbf{a}} v)(t) &= \int_t^1 (a_1 (T_1 v)(s) + b_1 (T_2 v)(s) + c_1 (T_3 v)(s) \\ &\quad + d_1 v(s)) ds, \quad v \in C[0, 1]. \end{aligned} \quad (8)$$

Clearly, $K_{\mathbf{a}} : P \rightarrow P$ is a completely continuous linear operator.

Lemma 1. For the operator $K_{\mathbf{a}}$ defined by (8), the spectral radius $r(K_{\mathbf{a}}) \neq 0$ and $K_{\mathbf{a}}$ had unique positive unit eigenfunction $\varphi_{\mathbf{a}}$ corresponding to its first eigenvalue $\lambda_1 = (r(K_{\mathbf{a}}))^{-1}$.

Proof. Take $v(t) = 1 - t$; clearly $v \in P$; by (3) we have

$$\begin{aligned} (T_1 v)(t) &\geq \int_0^t \frac{2ts - s^2}{2} (1 - s) ds \geq \int_0^t \frac{ts}{2} (1 - s) ds \\ &= \frac{t^3}{12} (3 - 2t) \geq \frac{t^3}{12}, \\ (T_2 v)(t) &\geq \int_0^t s (1 - s) ds = \frac{t^2}{6} (3 - 2t) \geq \frac{t^2}{6}. \end{aligned} \quad (9)$$

So we have

$$\begin{aligned} (K_{\mathbf{a}} v)(t) &\geq a_1 \int_t^1 (T_1 v_0)(s) ds + b_1 \int_t^1 (T_2 v_0)(s) ds \\ &\geq \left(\frac{a_1}{48} + \frac{b_1}{18} \right) v(t). \end{aligned} \quad (10)$$

Thus there exists a constant $c > 0$ such that $c(K_{\mathbf{a}} v)(t) \geq v(t)$, $t \in [0, 1]$. From Lemma 2 in [22], we know that the spectral radius $r(K_{\mathbf{a}}) \neq 0$ and $K_{\mathbf{a}}$ had a positive eigenfunction corresponding to its first eigenvalue $\lambda_1 = (r(K_{\mathbf{a}}))^{-1}$. Hence Lemma 1 holds.

It follows from the definition of operator $K_{\mathbf{a}}$ that the function $\varphi_{\mathbf{a}}$, corresponding to the first eigenvalue $\lambda_1 = (r(K_{\mathbf{a}}))^{-1}$ of the operator $K_{\mathbf{a}}$, belongs to $C^4[0, 1]$ and satisfies the equation

$$\begin{aligned} (T_1 \varphi_{\mathbf{a}})^{(4)}(t) &= \lambda_1 (a_1 (T_1 \varphi_{\mathbf{a}}) + b_1 (T_2 \varphi_{\mathbf{a}})' + c_1 (T_3 \varphi_{\mathbf{a}})'' \\ &\quad - d_1 (T_1 \varphi_{\mathbf{a}})'''), \quad t \in [0, 1], \\ (T_1 \varphi_{\mathbf{a}})(0) &= (T_1 \varphi_{\mathbf{a}})'(0) = (T_1 \varphi_{\mathbf{a}})''(1) = (T_1 \varphi_{\mathbf{a}})'''(1) \\ &= 0. \end{aligned} \quad (11)$$

In particular, for $\mathbf{a} = (a_1, b_1, 0, 0) \in \mathbb{R}_+^4$, the positive eigenfunction $\varphi_{\mathbf{a}}$ of the operator $K_{\mathbf{a}}$ corresponding to the first

eigenvalue $\lambda_1 = (r(K_a))^{-1}$ belongs to $C^4[0, 1]$ and satisfies the equation

$$\begin{aligned} (T_1 \varphi_a)^{(4)}(t) &= \lambda_1 (a_1 (T_1 \varphi_a) + b_1 (T_1 \varphi_a)'), \\ t &\in [0, 1], \\ (T_1 \varphi_a)(0) &= (T_1 \varphi_a)'(0) = (T_1 \varphi_a)''(1) \\ &= (T_1 \varphi_a)'''(1) = 0. \end{aligned} \quad (12)$$

□

Theorem 2. *If f satisfies (H_1) and the following conditions, then BVP (1) has at least one positive solution.*

(H_2) There are $\mathbf{a} = (a_1, b_1, c_1, d_1) \in \mathbb{R}_+^4$ with $a_1^2 + b_1^2 \neq 0$ and $c > 0$ such that $r(K_a) < 1$ and

$$\begin{aligned} f(t, x_1, x_2, x_3, x_4) &\leq a_1 x_1 + b_1 x_2 + c_1 x_3 + d_1 |x_4| + c, \\ (x_1, x_2, x_3, x_4) &\in \mathbb{R}_+^3 \times \mathbb{R}_-. \end{aligned} \quad (13)$$

(H_3) There are $\mathbf{b} = (a_2, b_2, c_2, d_2) \in \mathbb{R}_+^4$ with $a_2^2 + b_2^2 \neq 0$ and $r > 0$ such that $r(K_b) > 1$ and

$$\begin{aligned} f(t, x_1, x_2, x_3, x_4) &\geq a_1 x_1 + b_1 x_2 + c_1 x_3 + d_1 |x_4|, \\ (x_1, x_2, x_3, x_4) &\in [0, r]^3 \times [-r, 0]. \end{aligned} \quad (14)$$

Proof. Let $W_1 = \{v \in P : v = \lambda Av, \text{ for some } \lambda \in [0, 1]\}$. We will prove that W_1 is a bounded set. In fact, if $v \in W_1$, by the definition of W_1 , $v = \lambda Av$ for some $\lambda \in [0, 1]$. Hence we have

$$\begin{aligned} v(t) &\leq (Av)(t) \\ &= \int_t^1 f(s, (T_1 v)(s), (T_2 v)(s), (T_3 v)(s), -v(s)) ds, \end{aligned} \quad (15)$$

$$v \in P, \quad t \in [0, 1].$$

By (H_2) , we have $v(t) \leq (K_a v)(t) + c(1 - t)$, which can be rewritten in the form $(I - K_a)v \leq \tilde{v}_0$, where $\tilde{v}_0(t) = c(1 - t)$. The condition $r(K_a) < 1$ ensures that $I - K_a$ has a bounded inverse operator $(I - K_a)^{-1}$ which is given by $(I - K_a)^{-1} = I + K_a + K_a^2 + \dots + K_a^n + \dots$. It follows from $K_a(P) \subset P$ that $(I - K_a)^{-1}(P) \subset P$. So we have $v \leq (I - K_a)^{-1} \tilde{v}_0$ and W_1 is bounded. Select $R > \max\{r, \sup\{\|v\|_C : v \in W_1\}\}$; we have $v \neq \lambda Av$, $v \in \partial\Omega_R$, $\lambda \in [0, 1]$, where $\Omega_R = \{u \in C[0, 1] : \|u\|_C < R\}$. Now Lemma 2.3.1 in [26] implies

$$i(A, \Omega_R \cap P, P) = 1. \quad (16)$$

Let φ_b be the positive eigenvalue function of K_b (with replacement of $\mathbf{a}$ by $\mathbf{b}$ in Lemma 1). We may suppose that A has no fixed point on $\partial\Omega_r \cap P$ (otherwise, the proof is finished). Now we shall show that

$$v - Av \neq \lambda \varphi_b, \quad \forall v \in \partial\Omega_r \cap P, \quad \lambda \geq 0, \quad (17)$$

where $\Omega_r = \{u \in C[0, 1] : \|u\|_C < r\}$. In fact, if (17) is not valid, there exist $v_0 \in \partial\Omega_r \cap P$ and $\lambda_0 \geq 0$ such that

$v_0 - Av_0 = \lambda_0 \varphi_b$. Hence we have $\lambda_0 > 0$. Since $v_0 \in \partial\Omega_r \cap P$, by the definitions of P and T_i ($i = 1, 2, 3$), we have

$$\begin{aligned} 0 \leq v_1(t), (T_1 v)(t), (T_2 v)(t), -(T_3 v)(t) &\leq r, \\ t &\in [0, 1]. \end{aligned} \quad (18)$$

Hence from (H_3) , we obtain that

$$\begin{aligned} f(t, (T_1 v_0)(t), (T_2 v_0)(t), (T_3 v_0)(t), -v_0(t)) \\ \geq a_2 (T_1 v_0)(t) + b_2 (T_2 v_0)(t) + c_2 (T_3 v_0)(t) \\ + d_2 v_0(t) \end{aligned} \quad (19)$$

from which it follows that

$$v_0 = Av_0 + \lambda_0 \varphi_b \geq K_b v_0 + \lambda_0 \varphi_b. \quad (20)$$

Set $\lambda^* = \sup\{\lambda : v_0 \geq \lambda \varphi_b\}$. It is easy to see that $\lambda^* > 0$ and $v_0 \geq \lambda^* \varphi_b$. It follows from $K_b(P) \subset P$ that $K_b v_0 \geq \lambda^* K_b \varphi_b = \lambda^* r(K_b) \varphi_b$. Therefore, by (20),

$$v_0 = Av_0 + \lambda_0 \varphi_b \geq r(K_b) \lambda^* \varphi_b + \lambda_0 \varphi_b \quad (21)$$

which contradicts the definition of λ^* . Hence (17) is true and we have from Corollary 2.3.1 in [26] that

$$i(A, \Omega_r \cap P, P) = 0. \quad (22)$$

By (16) and (22), we have that

$$\begin{aligned} i(A, (\Omega_R \cap P) \setminus (\overline{\Omega_r} \cap P), P) \\ = i(A, \Omega_R \cap P, P) - i(A, \Omega_r \cap P, P) = 1. \end{aligned} \quad (23)$$

Then A has at least one fixed point in $(\Omega_R \cap P) \setminus (\overline{\Omega_r} \cap P)$ which means that BVP (1) has a positive solution. The proof of Theorem 2 is completed. □

Next, we consider the existence of solutions to BVP (1) when the nonlinearity f satisfies the superlinear growth condition. In this case, we assume that $f(t, x_0, x_1, x_2, x_3)$ is independent of x_2 and x_3 ; that is, we consider the following simplified BVP:

$$\begin{aligned} u^{(4)}(t) &= f(t, u(t), u'(t)) \\ u(0) &= u'(0) = u''(1) = u'''(1) = 0. \end{aligned} \quad (24)$$

As in [3], a positive solution of BVP (24) is equivalent to a nontrivial fixed point of B defined by

$$\begin{aligned} (Bu)(t) &= \int_0^1 G(t, s) f(s, u(s), u'(s)) ds, \\ u &\in C^1[0, 1], \end{aligned} \quad (25)$$

where $G(t, s)$ is Green's function of the homogeneous linear problem $u^{(4)}(t) = 0$, $u(0) = u'(0) = u''(1) = u'''(1) = 0$, which is explicitly given by

$$G(t, s) = \begin{cases} \frac{1}{6} t^2 (3s - t), & 0 \leq t \leq s \leq 1, \\ \frac{1}{6} s^2 (3t - s), & 0 \leq s \leq t \leq 1. \end{cases} \quad (26)$$

From (26) we can easily verify that $G(t, s)$ has the following properties:

$$\frac{1}{3}t^2s^2 \leq G(t, s) \leq \frac{1}{2}t^2 \left(\text{or } \frac{1}{2}s^2 \right), \quad t, s \in [0, 1]. \quad (27)$$

For $\mathbf{c} = (a_1, b_1) \in \mathbb{R}_+^2$ with $a_1^2 + b_1^2 \neq 0$ (we also use $\mathbf{c}$ denoting $(a_1, b_1, 0, 0)$). Define a linear integral operator L by

$$(L_{\mathbf{c}}u)(t) = \int_0^1 G(t, s) (a_1 u(s) + b_1 u'(s)) ds, \quad (28)$$

$$u \in C^1[0, 1].$$

In what follows we work on the Banach space $C^1[0, 1]$ and utilize the cone

$$P_3 = \left\{ u \in C^1[0, 1] : u(0) = 0, \quad u(t) \geq \frac{1}{2}t^2 \|u\|_C, \quad u'(t) \geq t \|u'\|_C, \quad 0 \leq t \leq 1 \right\}. \quad (29)$$

Lemma 3 (see [3, 6]). *If $u \in P_3$, then $\|u'\|_{C^1} = \|u\|_C$ and $(1/2)\|u\|_{C^1} \leq \|u\|_C \leq \|u\|_{C^1}$.*

Lemma 4 (see [3, 6]). *$B, L : P_3 \rightarrow P_3$ are two completely continuous operators.*

Based on the definition of operator $L_{\mathbf{c}}$, (12) can be rewritten in the following form:

$$T_1 \varphi_{\mathbf{c}} = (r(K_{\mathbf{c}}))^{-1} L_{\mathbf{c}}(T_1 \varphi_{\mathbf{c}}) \quad (30)$$

which means that the operator $L_{\mathbf{c}}$ has an eigenvalue $(r(K_{\mathbf{c}}))^{-1}$, and its related eigenfunction can be taken by $T_1 \varphi_{\mathbf{c}}$. In fact, we have $r(K_{\mathbf{c}}) = r(L_{\mathbf{c}})$ from equivalent differential equation (12).

Theorem 5. *If f satisfies (H_1) and the following conditions, then BVP (24) has at least one positive solution.*

(H_4) *There are $\mathbf{c} = (a_1, b_1) \in \mathbb{R}_+^2$ with $a_1^2 + b_1^2 \neq 0$ and $c > 0$ such that $r(L_{\mathbf{c}}) > 1$ and*

$$f(t, x_1, x_2) \geq a_1 x_1 + b_1 x_2 - c, \quad (31)$$

$$(x_1, x_2) \in \mathbb{R}^2, \quad t \in [0, 1].$$

(H_5) *There are $\mathbf{d} = (a_2, b_2) \in \mathbb{R}_+^2$ with $a_2^2 + b_2^2 \neq 0$ and $\delta > 0$ such that $r(L_{\mathbf{d}}) < 1$ and*

$$f(t, x_1, x_2) \leq a_2 x_1 + b_2 x_2, \quad (32)$$

$$(x_1, x_2) \in [0, \delta]^2, \quad t \in [0, 1].$$

Proof. Set $\varepsilon > 0$ such that $(1 - \varepsilon)r(L_{\mathbf{c}}) > 1$. Take $R > \max\{\delta, 30c/\varepsilon(2a_1 + 5b_1)\}$. For every $u \in \partial\Omega_R \cap P_3$, it follows from (27) and Lemma 3 that

$$\begin{aligned} Bu &\geq L_{\mathbf{c}}u - c \int_0^1 G(t, s) ds \\ &\geq (1 - \varepsilon)L_{\mathbf{c}}u + \varepsilon L_{\mathbf{c}}u - \frac{c}{2}t^2 \\ &\geq (1 - \varepsilon)L_{\mathbf{c}}u \\ &\quad + \varepsilon \int_0^1 \frac{1}{3}t^2s^2 \left(\frac{a_1}{2}s^2 \|u\|_{C^1} + b_1s \|u\|_{C^1} \right) ds - \frac{c}{2}t^2 \\ &\geq (1 - \varepsilon)L_{\mathbf{c}}u + t^2 \left(\varepsilon \|u\|_{C^1} \left(\frac{a_1}{30} + \frac{b_1}{12} \right) - \frac{c}{2} \right) \\ &\geq (1 - \varepsilon)L_{\mathbf{c}}u. \end{aligned} \quad (33)$$

We may suppose that B has no fixed point on $\partial\Omega_R \cap P_3$ (otherwise, the proof is finished). Now we show that

$$u - Bu \neq \mu T_1 \varphi_{\mathbf{c}}, \quad \forall u \in \partial\Omega_R \cap P_3, \quad \mu \geq 0, \quad (34)$$

where $\varphi_{\mathbf{c}}$ is the positive eigenfunction of operator $S_{\mathbf{c}}$ related to its first eigenvalue $(r(S_{\mathbf{c}}))^{-1}$, and $T_1 \varphi_{\mathbf{c}} \in P_3 \setminus \{0\}$ follows from (5).

In fact, if (34) is not valid, there exist $u_0 \in \partial\Omega_R \cap P_3$ and $\mu_0 \geq 0$ such that

$$u_0 - Bu_0 = \mu_0 T_1 \varphi_{\mathbf{c}}. \quad (35)$$

Hence we have $\mu_0 > 0$. Set $\mu^* = \sup\{\mu : u_0 \geq \mu T_1 \varphi_{\mathbf{c}}\}$. It is easy to see that $\mu^* > 0$ and $u_0 \geq \mu^* T_1 \varphi_{\mathbf{c}}$. Since $u_0 \in \partial\Omega_R \cap P_3$, by (33), we obtain that

$$u_0 = Bu_0 + \mu_0 T_1 \varphi_{\mathbf{c}} \geq (1 - \varepsilon)L_{\mathbf{c}}u_0 + \mu_0 T_1 \varphi_{\mathbf{c}}. \quad (36)$$

Therefore by (30), $u_0 \geq \mu^*(1 - \varepsilon)r(L_{\mathbf{c}})T_1 \varphi_{\mathbf{c}} > \mu^* T_1 \varphi_{\mathbf{c}}$, which contradicts the definition of μ^* . Hence (34) is true and we have from Corollary 2.3.1 in [26] that

$$i(A, \Omega_R \cap P_3, P_3) = 0. \quad (37)$$

Next, we prove that

$$u \neq \mu Bu, \quad u \in \partial\Omega_{\delta} \cap P_3, \quad \mu \in [0, 1]. \quad (38)$$

If this is false, then there exist $u_1 \in \partial\Omega_{\delta} \cap P_3$ and $\mu_1 \in [0, 1]$ such that $u_1 = \mu_1 Bu_1$. It follows from Lemma 3 that $0 \leq u_1(t), u_1'(t) \leq \delta$, $t \in [0, 1]$. Then by assumption (H_5) , we have $u_1 = \mu_1 Bu_1 \leq \mu_1 L_{\mathbf{d}}u_1 \leq L_{\mathbf{d}}u_1$. Notice that $r(L_{\mathbf{d}}) < 1$; we obtain $u_1 \leq (I - L_{\mathbf{d}})^{-1}0 = 0$, which is a contradiction with $u_1 \in \partial\Omega_{\delta} \cap P_3$. Hence, (38) holds. Then by Lemma 2.3.1 in [26], we have

$$i(A, \Omega_{\delta} \cap P_3, P_3) = 1. \quad (39)$$

By (37) and (39), we have that

$$\begin{aligned} &i(B, (\Omega_R \cap P_3) \setminus (\overline{\Omega_{\delta}} \cap P_3), P_3) \\ &= i(B, \Omega_R \cap P_3, P_3) - i(B, \Omega_{\delta} \cap P_3, P_3) = -1. \end{aligned} \quad (40)$$

Then B has at least one fixed point in $(\Omega_R \cap P_3) \setminus (\overline{\Omega_{\delta}} \cap P_3)$ which means that BVP (24) has a positive solution. The proof of Theorem 5 is completed. $\square$

Conflicts of Interest

The author declares that they have no conflicts of interest.

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