Research Article

General Holmstedt’s Formulae for the K-Functional

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1. Introduction

In recent papers [1–11] the classical Holmstedt formula for the K-functional [12] was extended to more general cases. See also [13] for more results about generalized Holmstedt’s formula and reiteration theorems not only for the K-method, but for the J-method as well. Here we consider the K-method in the most general case for quasi-normed spaces. Namely, let \((A_0, A_1)\) be a compatible couple of quasi-normed spaces, that is, both \(A_0\) and \(A_1\) are linearly and continuously embedded in some Hausdorff topological vector space. By definition, the K-interpolation space \((A_0, A_1)_\varphi\) has a quasi-norm

\[\|a\|_{(A_0, A_1)_\varphi} = \|K(t, a)\|_\varphi,\]

where \(K(t, a) = K(t, a; A_0, A_1)\) is the K-functional of Peetre [13, 14], defined for \(0 < t < \infty\), \(a \in A_0 + A_1\) as follows:

\[K(t, a) = \inf_{\alpha_0, \alpha_1} \left\{ \|a_0\|_{A_0} + t \|a_1\|_{A_1} \right\},\]

and \(\varphi\) is a quasi-normed space of Lebesgue measurable functions, defined on \((0, \infty)\), with monotone quasi norm as follows: \(|g| \leq |h|\) implies \(\|g\|_\varphi \leq \|h\|_\varphi\) such that \(\min(1, t) \in \varphi\). If \(\varphi = L^q_k(w)\), that is,

\[\|g\|_\varphi = \left(\int_0^\infty |w(t)|g(t)\frac{dt}{t}\right)^{1/q}, \quad 0 < q \leq \infty,\]

we write \((A_0, A_1)_{\varphi(t)}\) instead of \((A_0, A_1)_\varphi\); if \(q = \infty\), the above integral has to be replaced by \(\sup |w(t)|g(t)|\). Here \(w\) is a nonnegative Lebesgue measurable function defined on \((0, \infty)\) and called weight.

In [1, 2] the case when \(w(t) = t^{-\theta}v(t), 0 < \theta < 1\), and \(v\)-repeated logarithms was considered and this was extended to \(v\)-slowly varying in [3]. Certain limiting cases (when \(\theta = 0\)) are treated in [5, 8–10]. The case \(\varphi = L^1_k(w)\) with arbitrary \(w\) was investigated in [6]. In [4, 7, 11] the case when \(\varphi = E(t^{-\theta})\) and \(E\) is rearrangement invariant Banach space on \((0, \infty)\) with the Haar measure \(dt/t\) is treated in detail. Note that in the case \(w(t) = t^{-\theta}v(t), 0 < \theta < 1\), \(v\)-slowly varying, the K- and J-methods are equivalent, but this is not true in the limiting cases \(\theta = 0\) or \(\theta = 1\). The problem of the relation between both real interpolation methods for Banach couples is treated in [13], where general theorems are proven and certain applications are given. More results about the relation between K- and J-methods in the limiting cases are obtained in [15, 16].

We use the notations \(a_1 \leq a_2\) or \(a_2 \geq a_1\) for nonnegative functions or functionals to mean that the quotient \(a_1/a_2\) is bounded; also, \(a_1 \approx a_2\) means that \(a_1 \leq a_2\) and \(a_1 \geq a_2\). We say that \(a_1\) is equivalent to \(a_2\) if \(a_1 \approx a_2\). Recall that the weight \(w\) is called slowly varying if for every \(\varepsilon > 0\) the function \(t^{\varepsilon}w(t)\) is equivalent to a nondecreasing one and the function \(t^{-\varepsilon}w(t)\) is equivalent to a nonincreasing one.

The main results are announced in [17].
2. Formulae for the $K$-Functional

Using the Holmstedt argument, we can prove general formulae for the $K$-functional. Let $K(t, a) = K(t, a; A_0, A_1)$ and let

$$I(t, a) := \|\chi_{(0,t)}(u) K(u, a)\|_{\Phi_0},$$

$$g_0(t) := t \|\chi_{(0,\infty)}(u)\|_{\Phi_0},$$

(4)

$$h_0(t) := \|u\chi_{(0,t)}(u)\|_{\Phi_0}.$$  

\textbf{Theorem 1} (case $((A_0, A_1)_{\Phi_0}, A_1))$. If $a \in (A_0, A_1)_{\Phi_0} + A_1$, then

$$K(\rho(t), a; (A_0, A_1)_{\Phi_0}, A_1) = I(t, a) + \rho(t) \frac{K(t, a)}{t},$$

(5)

where $\rho(t) \equiv g_0(t) + h_0(t)$.

\textbf{Proof.} Let $w = g_0(t) + h_0(t)$, and $B_0 := (A_0, A_1)_{\Phi_0}$, and $B_1 := A_1$. Using monotonicity of $K$, we get

$$I(t, a) \geq h_0(t) \frac{K(t, a)}{t},$$

(6)

$$\|a\|_{B_0} \geq (g_0(t) + h_0(t)) \frac{K(t, a)}{t}.$$  

(7)

Let $a = a_0 + a_1, a_j \in A_j$, and $j = 0, 1$. Then

$$K(t, a) \leq K(t, a_0) + K(t, a_1) \leq \frac{t}{w(t)} \|a_0\|_{B_0} + \frac{t}{w(t)} \|a_1\|_{B_1},$$

whence

$$w(t) \frac{K(t, a)}{t} \leq K(w(t); a; B_0, B_1).$$

(9)

Also,

$$I(t, a) \leq \|\chi_{(0,t)}(u) K(u, a_0)\|_{\Phi_0} + \|\chi_{(0,t)}(u) K(u, a_1)\|_{\Phi_0} \leq \|a_0\|_{B_0} + h_0(t) \|a_1\|_{B_1}.$$  

(10)

Hence

$$I(t, a) \leq K(w(t); a; B_0, B_1).$$

(11)

Now we estimate $K(w(t), a; B_0, B_1)$ from above. To this end, following Holmstedt, we choose decomposition $a = a_0 + a_1$ such that $K(t, a) \approx \|a_0\|_{A_0} + t \|a_1\|_{A_1}$. Then

$$K(u, a_0) \leq \|a_0\|_{A_0} \leq K(t, a),$$

(12)

$$K(u, a_1) \leq u \|a_1\|_{A_1} \leq u \frac{K(t, a)}{t}.$$  

Further,

$$\|a_0\|_{B_0} = \|K(u, a_0)\|_{\Phi_0} \leq \|\chi_{(0,t)}(u) K(u, a_0)\|_{\Phi_0} + \|\chi_{(0,\infty)}(u) K(u, a_0)\|_{\Phi_0},$$

(13)

Denote by $C$ the first term on the right-hand side and by $D$ the second one. We have

$$C \leq I(t, a) + \|\chi_{(0,t)}(u) K(u, a)\|_{\Phi_0},$$

$$\leq I(t, a) + h_0(t) \frac{K(t, a)}{t},$$

(14)

$$D \leq g_0(t) \frac{K(t, a)}{t},$$

whence

$$\|a_0\|_{B_0} \leq I(t, a) + w(t) \frac{K(t, a)}{t}.$$  

(15)

Also,

$$w(t) \|a_1\|_{B_1} = w(t) \|a_1\|_{A_1} \leq w(t) \frac{K(t, a)}{t}.$$  

Thus

$$K(w(t); a; B_0, B_1) \leq I(t, a) + w(t) \frac{K(t, a)}{t}.$$  

(17)

\Box

\textbf{Remark 2.} If the spaces $(A_0, A_1)_{\Phi_0}$ and $A_1$ are too close, then formula (5) might be useless. For example, if $\Phi_0 = L^\infty(1/t)$, then $g_0(t) = 1 = h_0(t)$. In applications we require $\rho$ to be increasing (strictly).

Let

$$J(t, a) := \|\chi_{(0,t)}(u) K(u, a)\|_{\Phi_1},$$

$$g_1(t) := t \|\chi_{(0,\infty)}(u)\|_{\Phi_1},$$

(18)

$$h_1(t) := \|u\chi_{(0,t)}(u)\|_{\Phi_1}.$$  

\textbf{Theorem 3} (case $((A_0, A_0, A_1)_{\Phi_0}, A_1))$. If $a \in A_0 + (A_0, A_1)_{\Phi_0}$, then

$$K(\rho(t), a; (A_0, A_0, A_1)_{\Phi_0}, A_1) = \rho(t) J(t, a) + K(t, a),$$

(19)

where $\rho(t) = t/g_1(t) + h_1(t)$.

\textbf{Proof.} Let $w(t) := t(g_1(t) + h_1(t))$, $B_0 := A_0$, and $B_1 := (A_0, A_1)_{\Phi_0}$. Using monotonicity of $K$, we get

$$J(t, a) \geq \rho(t) \frac{K(t, a)}{t},$$

(20)

$$\|a\|_{B_0} \geq \left( \frac{g_1(t) + h_1(t)}{t} \right) \frac{K(t, a)}{t}.$$  

(21)

Let $a = a_0 + a_1, a_j \in A_j$, and $j = 0, 1$. Then

$$K(t, a) \leq K(t, a_0) + K(t, a_1) \leq \|a_0\|_{B_0} + w(t) \|a_1\|_{B_1},$$

whence

$$K(t, a) \leq K(w(t); a; B_0, B_1).$$

(23)
Also,
\[
J(t, a) \leq \|x(t, \infty) (u) K(u, a_0)\|_{\Phi_1} + \|x(t, \infty) (u) K(u, a_1)\|_{\Phi_1} \leq \frac{g_1(t)}{t} \|a_0\|_{B_0} + \|a_1\|_{B_1}.
\]

Hence
\[
w(t) J(t, a) \leq K(w(t), a; B_0, B_1).
\] (25)

Now we estimate \(K(w(t), a; B_0, B_1)\) from above using decomposition \(a = a_0 + a_1\) such that (12) are satisfied. Then
\[
\|a_0\|_{B_0} = \|a_0\|_{A_0} \leq K(t, a).
\] (26)

We have
\[
\|a_1\|_{B_1} \leq \|x(0, \infty) (u) K(u, a_1)\|_{\Phi_1} + \|x(t, \infty) (u) K(u, a_1)\|_{\Phi_1} = C + D,
\]
\[
C \leq h_1(t) \frac{K(t, a)}{t},
\] (27)
\[
D \leq J(t, a) + g_1(t) \frac{K(t, a)}{t},
\]
whence
\[
w(t) \|a_1\|_{B_1} \leq w(t) J(t, a) + K(t, a).
\] (28)

Hence
\[
K(w(t), a; B_0, B_1) \leq w(t) J(t, a) + K(t, a).
\] (29)

Case 2. If
\[
\frac{g_0}{g_1} \leq \rho \leq \frac{h_0}{h_1}, \quad h \leq \rho \leq g,
\] (33)

then
\[
K(\rho(t), a; B_0, B_1) \approx I(t, a) + \rho(t) J(t, a)
\]
\[
+ g_0(t) \frac{K(t, a)}{t}.
\] (34)

Case 3. If
\[
\frac{g_0}{g_1} \leq \rho \leq \frac{g_0 + h_0}{h_1}, \quad h \leq \rho \leq g,
\] (35)

then
\[
K(\rho(t), a; B_0, B_1) \approx I(t, a) + \rho(t) J(t, a)
\]
\[
+ \rho(t) h_1(t) \frac{K(t, a)}{t}.
\] (36)

Case 4. If at least one of the conditions
\[
g_0/g_1 \leq \rho \leq g_1, \quad h_0 \leq \rho \leq g,
\]
\[
g_0 \leq h_0, \quad \rho \leq h_0/h_1, \quad h \leq \rho \leq g,
\]
\[
g_0 \leq h_0, \quad h_1 \leq g, \quad h \leq \rho \leq g
\]
is satisfied, then
\[
K(\rho(t), a; B_0, B_1) \approx I(t, a) + \rho(t) J(t, a).
\] (37)

Proof. From (6) and (20) it follows that
\[
\frac{g_0(t)}{t} \frac{K(t, a)}{t} \leq I(t, a) \quad \text{if} \quad g_0(t) \leq h_0(t),
\]
\[
\frac{g_0(t)}{t} \frac{K(t, a)}{t} \leq \rho(t) J(t, a) \quad \text{if} \quad \frac{g_0(t)}{g_1(t)} \leq \rho,
\] (38)
\[
\rho(t) h_1(t) \frac{K(t, a)}{t} \leq I(t, a) \quad \text{if} \quad \rho \leq \frac{h_0(t)}{h_1(t)},
\]
\[
\rho(t) h_1(t) \frac{K(t, a)}{t} \leq \rho(t) J(t, a) \quad \text{if} \quad h_1(t) \leq g_1(t).
\]

Let \(a = a_0 + a_1, a_j \in A_j, \text{ and } j = 0, 1\). Then, using estimates (7) and (21), we have
\[
K(t, a) \leq K(t, a_0) + K(t, a_1)
\]
\[
\leq \frac{t}{g_0(t) + h_0(t)} \|q_0\|_{B_0}
\]
\[
+ \frac{t}{g_1(t) + h_1(t)} \|q_1\|_{B_1},
\] (39)
whence
\[
g_0(t) \frac{K(t,a)}{t} \leq K(\rho(t),a;B_0,B_1)
\]
if \( \frac{g_0}{g_1 + h_0} \leq \rho, \)
\[
\rho(t) h_1(t) \frac{K(t,a)}{t} \leq K(\rho(t),a;B_0,B_1)
\]
if \( \rho \leq \frac{g_0 + h_0}{h_1}. \)

Also,
\[
I(t,a) \leq \| \chi_{(0,\ell)}(u) K(u,a_0) \|_{\Phi_j} + \| \chi_{(0,\ell)}(u) K(u,a_1) \|_{\Phi_j} \leq \| a_0 \|_{B_0} + h(t) \| a_1 \|_{B_1}, \]
\[
J(t,a) \leq \| \chi_{(t,\infty)}(u) K(u,a_0) \|_{\Phi_j} + \| \chi_{(t,\infty)}(u) K(u,a_1) \|_{\Phi_j} \leq \frac{1}{g(t)} \| a_0 \|_{B_0} + \| a_1 \|_{B_1}. \]

Hence,
\[
I(t,a) \leq K(\rho(t),a;B_0,B_1) \quad \text{if } h \leq \rho,
\]
\[
\rho(t) J(t,a) \leq K(\rho(t),a;B_0,B_1) \quad \text{if } \rho \leq g. \]

To estimate \( K(\rho(t),a;B_0,B_1) \) from above we use the same decomposition as before with properties (12). Then
\[
\| a_0 \|_{B_0} \leq \| \chi_{(0,\ell)}(u) K(u,a_0) \|_{\Phi_j} + \| \chi_{(0,\ell)}(u) K(u,a_1) \|_{\Phi_j} \leq \| a_0 \|_{B_0} + h(t) \| a_1 \|_{B_1},
\]
whence
\[
\| a_0 \|_{B_0} \leq I(t,a) + (g_0(t) + h_0(t)) \frac{K(t,a)}{t} \leq I(t,a) + g_0(t) \frac{K(t,a)}{t}. \]

Also,
\[
\| a_1 \|_{B_1} \leq \| \chi_{(0,\ell)}(u) K(u,a_1) \|_{\Phi_j} + \| \chi_{(t,\infty)}(u) K(u,a_1) \|_{\Phi_j} = C + D,
\]
whence
\[
\rho(t) \| a_1 \|_{B_1} \leq \rho(t) J(t,a) + \rho(t) h_1(t) \frac{K(t,a)}{t} \leq J(t,a) + \rho(t) h_1(t) \frac{K(t,a)}{t}. \]

Therefore for all weights \( \rho \) we have
\[
K(\rho(t),a;B_0,B_1) \leq I(t,a) + \rho(t) J(t,a) + (g_0(t) + \rho(t) h_1(t)) \frac{K(t,a)}{t}. \]

We give examples that are not entirely covered by the previous papers.

**Example 5** (distant spaces). Let
\[
\| g \|_{\Phi_j} = \left( \int_0^1 \left[ t^{-\eta_j} v_j(t) g(t) \right]^{p_j} \frac{dt}{t} \right)^{1/p_j} \]
\[
+ \left( \int_1^\infty \left[ t^{-\eta_j} \omega_j(t) g(t) \right]^{q_j} \frac{dt}{t} \right)^{1/q_j}, \]
where \( 0 \leq \eta_0 < \theta_1 \leq 1, 0 \leq \theta_1 < \eta_1 \leq 1, 0 < p_j, q_j \leq \infty, \) and \( v_j, \omega_j \) are slowly varying weights and \( j = 0, 1. \) We call these spaces distant if \( \theta_0 < \theta_1 \) or \( \eta_0 < \eta_1, \) in opposition to the case \( \theta_0 = \theta_1 \) and \( \eta_0 = \eta_1. \) We have
\[
g_j(t) \]
\[
= \left\{ \begin{array}{ll}
t \left( 1 + \left( \int_0^t \left[ s^{-\eta_j} v_j(s) \right]^{p_j} \frac{ds}{s} \right)^{1/p_j} \right), & 0 < t < 1, \\
\int_1^t \left[ s^{-\eta_j} \omega_j(s) \right]^{q_j} \frac{ds}{s} \right)^{1/q_j}, & t \geq 1, \end{array} \right.
\]
\[
h_j(t) \]
\[
= \left\{ \begin{array}{ll}
\left( \int_0^t \left[ s^{-1-\eta_j} v_j(s) \right]^{p_j} \frac{ds}{s} \right)^{1/p_j}, & 0 < t < 1, \\
1 + \left( \int_1^t \left[ s^{-1-\eta_j} \omega_j(s) \right]^{q_j} \frac{ds}{s} \right)^{1/q_j}, & t \geq 1. \end{array} \right.
\]

These integrals are convergent due to the property \( \min(1,t) \in \Phi_j, j = 0, 1. \) Moreover,
\[
g_1(t) = \left\{ \begin{array}{ll}
t^{1-\eta_0} v_0(t), & 0 < t < 1, \\
t^{1-\eta_1} \omega_0(t), & t \geq 1, \end{array} \right.
\]
\[
h_0(t) = \left\{ \begin{array}{ll}
t^{1-\theta_0} v_0(t), & 0 < t < 1, \\
t^{1-\theta_1} \omega_0(t), & t \geq 1. \end{array} \right.
\]
Also,

\[ g(t) \approx \begin{cases} \frac{t^{\theta_1}}{v_1(t)} \left( 1 + \left( \int_0^1 \left[ s^{-\theta_1} v_0(s) \right]^{p_1} \frac{ds}{s} \right)^{1/p_1} \right), & 0 < t < 1, \\ \frac{t^{\theta_1}}{v_1(t)} \left( \int_0^\infty \left[ s^{-\theta_1} v_1(s) \right]^{p_1} \frac{ds}{s} \right)^{-1/p_1}, & t \geq 1, \end{cases} \] for \( 0 < t < 1 \) and

\[ h(t) \approx \begin{cases} \frac{t^{1-\theta_0}}{w_0(t)} \left( 1 + \left( \int_0^1 \left[ s^{1-\theta_0} w_0(s) \right]^{q_1} \frac{ds}{s} \right)^{1/q_1} \right), & 0 < t < 1, \\ \frac{t^{1-\theta_0}}{w_0(t)} \left( 1 + \left( \int_1^\infty \left[ s^{1-\theta_1} w_0(s) \right]^{q_1} \frac{ds}{s} \right)^{1/q_1} \right)^{-1}, & t \geq 1. \end{cases} \] for \( t \geq 1 \). Hence,

\[ h \leq \frac{g_0}{h_1} \leq g \]

and therefore

\[ K \left( \rho(t), a; B_0, B_1 \right) = I(t, a) + \rho(t) J(t, a) + g_0(t) h_1(t), \]

where \( B_0 = (A_0, A_1)_{\Phi_0}, B_1 = (A_0, A_1)_{\Phi_1}, \rho(t) \approx g_0(t)/h_1(t). \)

Also,

\[ K \left( \rho(t), a; (A_0, A_1)_{\Phi_0}, A_1 \right) = I(t, a) + g_0(t) \frac{K(t, a)}{t}, \]

where \( \rho(t) = g_0(t). \)

In particular,

\[ g_j(t) \geq h_j(t), \quad 0 < t < 1, \]

\[ g_j(t) \leq h_j(t), \quad t \geq 1. \]
On the other hand, since $1/p_0 \leq 1/p_1$, we have

$$\frac{g_0}{g_1 + h_1} \leq h \leq \frac{h_0}{h_1}$$

(63)

and therefore

$$K\left(\rho(t), a; (A_0, A_1)_{\Psi}, (A_0, A_1)_{\Phi}\right)$$

$$= I(t, a) + \rho(t) f(t, a) + g_0(t) \frac{K(t, a)}{t},$$

(64)

where

$$\rho(t) = \begin{cases} (1 - \log t)^{a_0 - a_1 + 1/p_0 - 1/p_1}, & 0 < t < 1, \\ (1 + \log t)^{b_0 - b_1 + 1/q_0 - 1/q_1}, & t \geq 1. \end{cases}$$

(65)

3. Reiteration

The formulæ for the $K$-functionally imply immediately theorems of reiteration or stability of the $K$-method. In particular, we recover many classical results. For another type of general reiteration theorems see [13]. For example, Theorems 1 and 3 imply the following results.

**Theorem 7.** Let $(A_0, A_1)$ be a compatible couple of quasi-normed spaces. Then

$$\left(A_0, A_1\right) \rho = (A_0, A_1)\Psi,$$

(66)

where

$$\|g\| = \|\chi_{(0,\rho^{-1}(t))} (u) g(u)\|_{\Phi} + \frac{tg(\rho^{-1}(t))}{\rho^{-1}(t)}$$

(67)

and $\rho$ is the same as in Theorem 1 and $\rho(0) = 0, \rho(\infty) = \infty,$ and $\rho(1) = 1; \rho$ is increasing.

**Proof.** We only need to check that min$(1, t) \in \Psi.$ Since

$$\|\chi_{(0,1)}(t) \chi_{(0,\rho^{-1}(t))} (u) t\|_{\Phi} = \|\chi_{(0,1)}(t) h_0 (\rho^{-1}(t))\|_{\Phi}$$

(68)

and $h_0 \leq \rho$, therefore $h_0 (\rho^{-1}(t)) \leq t$; we see that the above quantity is finite. Also

$$\|\chi_{(1,\rho)} (t) \chi_{(0,1)} (u)\|_{\Phi} = \|\chi_{(1,\rho)}\|_{\Phi} h_0 (1) < \infty,$$

$$\|\chi_{(1,\rho)} (t) \chi_{(1,\rho^{-1}(t))}\|_{\Phi} \leq \|\chi_{(1,\rho)}\|_{\Phi} \|\chi_{(1,\rho)}\|_{\Phi}$$

(69)

$< \infty.$

Further, for $t > 1$, we have $g_0(t) \leq t$ and $h_0(t) \leq 1 + \|u\chi_{(1,t)}(u)\|_{\Phi} \leq t$; hence $t \leq \rho^{-1}(t)$. Therefore

$$\|\chi_{(1,\rho)} (t) t\|_{\Phi} \leq \|\chi_{(1,\rho)}\|_{\Phi} < \infty.$$ 

(70)

**Theorem 8.** Let $(A_0, A_1)$ be a compatible couple of quasi-normed spaces. Then

$$\left(A_0, (A_0, A_1)_{\Psi}\right) = (A_0, A_1)_{\Psi},$$

(71)

where

$$\|g\| = \|t \chi_{(\rho^{-1}(t),\infty)} g (\rho^{-1}(t))\|_{\Phi} + \|g(\rho^{-1}(t))\|_{\Phi}$$

(72)

and $\rho$ is the same as in Theorem 3 and $\rho(0) = 0, \rho(\infty) = \infty,$ and $\rho(1) = 1; \rho$ is increasing.

**Proof.** We only need to check that min$(1, t) \in \Psi.$ We have

$$\|\chi_{(0,1)}(t) \chi_{(\rho^{-1}(t),\infty)} (u)\|_{\Phi} < \infty$$

(73)

and, using also $\rho(u) \leq u/g_1(u)$,

$$\|\chi_{(1,\rho)} (t) \chi_{(\rho^{-1}(t),\infty)} (u)\|_{\Phi} < \infty.$$ 

(74)

As above, we check that $g_1(t) \leq 1$ and $h_1(t) \leq 1$ for $0 < t < 1$. Hence $\rho(t) \geq t$ for $0 < t < 1$. Then

$$\|\chi_{(0,1)}(t) t\|_{\Phi} < \infty.$$ 

(75)

In some cases the quasi norm of $\Psi$ can be simplified.

**Example 9 (distant spaces).** Let

$$\|g\| = \left(\int_0^1 \left[v_0 (t) g(t)\right]^{p_0} \frac{dt}{t}\right)^{1/p_0}$$

(76)

$$+ \left(\int_0^\infty \left[w_0 (t) g(t)\right]^{q_0} \frac{dt}{t}\right)^{1/q_0},$$

where $0 < p_0, q_0 < \infty$ and $v_0, w_0$ are slowly varying weights and let

$$\|g\| = \left(\int_0^1 \left[t^{-1} v(t) g(t)\right]^p \frac{dt}{t}\right)^{1/p}$$

(77)

$$+ \left(\int_1^\infty \left[t^{-1} w(t) g(t)\right]^q \frac{dt}{t}\right)^{1/q},$$

where $0 < p, q < \infty$ and $v, w$ are slowly varying weights. Then

$$\left(A_0, (A_0, A_1)_{\Phi}\right) = (A_0, A_1)_{\Psi},$$

$$\|g\| = \left(\int_0^1 \left[t^{-1} v(\rho(t)) g(t)\right]^p \frac{dt}{t}\right)^{1/p}$$

(78)

$$+ \left(\int_1^\infty \left[t^{-1} w(\rho(t)) g(t)\right]^q \frac{dt}{t}\right)^{1/q},$$
where

\[
\rho(t) = \begin{cases} 
  t \left(1 + \int_0^t \sqrt[p]{u} (s) \frac{ds}{s} \right)^{1/p} , & 0 < t \leq 1, \\
  t \left(\int_0^\infty \sqrt[q]{w}(s) \frac{ds}{s} \right)^{1/q}, & t \geq 1.
\end{cases}
\]

Indeed, we have

\[
K(\rho(t),a;A_0,A_1) \approx I(t,a) + \frac{\rho(t)}{t} K(t,a;A_0,A_1) \tag{79}
\]

for \(0 < t < 1\) and

\[
K(\rho(t),a;A_0,A_1) \approx I_1(t,a)
\]

for \(t \geq 1\), where

\[
I(t,a) = \left(\int_0^t (\sqrt[p]{v(u)} K(u,a) - u) \frac{du}{u}\right)^{1/p}
\]

and

\[
I_1(t,a) = \left(\int_1^t (w(u) K(u,a)) \frac{du}{u}\right)^{1/q}.
\]

Let \(p(1) = 1\). Then

\[
\|a\|_{(A_0,A_1)_{\rho(1)},A_1} = B_0^{1/p} + B_1^{1/p} + C_0^{1/q} + C_1^{1/q} + C_2^{1/q},
\]

where

\[
B_0 = \int_0^1 \left(\frac{\sqrt[p]{v(t)}}{\sqrt[p]{\rho(t)}}\right)^p (I(t,a))^p \frac{d\rho}{\rho},
\]

\[
B_1 = \int_0^1 \left(\frac{\sqrt[p]{v(t)}}{\sqrt[p]{\rho(t)}}\right)^p \left(\frac{\sqrt[p]{\rho(t)}}{t} K(t,a)\right)^p \frac{d\rho}{\rho},
\]

\[
C_0 = \left(\int_0^1 \sqrt[p]{v(s)} K(s,a) \frac{ds}{s}\right)^{1/p},
\]

\[
C_1 = \left(\int_1^\infty \sqrt[q]{w}(t) \frac{dt}{t}\right)^{q},
\]

\[
C_2 = \left(\int_1^\infty \sqrt[q]{w}(t) \frac{dt}{t}\right)^{q} \left(\frac{I_1(t,a)}{\sqrt[p]{\rho(t)}}\right)^{q}\frac{d\rho}{\rho},
\]

We can choose equivalent \(\rho\) so that \(d\rho/\rho = dt/t\) (see [6]); hence

\[
B_1 \approx \int_0^1 \left(\frac{\sqrt[p]{v(\rho)} K(u,a)}{u}\right)^p \frac{du}{u},
\]

\[
C_2 \approx \int_1^\infty \left(\frac{\sqrt[p]{w(\rho)} K(u,a)}{u}\right)^q \frac{du}{u}.
\]

It is sufficient to prove that

\[
B_0 + C_0 \leq \int_0^1 \left(\frac{\sqrt[p]{v(\rho)} K(u,a)}{u}\right)^p \frac{du}{u},
\]

\[
C_1 \leq \int_1^\infty \left(\frac{\sqrt[p]{w(\rho)} K(u,a)}{u}\right)^q \frac{du}{u}.
\]

To estimate \(C_0\) for \(p \leq p_0\), we use monotonicity of the interpolation scale, while for \(p > p_0\) we apply Hölder’s inequality. Namely, if \(p \leq p_0\),

\[
\int_0^1 \left[\frac{\sqrt[p]{v(\rho)} K(u,a)}{u}\right]^p \frac{du}{u} \leq \left(\int_0^1 \left[\frac{\sqrt[p]{v(\rho)} K(u,a)}{u}\right]^{p/p} \frac{du}{u}\right)^{p/p} \tag{88}
\]

and if \(p > p_0\),

\[
\int_0^1 \left[\frac{\sqrt[p]{v(\rho)} K(u,a)}{u}\right]^p \frac{du}{u} \leq \left(\int_0^1 \left[\frac{\sqrt[p]{v(\rho)} K(u,a)}{u}\right]^p \frac{du}{u}\right)^{p/p} \tag{89}
\]

Next we check only the inequality for \(B_0\) and the inequality for \(C_1\) being similar. For \(p_0 = p_0\), it follows from Fubini’s theorem. \(B_0\) can be estimated from above if \(p > p_0\) by the Muckenhoupt result [18].

Let \(v_i \equiv w_1^{1-s}(\phi \int_0^1 \sqrt[p]{w}(u) du)\), where \(1 < s < \infty\), \(q > 0\), and \(w_1 > 0\). Then

\[
\int_0^1 \left(\int_0^\infty \sqrt[p]{v(u) h(u) du}^s \right) \sqrt[p]{w}(t) dt \leq \int_0^1 h^s(t) v_1(t) dt. \tag{90}
\]

We apply this for \(v(t) = v_1(t) h(t) K(t,a)\),

\[
w_1(t) = \left(\frac{(\sqrt[p]{v(t)})/\sqrt[p]{\rho(t)}}{\sqrt[p]{\rho(t)}}\right)^{\rho(t)},
\]

\[
s = \frac{p}{p_0}. \tag{91}
\]

Then \(v_1(t) = [\sqrt[p]{v(t)} v(\rho(t))/\sqrt[p]{\rho(t)})^{\rho(t)} t^{-1}\) and hence

\[
B_0 \leq \int_0^1 \left(\frac{\sqrt[p]{v(t) K(t,a)}}{u}\right)^p \frac{du}{u}. \tag{92}
\]
Further, if $p < p_0$, we estimate $B_0$ from above using integration by parts. We have

$$B_0 \approx \int_0^1 (I(a,t))^p \, d(\int_0^1 \left( \frac{v(t)}{\rho(t)} \right)^p \, d\rho).$$

(93)

Indeed, let $a \in (A_0,A_1)$. Using that $v(t)/\rho$ is equivalent to a decreasing function, we have

$$S := (I(a,t))^{1/p} \int_0^t \left( \frac{v(t)}{\rho(t)} \right)^p \, du,$$

and since $t v_0(t) \leq \rho(t), \ p < p_0,

$$S \leq \int_0^t \left( \frac{v(t) K(u,a) v_0(u)}{\rho(u)} \right)^p \, du,$$

(94)

The integral above has a limit zero when $t \to 0$. Thus (93) is true. Further,

$$B_0 \leq \int_0^1 (I(a,t))^{1/p} \left( \int_0^t \left( \frac{v(t)}{\rho(t)} \right)^p \, dt \right),$$

(95)

or

$$B_0 \leq \int_0^1 \left( v_0(t) K(t,a) \right)^p \left( \frac{v(t)}{\rho(t)} \right)^p \, dt.$$

(96)

Since $\rho(t) = g_0(t) \geq t v_0(t)$, we get (86).

Now we consider examples of nondistant spaces: $B := (A_p(A_0,A_1)_{w_1,(s)},h(t,a),u_1$-slowly varying, $0 < p < \infty$, $0 < q < \infty$.

In order to handle this case, we need the following result.

**Lemma 10.** One has

$$\int_0^\infty \left( \int_0^\infty g(t) \psi(u) \, du \right)^s w_0(t) \, dt,$$

$$= \int_0^\infty g^s(t) v_0(t) \, dt;$$

(98)

$\varphi \geq 0, g \geq 0, w_0 \geq 0, v_0 \geq 0$, and $0 < s < \infty$, if the following conditions are satisfied:

$$\int_0^\infty \psi(u) g(u) \, du \geq s \int_0^\infty \psi(u) \, du,$$

$$\int_0^\infty \psi(u) \, du \leq \int_0^\infty w_0(u) \, du,$$

(99)

(100)

where

$$v_0(t) = \varphi(t) \left( \int_0^\infty \varphi(u) \, du \right)^{s-1} \int_0^t w_0(u) \, du,$$

for $0 < s \leq 1$.

$$v_0(t) = w_0^{1-s}(t) \left( \varphi(t) \int_0^t w_0(u) \, du \right)^s,$$

for $1 \leq s < \infty$.

Proof. The estimate (98) from above is a consequence of Lemmas 3.2, 3.3 in [6]. Using (99), we get an estimate from below with $\int_0^\infty g^s(t) u(t) \varphi(u) \, du$. Now condition (100) ensures the equivalence in (98).

Theorem 11. Let $w_1$ be a slowly varying weight and let $\rho(t) = 1/\int_0^\infty w_1^q(\psi(u) \, du, 0 < \rho < \infty$, satisfying $p(0) = 0, \rho(\infty) = \infty$. If

$$\int_0^t h^q(s) \frac{ds}{s} = h^q(t) \frac{t^q}{q}, \quad 0 < q < \infty,$$

(102)

then

$$B := (A_p(A_0,A_1)_{w_1,(s)},h(t,a),u_1 \text{-slowly varying, 0 < p < \infty, satisfying p(0) = 0, \rho(\infty) = \infty}.$$

$$B = (A_p(A_0,A_1)_{w_1,(s)},h(t,a),u_1 \text{-slowly varying, 0 < p < \infty, satisfying p(0) = 0, \rho(\infty) = \infty})$$

(103)

Proof. We have

$$K(\rho(t,a),w_1,(s),h(t,a)) = \rho(t) \int_0^\infty u^q(s) \, ds \left( \int_0^\infty w_1^q(s) \, ds \right)^{1/p};$$

(104)

hence

$$\|a\|_B^p \approx \int_0^\infty h^q(\rho(t)) \, dt$$

$$\int_0^\infty \left( \int_0^\infty w_1^q(s) \, ds \right)^s \, ds \approx \int_0^\infty h^q(\rho(t)) \, dt.$$
For example, condition (102) is true if $h(t) = t^{-\theta}a(t)$, $0 \leq \theta < 1$, $a$-slowly varying.

**Competing Interests**

The authors declare that they have no competing interests.

**References**


