1. Introduction

Let \( p \in \mathbb{N} = \{1, 2, \ldots\} \) and denote \( \mathcal{A}_p \) as the class of multivalent functions of the form
\[
f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,
\]
which are analytic in the open unit disk:
\[
U = \{z \in \mathbb{C} : |z| < 1\}.
\]
For two parameters \( \alpha \in [-p, p] \) and \( \beta \geq 0 \), function \( f(z) \in \mathcal{A}_p \) is said to be in class \( \mathcal{UST}(p, \alpha, \beta) \) of \( p \)-valent \( \beta \)-uniformly star-like functions of order \( \alpha \) in \( U \), if and only if
\[
\Re \left( \frac{zf'(z)}{f(z)} - \alpha \right) \geq \beta \left| \frac{zf'(z)}{f(z)} - p \right|,
\]
where \( \Re(.) \) denotes taking the real part of argument. On the other hand, function \( f(z) \in \mathcal{A}_p \) is said to be in class \( \mathcal{UCV}(p, \alpha, \beta) \) of \( p \)-valent \( \beta \)-uniformly convex functions of order \( \alpha \) in \( U \), if and only if
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} - \alpha \right) \geq \beta \left| 1 + \frac{zf''(z)}{f'(z)} - p \right|.
\]
We note from (3) and (4) that
\[
f(z) \in \mathcal{UCV}(p, \alpha, \beta) \iff \frac{zf'(z)}{p} \in \mathcal{UST}(p, \alpha, \beta).
\]

The classes \( \mathcal{UST}(p, \alpha, \beta) \) and \( \mathcal{UCV}(p, \alpha, \beta) \) were introduced recently by Khairnar and More [1]. Various subclasses of analytic and univalent or multivalent functions were studied in many papers (see, e.g., [2–4]). Recently, Nishiwaki and Owa in [5] introduced two classes \( \mathcal{MD}(\alpha, \beta) \) consisting of all functions \( f(z) \in \mathcal{A}_1 \), which satisfy
\[
\Re \left( \frac{zf'(z)}{f(z)} - \alpha \right) < \beta \left| \frac{zf'(z)}{f(z)} - 1 \right|,
\]
and \( \mathcal{ND}(\alpha, \beta) \) consisting of all functions \( f(z) \in \mathcal{A}_1 \), which satisfy
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} - \alpha \right) < \beta \left| \frac{zf''(z)}{f'(z)} \right|.
\]
where \( \alpha \geq 1 \) and \( \beta \leq 0 \). We notice from definitions of these classes that

\[
f(z) \in \mathcal{N}(\alpha, \beta) \iff z^\alpha f(z) \in \mathcal{M}(\alpha, \beta).
\]

For each \( f(z) \in \mathcal{A}_p \), it is easily seen upon differentiating both sides of (1) \( q \) times with respect to \( z \) that

\[
f^{(q)}(z) = \delta(p, q) z^{p-q} \sum_{n=p+1}^{\infty} \delta(n, q) a_n z^{n-q},
\]

where \( p > q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), and \( \delta(p, q) \) denotes \( q \)-permutations of \( p \) objects; that is,

\[
\delta(p, q) = \frac{p!}{(p-q)!}.
\]

Let \( q, m \in \mathbb{N}_0 \) and \( p \in \mathbb{N} \) such that \( p > q + m \), and assume that

\[
-\delta(p - q, m) \leq \alpha < \delta(p - q, m), \quad \beta \geq 0.
\]

Srivastava et al. in [6] introduced a subclass of the \( \mathcal{M}(\alpha, \beta) \) consisting of functions related to close-to-convexity, starlikeness, and convexity. K. I. Noor defined some subclasses that include extreme points, and integral means inequalities for this subclass \( \mathcal{M}_m(p, q, \alpha, \beta) \) of multivalent functions involving higher-order derivatives.

### 2. Coefficient Inequalities

We derive sufficient conditions for \( f(z) \) which are given by using coefficient inequalities.

**Theorem 2.** Let \( f(z) \) be a function of form (1). If the coefficients of \( f(z) \) satisfy

\[
\sum_{n=p+1}^{\infty} \Phi(n, p, q, m, \alpha, \beta) |a_n| \leq \delta(p, q)(\alpha - |2\delta(p - q, m) - \alpha|),
\]

where

\[
\Phi(n, p, q, m, \alpha, \beta) = \delta(n, q) 
\]

\[
\cdot \left| \delta(n - q, m) + \delta(p - q, m) - \alpha \right| 
\]

\[
+ \left| \delta(n - q, m) - \delta(p - q, m) - \alpha \right| 
\]

\[
- 2\beta \left| \delta(n - q, m) - \delta(p - q, m) \right|,
\]

then, \( f(z) \) is in class \( \mathcal{M}_m(p, q, \alpha, \beta) \).

**Proof.** Suppose that inequality (14) holds, and denote

\[
F(z) = \frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)} - \alpha 
\]

\[
- \beta \frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)} - \delta(p - q, m),
\]

From the definition, we can verify that

\[
\left| \frac{F(z) + \delta(p - q, m)}{F(z) - \delta(p - q, m)} \right| < 1,
\]

then \( f(z) \in \mathcal{M}_m(p, q, \alpha, \beta) \). In fact, denoting

\[
\Psi^+(z; f, m, q, \alpha, \beta) = z^m f^{(q+m)}(z) - \alpha f^{(q)}(z) 
\]

\[
\pm \delta(p - q, m) f^{(q)}(z) 
\]

\[
- \beta e^{i\theta} z^m f^{(q+m)}(z) - \delta(p - q, m) f^{(q)}(z),
\]

\[
\Sigma^+(z; m, q, \alpha) = \sum_{n=p+1}^{\infty} \delta(n, q) 
\]

\[
\cdot \left( \delta(n - q, m) \pm \delta(p - q, m) - \alpha \right) a_n z^{n-p},
\]

\[
\Xi^+(m, p, q, \alpha) = \sum_{n=p+1}^{\infty} \delta(n, q) 
\]

\[
\cdot \left| \delta(n - q, m) \pm \delta(p - q, m) - \alpha \right| a_n,
\]

and

\[
\Xi^-(m, p, q, \alpha) = \sum_{n=p+1}^{\infty} \delta(n, q) 
\]

\[
\cdot \left| \delta(n - q, m) \mp \delta(p - q, m) - \alpha \right| a_n,
\]

we obtain several properties including the coefficient inequalities, distortion theorems, extreme points, and integral means inequalities for this subclass.
we have
\[
\frac{|F(z) + \delta(p-q,m)|}{|F(z) - \delta(p-q,m)|} = \frac{\Psi^+(z; f, m, p, q, \alpha, \beta)}{\Psi^-(z; f, m, p, q, \alpha, \beta)}
\]
\[
= \frac{\delta(p,q)(2\delta(p-q,m) - \alpha) + \Sigma^+(z; m, p, q, \alpha) - \beta e^\theta |\Sigma^-(z; m, p, q, 0)|}{\delta(p,q) - \Sigma^-(z; m, p, q, \alpha) + \beta e^\theta |\Sigma^+(z; m, p, q, 0)|}
\]
\[
\leq \frac{\delta(p,q)\Sigma^+(m, p, q, \alpha) + \Sigma^-(m, p, q, 0)}{\Sigma^-(m, p, q, \alpha) + \Sigma^+(m, p, q, 0)}.
\]

Here, we use technology \( f(z) = e^{i\theta}|f(z)| \). If (14) satisfies, we drive that the last expression above is bounded by 1 which implies \( f(z) \in \mathcal{M}_m(\alpha, \beta) \). Thus, the proof of Theorem 2 is completed.

**Example 3.** Function \( f(z) \) given by
\[
f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{\delta(p,q)(\alpha - 2\delta(p-q,m) - |\alpha|)}{(n+\gamma)(n+\gamma+1)} \Phi(n, p, q, m, \alpha, \beta) z^n
\]
belongs to class \( \mathcal{M}_m(\alpha, \beta) \) for \( \gamma > -(p+1) \) and \( \xi_n \in \mathbb{C} \) with \( |\xi_n| = 1 \).

As a special case of Theorem 2, as in [2], we can obtain the following corollary.

**Corollary 4.** Function \( f(z) \in A_1 \) is in class \( \mathcal{M}_m(\alpha, \beta) \), if
\[
\sum_{n=2}^{\infty} \left| n+1 - \alpha \right| + \left| n-1 - \alpha \right| - 2\beta (n-1) \cdot |a_n| \leq \alpha - |2 - \alpha|.
\]

In view of Theorem 2, we introduce subclass \( \mathcal{M}_m^*(\alpha, \beta) \) which consists of functions of the form
\[
f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n
\]

whose Taylor-Maclaurin coefficients \( a_n \) are nonnegative and satisfy inequality (14). By the coefficient inequalities for classes \( \mathcal{M}_m^*(\alpha, \beta) \), we have the following theorem.

**Theorem 5.** If \( \beta_1 \leq \beta_2 \leq 0 \), then
\[
\mathcal{M}_m^*(p, q; \alpha, \beta_1) \subset \mathcal{M}_m^*(p, q; \alpha, \beta_2).
\]

Since \( \mathcal{M}_1(1, 0; \alpha, \beta) = \mathcal{M}(\alpha, \beta) \), we get the following corollary, which is a theorem in [2].

**Corollary 6.** If \( \beta_1 \leq \beta_2 \leq 0 \), then
\[
\mathcal{M}(\alpha, \beta_1) \subset \mathcal{M}(\alpha, \beta_2).
\]

### 3. Distortion Theorems

**Lemma 7.** If \( f(z) \in \mathcal{M}_m^*(p, q; \alpha, \beta) \), then there exists \( p_0 \in \mathbb{N} \) such that
\[
\sum_{n=p_0+1}^{\infty} a_n \leq A_{p_0},
\]
where
\[
A_{p_0} = \frac{\delta(p,q)(\alpha - 2\delta(p-q,m) - |\alpha|)}{\Phi(p_0+1, p, q, m, \alpha, \beta)} - \sum_{n=p_0+1}^{\infty} \phi(n, p, q, m, \alpha, \beta) a_n.
\]

and \( \Phi(n, p, q, m, \alpha, \beta) \) is given in (15).

**Proof.** From the definition of \( \Phi(n, p, q, m, \alpha, \beta) \), there exists \( p_0 \in \mathbb{N} \) such that function \( \Phi(n, p, q, m, \alpha, \beta) \) is increasing with respect to \( n \) when \( n > p_0 \). According to Theorem 2, we have
\[
\sum_{n=p_0+1}^{\infty} \Phi(n, p, q, m, \alpha, \beta) a_n \leq \delta(p,q)(\alpha - 2\delta(p-q,m) - |\alpha|)
\]
From
\[
\Phi(p_0+1, p, q, m, \alpha, \beta) \sum_{n=p_0+1}^{\infty} a_n \leq \sum_{n=p_0+1}^{\infty} \Phi(n, p, q, m, \alpha, \beta) a_n,
\]

we have
\[
\Phi(p_0 + 1, p, q, m, \alpha, \beta) \sum_{n=p+1}^{\infty} a_n
\]
\[
\leq \delta(p, q) (\alpha - |2\delta(p - q, m) - \alpha|) - \sum_{n=p+1}^{p} \Phi(n, p, q, m, \alpha, \beta) a_n.
\]
This implies that inequality (25) holds.

Using the same argument, we obtain the following inequality.

**Lemma 8.** If \( f(z) \in M \mathcal{D}_m^{*}(p, q; \alpha, \beta) \), then there exists \( p_0 \in \mathbb{N} \) such that
\[
\sum_{n=p_0+1}^{\infty} n a_n \leq B_{p_0},
\]
where

\[
B_{p_0} = \frac{(p_0 + 1) \left[ \delta(p, q) (\alpha - |2\delta(p - q, m) - \alpha|) - \sum_{n=p_0+1}^{p_0} \Phi(n, p, q, m, \alpha, \beta) a_n \right]}{\Phi(p_0 + 1, p, q, m, \alpha, \beta)},
\]
and \( \Phi(n, p, q, m, \alpha, \beta) \) is defined by (15).

**Theorem 9.** Let \( f(z) \) be a function in class \( M \mathcal{D}_m^{*}(p, q; \alpha, \beta) \). Then, for \( |z| = r < 1 \),

\[
|f(z)| \leq r^p + \sum_{n=p+1}^{p} a_n r^n + A_p r^{p+1},
\]
\[
|f(z)| \geq r^p - \sum_{n=p+1}^{p} a_n r^n - A_p r^{p+1},
\]
where \( A_p \) and \( B_p \) are given in Lemmas 7 and 8, respectively.

**Proof.** Let \( f(z) \) be a function of form (22). For \( |z| = r < 1 \), using Lemma 7, we have
\[
|f(z)| \leq |z|^p + \sum_{n=p+1}^{p} |a_n| \cdot |z|^n + \sum_{n=p+1}^{\infty} |a_n| \cdot |z|^n
\]
\[
\leq |z|^p + \sum_{n=p+1}^{p} |a_n| \cdot |z|^n + |z|^{p+1} \sum_{n=p+1}^{\infty} |a_n|
\]
\[
\leq r^p + \sum_{n=p+1}^{p} a_n r^n + A_p r^{p+1},
\]
\[
|f(z)| \geq |z|^p - \sum_{n=p+1}^{p} |a_n| \cdot |z|^n - |z|^{p+1} \sum_{n=p+1}^{\infty} |a_n|
\]
\[
\geq r^p - \sum_{n=p+1}^{p} a_n r^n - A_p r^{p+1}.
\]

Using the same argument, we can prove the following result.

**Theorem 10.** Let \( f(z) \) be a function in class \( M \mathcal{D}_m^{*}(p, q; \alpha, \beta) \). Then, for \( |z| = r < 1 \),
\[
|f'(z)| \leq pr^{p-1} + \frac{p}{p_n} a_n r^{n-1} + B_p r^n,
\]
\[
|f''(z)| \geq pr^{p-1} - \frac{p}{p_n} a_n r^{n-1} - B_p r^n,
\]
where \( A_p \) and \( B_p \) are given in Lemmas 7 and 8, respectively.

### 4. Extreme Points

**Theorem 11.** Let \( f_p(z) = z^p \) and, for each \( n = p + 1, p + 2, \ldots \), define
\[
f_n(z) = z^p + \frac{\delta(p, q) (\alpha - |2\delta(p - q, m) - \alpha|)}{\Phi(n, p, q, m, \alpha, \beta)} z^n,
\]
where \( \Phi(n, p, q, m, \alpha, \beta) \) is defined by (15). Then \( f(z) \in M \mathcal{D}_m^{*}(p, q; \alpha, \beta) \) if and only if it can be expressed in the form
\[
f(z) = \sum_{n=p}^{\infty} \lambda_n f_n(z),
\]
where \( \lambda_n \geq 0 \) for all \( n = p, p + 1, \ldots \), and \( \sum_{n=p}^{\infty} \lambda_n = 1 \).

**Proof.** Suppose that
\[
f(z) = \sum_{n=p}^{\infty} \lambda_n f_n(z) = \lambda_p f_p(z) + \sum_{n=p+1}^{\infty} \lambda_n f_n(z)
\]
\[
= z^p + \sum_{n=p+1}^{\infty} \frac{\delta(p, q) (\alpha - |2\delta(p - q, m) - \alpha|)}{\Phi(n, p, q, m, \alpha, \beta)} z^n.
\]
Then,
\[ \sum_{n=p+1}^{\infty} \Phi(n, p, q, m, \alpha, \beta) \lambda_n \]
\[ = \sum_{n=p+1}^{\infty} \frac{\delta(p, q) (\alpha - 2\delta(p - q, m) - \alpha)}{\Phi(n, p, q, m, \alpha, \beta)} \lambda_n \]
\[ = \sum_{n=p+1}^{\infty} \lambda_n \Phi(n, p, q, m, \alpha, \beta) \delta(p, q) (\alpha - 2\delta(p - q, m) - \alpha) \]
\[ = \delta(p, q) \Phi(n, p, q, m, \alpha, \beta) \lambda_n \]
\[ \leq \delta(p, q) \Phi(n, p, q, m, \alpha, \beta) \lambda_n \]
and
\[ \lambda_n \]
we denote
\[ \lambda_n = \frac{\Phi(n, p, q, m, \alpha, \beta)}{\delta(p, q) (\alpha - 2\delta(p - q, m) - \alpha)} a_n \]
\[ n = p + 1, p + 2, \ldots \]}

Thus, it follows from Theorem 2 that \( f(z) \in \mathcal{M} \mathcal{D}_m^*(p, q; \alpha, \beta) \). Conversely, suppose that \( f(z) \in \mathcal{M} \mathcal{D}_m^*(p, q; \alpha, \beta) \). Since
\[ a_n \leq \frac{\delta(p, q) (\alpha - 2\delta(p - q, m) - \alpha)}{\Phi(n, p, q, m, \alpha, \beta)} \]
\[ n = p + 1, p + 2, \ldots \]}

And \( \lambda_p = 1 - \sum_{n=p+1}^{\infty} \lambda_n \). By a simple calculation, we get \( f(z) \in \mathcal{M} \mathcal{D}_m^*(p, q; \alpha, \beta) \).

**Corollary 12.** The extreme points of \( \mathcal{M} \mathcal{D}_m^*(p, q; \alpha, \beta) \) are functions \( f_p(z) = z^p \) and
\[ f_n(z) = \frac{\Phi(n, p, q, m, \alpha, \beta)}{\delta(p, q) (\alpha - 2\delta(p - q, m) - \alpha)} \sum_{n=1}^{\infty} a_n z^{n-p} \]
for each \( n = p + 1, p + 2, \ldots \) \( \square \)

**5. Integral Means Inequalities**

Assume that two functions \( f(z) \) and \( g(z) \) are analytic in \( \mathcal{U} \). We say that \( f(z) \) is subordinate to \( g(z) \), written as \( f(z) \leq g(z) \), if there exists an analytic function \( w(z) \) in \( \mathcal{U} \) with \( w(0) = 1 \) and \( |w(z)| < 1 \) such that \( f(z) = g(w(z)) \).

**Lemma 13** (see [11]). If \( f(z) \) and \( g(z) \) are analytic in \( \mathcal{U} \) with \( f(z) < g(z) \), then, for \( \mu > 0 \) and \( z = re^{i\theta} \) \((0 < r < 1)\),
\[ \int_0^{2\pi} |f(z)|^\mu d\theta \leq \int_0^{2\pi} |g(z)|^\mu d\theta . \]
\[ \int_0^{2\pi} |f(z)|^\mu d\theta \leq \int_0^{2\pi} |g(z)|^\mu d\theta . \]

**Theorem 14.** Let \( f(z) \in \mathcal{M} \mathcal{D}_m^*(p, q; \alpha, \beta) \) and \( f_n(z) \) be given by (35). Suppose that
\[ \sum_{n=p+1}^{\infty} a_n \leq \frac{\Phi(n, p, q, m, \alpha, \beta)}{\delta(p, q) (\alpha - 2\delta(p - q, m) - \alpha)} \]
\[ \lambda_n \]

If there exists function \( w(z) \), \( z \in \mathcal{U} \), that satisfied the condition
\[ w(z) = \left( \frac{\Phi(n, p, q, m, \alpha, \beta)}{\delta(p, q) (\alpha - 2\delta(p - q, m) - \alpha)} \right)^{1/(n-p)} \sum_{n=p+1}^{\infty} a_n z^{n-p} \]
then, for \( z = re^{i\theta} \), \( 0 < r < 1 \), we have
\[ \int_0^{2\pi} |f(z)|^\mu d\theta \leq \int_0^{2\pi} |f_n(z)|^\mu d\theta . \]
\[ \int_0^{2\pi} |f(z)|^\mu d\theta \leq \int_0^{2\pi} |f_n(z)|^\mu d\theta . \]

**Proof.** In order to obtain the result, it is necessary to prove the following inequality:
\[ \int_0^{2\pi} \left| 1 + \sum_{n=p+1}^{\infty} a_n z^{n-p} \right|^\mu d\theta \leq \int_0^{2\pi} \left| 1 + \sum_{n=p+1}^{\infty} a_n z^{n-p} \right|^\mu d\theta . \]
\[ \int_0^{2\pi} \left| 1 + \sum_{n=p+1}^{\infty} a_n z^{n-p} \right|^\mu d\theta \leq \int_0^{2\pi} \left| 1 + \sum_{n=p+1}^{\infty} a_n z^{n-p} \right|^\mu d\theta . \]

From Lemma 13, it is sufficient to verify the subordination:
\[ 1 + \sum_{n=p+1}^{\infty} a_n z^{n-p} \]
\[ < 1 + \sum_{n=p+1}^{\infty} a_n z^{n-p} \]
\[ = 1 + \frac{\delta(p, q) (\alpha - 2\delta(p - q, m) - \alpha)}{\Phi(n, p, q, m, \alpha, \beta)} w(z)^{n-p} . \]
\[ = 1 + \frac{\delta(p, q) (\alpha - 2\delta(p - q, m) - \alpha)}{\Phi(n, p, q, m, \alpha, \beta)} w(z)^{n-p} . \]

We find that
\[ w(z) = \left( \frac{\Phi(n, p, q, m, \alpha, \beta)}{\delta(p, q) (\alpha - 2\delta(p - q, m) - \alpha)} \right)^{1/(n-p)} \sum_{n=p+1}^{\infty} a_n z^{n-p} \]
\[ \sum_{n=p+1}^{\infty} a_n z^{n-p} \]
\[ = \left| w(z) \right|^{1/(n-p)} \]
which readily yields \( w(0) = 0 \) and
\[ \left| w(z) \right|^{1/(n-p)} \]
\[ \leq \left| w(z) \right|^{1/(n-p)} \]
\[ \leq |z| \sum_{n=p+1}^{\infty} a_n \]
\[ \leq |z| \sum_{n=p+1}^{\infty} a_n \]
\[ \leq |z| < 1 . \]
This means that the hypotheses of \( w(z) \) are satisfied and the theorem is proved.

**Competing Interests**

The authors declare that there are no competing interests regarding the publication of this paper.

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