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Research Article

Certain Subclasses of Multivalent Functions Defined by Higher-Order Derivative

Xiaofei Li,[1] Deng Ding,[1] Liping Xu,[2] Chuan Qin,[3] and Songbo Hu[4]

[1] *Department of Mathematics, University of Macau, Taipa 999078, Macau*
[2] *School of Information and Mathematics, Yangtze University, Jingzhou 434000, China*
[3] *Engineering and Technology College, Yangtze University, Xueyuan Rd., Jingzhou 434000, China*
[4] *School of Health Sciences, Wuhan University, Wuhan 430072, China*

Correspondence should be addressed to Liping Xu; xlp211@126.com

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In this paper, we define and study some subclasses of multivalent analytic functions of higher order in the unit disc. These classes generalize some classes previously studied. We obtain coefficient inequalities, distortion theorems, extreme points, and integral mean inequalities. We derive some results as special cases.

1. Introduction

Let $p \in \mathbb{N} = \{1, 2, \ldots\}$ and denote $\mathcal{A}_p$ as the class of multivalent functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk:

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}. \tag{2}$$

For two parameters $\alpha \in [-p, p]$ and $\beta \geq 0$, function $f(z) \in \mathcal{A}_p$ is said to be in class $\mathcal{UST}(p, \alpha, \beta)$ of p-valent β-uniformly star-like functions of order α in $\mathbb{U}$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) \geq \beta\left|\frac{zf'(z)}{f(z)} - p\right|, \tag{3}$$

where $\Re(\cdot)$ denotes taking the real part of argument. On the other hand, function $f(z) \in \mathcal{A}_p$ is said to be in class $\mathcal{UCV}(p, \alpha, \beta)$ of p-valent β-uniformly convex functions of order α in $\mathbb{U}$, if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) \geq \beta\left|1 + \frac{zf''(z)}{f'(z)} - p\right|. \tag{4}$$

We note from (3) and (4) that

$$f(z) \in \mathcal{UCV}(p, \alpha, \beta) \Longleftrightarrow \frac{zf'(z)}{p} \in \mathcal{UST}(p, \alpha, \beta). \tag{5}$$

The classes $\mathcal{UST}(p, \alpha, \beta)$ and $\mathcal{UCV}(p, \alpha, \beta)$ were introduced recently by Khairnar and More [1]. Various subclasses of analytic and univalent or multivalent functions were studied in many papers (see, e.g., [2–4]). Recently, Nishiwaki and Owa in [5] introduced two classes $\mathcal{MD}(\alpha, \beta)$ consisting of all functions $f(z) \in \mathcal{A}_1$, which satisfy

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) < \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, \tag{6}$$

and $\mathcal{ND}(\alpha, \beta)$ consisting of all functions $f(z) \in \mathcal{A}_1$, which satisfy

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) < \beta\left|\frac{zf''(z)}{f'(z)}\right|, \tag{7}$$

where $\alpha \geq 1$ and $\beta \leq 0$. We notice from definitions of these classes that

$$f(z) \in \mathcal{ND}(\alpha,\beta) \Longleftrightarrow zf'(z) \in \mathcal{MD}(\alpha,\beta). \tag{8}$$

For each $f(z) \in \mathcal{A}_p$, it is easily seen upon differentiating both sides of (1) q times with respect to z that

$$f^{(q)}(z) = \delta(p,q) z^{p-q} + \sum_{n=p+1}^{\infty} \delta(n,q) a_n z^{n-q}, \tag{9}$$

where $p > q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\delta(p,q)$ denotes q-permutations of p objects; that is,

$$\delta(p,q) = \frac{p!}{(p-q)!}. \tag{10}$$

Let $q, m \in \mathbb{N}_0$ and $p \in \mathbb{N}$ such that $p > q+m$, and assume that

$$-\delta(p-q,m) \leq \alpha < \delta(p-q,m), \quad \beta \geq 0. \tag{11}$$

Srivastava et al. in [6] introduced a subclass of the p-valent function class $\mathcal{US}_m(p,q;\alpha,\beta)$ consisting of functions $f(z)$ of form (1), which satisfy the following analytic criterion: for all $z \in \mathbb{U}$,

$$\Re\left(\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)} - \alpha\right) \geq \beta\left|\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)} - \delta(p-q,m)\right|. \tag{12}$$

Recently, many papers have discussed such aspects of analytic univalent or multivalent functions. For example, Aouf et al. in [7, 8] discussed a subclass of p-valent analytic function with negative coefficient by using higher-order derivatives and investigated many properties of distortion theorems, closure theorems, modified Hadamard products, and radii of close-to-convexity, starlikeness, and convexity. K. I. Noor and M. A. Noor in [9] defined some subclasses of analytic functions related to k-uniformly close-to-convex functions of higher order and studied the following: rate of growth of coefficients, inclusion relations, radius problems, and necessary conditions for univalency. Seoudy in [10] defined a class of p-valent functions defined by certain linear operator and obtained subordination and inclusion properties.

Following Srivastava et al. [6] and Nishiwaki and Owa [5], we define a new subclass $\mathcal{MD}_m(p,q;\alpha,\beta)$ of multivalent functions involving higher-order derivative.

Definition 1. Let $q, m \in \mathbb{N}_0$ and $p \in \mathbb{N}$ such that $p > q+m$, and α and β, two real numbers, satisfy that $\alpha \geq \delta(p-q,m)$ and $\beta \leq 0$. Function $f(z) \in \mathcal{MD}_m(p,q;\alpha,\beta)$, if and only if $f(z) \in \mathcal{A}_p$ and satisfies the following inequality:

$$\Re\left(\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)} - \alpha\right) < \beta\left|\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)} - \delta(p-q,m)\right|. \tag{13}$$

From the above definition, it is clear that $\mathcal{MD}_1(1,0;\alpha,\beta) = \mathcal{MD}(\alpha,\beta)$. In this paper, we obtain several properties including the coefficient inequalities, distortion theorems, extreme points, and integral means inequalities for this subclass $\mathcal{MD}_m(p,q;\alpha,\beta)$ of multivalent functions involving higher-order derivative.

2. Coefficient Inequalities

We derive sufficient conditions for $f(z)$ which are given by using coefficient inequalities.

Theorem 2. *Let $f(z)$ be a function of form (1). If the coefficients of $f(z)$ satisfy*

$$\sum_{n=p+1}^{\infty} \Phi(n,p,q,m,\alpha,\beta)|a_n| \leq \delta(p,q)(\alpha - |2\delta(p-q,m) - \alpha|), \tag{14}$$

where

$$\begin{aligned}\Phi(n,p,q,m,\alpha,\beta) &= \delta(n,q) \\ &\cdot [|\delta(n-q,m) + \delta(p-q,m) - \alpha| \\ &+ |\delta(n-q,m) - \delta(p-q,m) - \alpha| \\ &- 2\beta|\delta(n-q,m) - \delta(p-q,m)|],\end{aligned} \tag{15}$$

then, $f(z)$ is in class $\mathcal{MD}_m(p,q;\alpha,\beta)$.

Proof. Suppose that inequality (14) holds, and denote

$$F(z) = \frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)} - \alpha - \beta\left|\frac{z^m f^{(q+m)}(z)}{f^{(q)}(z)} - \delta(p-q,m)\right|. \tag{16}$$

From the definition, we can verify that if

$$\left|\frac{F(z) + \delta(p-q,m)}{F(z) - \delta(p-q,m)}\right| < 1, \tag{17}$$

then $f(z) \in \mathcal{MD}_m(p,q;\alpha,\beta)$. In fact, denoting

$$\begin{aligned}\Psi^{\pm}(z;f,m,p,q,\alpha,\beta) &= z^m f^{(q+m)}(z) - \alpha f^{(q)}(z) \\ &\pm \delta(p-q,m) f^{(q)}(z) \\ &- \beta e^{i\theta}\left|z^m f^{(q+m)}(z) - \delta(p-q,m) f^{(q)}(z)\right|, \\ \Sigma^{\pm}(z;m,p,q,\alpha) &= \sum_{n=p+1}^{\infty} \delta(n,q) \\ &\cdot (\delta(n-q,m) \pm \delta(p-q,m) - \alpha) a_n z^{n-p}, \\ \Xi^{\pm}(m,p,q,\alpha) &= \sum_{n=p+1}^{\infty} \delta(n,q) \\ &\cdot |\delta(n-q,m) \pm \delta(p-q,m) - \alpha| \cdot |a_n|,\end{aligned} \tag{18}$$

we have

$$\begin{aligned}\left|\frac{F(z)+\delta(p-q,m)}{F(z)-\delta(p-q,m)}\right| &= \left|\frac{\Psi^{+}(z;f,m,p,q,\alpha,\beta)}{\Psi^{-}(z;f,m,p,q,\alpha,\beta)}\right| \\ &= \left|\frac{\delta(p,q)(2\delta(p-q,m)-\alpha)+\Sigma^{+}(z;m,p,q,\alpha)-\beta e^{i\theta}\left|\Sigma^{-}(z;m,p,q,0)\right|}{-\left[\alpha\delta(p,q)-\Sigma^{-}(z;m,p,q,\alpha)+\beta e^{i\theta}\left|\Sigma^{-}(z;m,p,q,0)\right|\right]}\right| \\ &\leq \frac{\delta(p,q)\left|2\delta(p-q,m)-\alpha\right|+\Xi^{+}(m,p,q,\alpha)-\beta\Xi^{-}(m,p,q,0)}{\alpha\delta(p,q)-\Xi^{-}(m,p,q,\alpha)+\beta\Xi^{-}(m,p,q,0)}.\end{aligned} \tag{19}$$

Here, we use technology $f(z) = e^{i\theta}|f(z)|$. If (14) satisfies, we drive that the last expression above is bounded by 1 which implies $f(z) \in \mathcal{MD}_m(p,q;\alpha,\beta)$. Thus, the proof of Theorem 2 is completed. □

Example 3. Function $f(z)$ given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)(\gamma+p+1)\xi_n}{(n+\gamma)(n+\gamma+1)\Phi(n,p,q,m,\alpha,\beta)} z^n \tag{20}$$

belongs to class $\mathcal{MD}_m(p,q;\alpha,\beta)$ for $\gamma > -(p+1)$ and $\xi_n \in \mathbb{C}$ with $|\xi_n| = 1$.

As a special case of Theorem 2, as in [2], we can obtain the following corollary.

Corollary 4. *Function $f(z) \in \mathcal{A}_1$ is in class $\mathcal{MD}(\alpha,\beta)$, if*

$$\sum_{n=2}^{\infty}\left[|n+1-\alpha|+|n-1-\alpha|-2\beta(n-1)\right]\cdot|a_n| \leq \alpha-|2-\alpha|. \tag{21}$$

In view of Theorem 2, we introduce subclass $\mathcal{MD}_m^*(p,q;\alpha,\beta)$ which consists of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \tag{22}$$

whose Taylor-Maclaurin coefficients a_n are nonnegative and satisfy inequality (14). By the coefficient inequalities for classes $\mathcal{MD}_m^*(p,q;\alpha,\beta)$, we have the following theorem.

Theorem 5. *If $\beta_1 \leq \beta_2 \leq 0$, then*

$$\mathcal{MD}_m^*(p,q;\alpha,\beta_1) \subset \mathcal{MD}_m^*(p,q;\alpha,\beta_2). \tag{23}$$

Since $\mathcal{MD}_1(1,0;\alpha,\beta) = \mathcal{MD}(\alpha,\beta)$, we get the following corollary, which is a theorem in [2].

Corollary 6. *If $\beta_1 \leq \beta_2 \leq 0$, then*

$$\mathcal{MD}(\alpha,\beta_1) \subset \mathcal{MD}(\alpha,\beta_2). \tag{24}$$

3. Distortion Theorems

Lemma 7. *If $f(z) \in \mathcal{MD}_m^*(p,q;\alpha,\beta)$, then there exists $p_0 \in \mathbb{N}$ such that*

$$\sum_{n=p_0+1}^{\infty} a_n \leq A_{p_0}, \tag{25}$$

where

$$A_{p_0} = \frac{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)-\sum_{n=p+1}^{p_0}\Phi(n,p,q,m,\alpha,\beta)a_n}{\Phi(p_0+1,p,q,m,\alpha,\beta)}, \tag{26}$$

and $\Phi(n,p,q,m,\alpha,\beta)$ is given in (15).

Proof. From the definition of $\Phi(n,p,q,m,\alpha,\beta)$, there exists $p_0 \in \mathbb{N}$ such that function $\Phi(n,p,q,m,\alpha,\beta)$ is increasing with respect to n when $n > p_0$. According to Theorem 2, we have

$$\begin{aligned}\sum_{n=p_0+1}^{\infty}\Phi(n,p,q,m,\alpha,\beta)a_n &\leq \delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|) \\ &\quad -\sum_{n=p+1}^{p_0}\Phi(n,p,q,m,\alpha,\beta)a_n.\end{aligned} \tag{27}$$

From

$$\begin{aligned}\Phi(p_0+1,p,q,m,\alpha,\beta)\sum_{n=p_0+1}^{\infty}a_n &\leq \sum_{n=p_0+1}^{\infty}\Phi(n,p,q,m,\alpha,\beta)a_n,\end{aligned} \tag{28}$$

we have

$$\begin{aligned}\Phi(p_0+1,p,q,m,\alpha,\beta)\sum_{n=p_0+1}^{\infty}a_n &\le \delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)\\ &\quad-\sum_{n=p+1}^{p_0}\Phi(n,p,q,m,\alpha,\beta)a_n.\end{aligned}\tag{29}$$

This implies that inequality (25) holds. □

Using the same argument, we obtain the following inequality.

Lemma 8. *If $f(z)\in\mathcal{MD}_m^*(p,q;\alpha,\beta)$, then there exists $p_0\in\mathbb{N}$ such that*

$$\sum_{n=p_0+1}^{\infty}na_n\le B_{p_0},\tag{30}$$

where

$$B_{p_0}=\frac{(p_0+1)\left[\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)-\sum_{n=p+1}^{p_0}\Phi(n,p,q,m,\alpha,\beta)a_n\right]}{\Phi(p_0+1,p,q,m,\alpha,\beta)},\tag{31}$$

and $\Phi(n,p,q,m,\alpha,\beta)$ is defined by (15).

Theorem 9. *Let $f(z)$ be a function in class $\mathcal{MD}_m^*(p,q;\alpha,\beta)$. Then, for $|z|=r<1$,*

$$\begin{aligned}|f(z)|&\le r^p+\sum_{n=p+1}^{p_0}a_nr^n+A_{p_0}r^{p_0+1},\\ |f(z)|&\ge r^p-\sum_{n=p+1}^{p_0}a_nr^n-A_{p_0}r^{p_0+1},\end{aligned}\tag{32}$$

where A_{p_0} and B_{p_0} are given in Lemmas 7 and 8, respectively.

Proof. Let $f(z)$ be a function of form (22). For $|z|=r<1$, using Lemma 7, we have

$$\begin{aligned}|f(z)|&\le|z|^p+\sum_{n=p+1}^{p_0}|a_n|\cdot|z|^n+\sum_{n=p_0+1}^{\infty}|a_n|\cdot|z|^n\\ &\le|z|^p+\sum_{n=p+1}^{p_0}|a_n|\cdot|z|^n+|z|^{p_0+1}\sum_{n=p_0+1}^{\infty}|a_n|\\ &\le r^p+\sum_{n=p+1}^{p_0}a_nr^n+A_{p_0}r^{p_0+1},\\ |f(z)|&\ge|z|^p-\sum_{n=p+1}^{p_0}|a_n|\cdot|z|^n-\sum_{n=p_0+1}^{\infty}|a_n|\cdot|z|^n\\ &\ge|z|^p-\sum_{n=p+1}^{p_0}|a_n|\cdot|z|^n-|z|^{p_0+1}\sum_{n=p_0+1}^{\infty}|a_n|\\ &\ge r^p-\sum_{n=p+1}^{p_0}a_nr^n-A_{p_0}r^{p_0+1}.\end{aligned}\tag{33}$$

□

Using the same argument, we can prove the following result.

Theorem 10. *Let $f(z)$ be a function in class $\mathcal{MD}_m^*(p,q;\alpha,\beta)$. Then, for $|z|=r<1$,*

$$\begin{aligned}|f'(z)|&\le pr^{p-1}+\sum_{n=p+1}^{p_0}na_nr^{n-1}+B_{p_0}r^{p_0},\\ |f'(z)|&\ge pr^{p-1}-\sum_{n=p+1}^{p_0}na_nr^{n-1}-B_{p_0}r^{p_0},\end{aligned}\tag{34}$$

where A_n and B_n are given in Lemmas 7 and 8, respectively.

4. Extreme Points

Theorem 11. *Let $f_p(z)=z^p$ and, for each $n=p+1,p+2,\ldots$, define*

$$f_n(z)=z^p+\frac{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)}{\Phi(n,p,q,m,\alpha,\beta)}z^n,\tag{35}$$

where $\Phi(n,p,q,m,\alpha,\beta)$ is defined by (15). Then $f(z)\in\mathcal{MD}_m^*(p,q;\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=p}^{\infty}\lambda_nf_n(z),\tag{36}$$

where $\lambda_n\ge0$ for all $n=p,p+1,\ldots$, and $\sum_{n=p}^{\infty}\lambda_n=1$.

Proof. Suppose that

$$\begin{aligned}f(z)&=\sum_{n=p}^{\infty}\lambda_nf_n(z)=\lambda_pf_p(z)+\sum_{n=p+1}^{\infty}\lambda_nf_n(z)\\ &=z^p\\ &\quad+\sum_{n=p+1}^{\infty}\lambda_n\frac{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)}{\Phi(n,p,q,m,\alpha,\beta)}z^n.\end{aligned}\tag{37}$$

Then,

$$\begin{aligned}
&\sum_{n=p+1}^{\infty} \Phi(n,p,q,m,\alpha,\beta)\lambda_n \\
&\quad\cdot \frac{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)}{\Phi(n,p,q,m,\alpha,\beta)} \\
&\quad= \sum_{n=p+1}^{\infty} \lambda_n \delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|) \\
&\quad= \delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)(1-\lambda_p) \\
&\quad\le \delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|).
\end{aligned} \tag{38}$$

Thus, it follows from Theorem 2 that $f(z) \in \mathcal{MD}_m^*(p,q;\alpha,\beta)$.

Conversely, suppose that $f(z) \in \mathcal{MD}_m^*(p,q;\alpha,\beta)$. Since

$$a_n \le \frac{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)}{\Phi(n,p,q,m,\alpha,\beta)}, \quad n = p+1, p+2, \ldots, \tag{39}$$

we denote

$$\lambda_n = \frac{\Phi(n,p,q,m,\alpha,\beta)}{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)} a_n, \quad n = p+1, p+2, \ldots. \tag{40}$$

And $\lambda_p = 1-\sum_{n=p+1}^{\infty} \lambda_n$. By a simple calculation, we get $f(z) \in \mathcal{MD}_m^*(p,q;\alpha,\beta)$. □

Corollary 12. *The extreme points of $\mathcal{MD}_m^*(p,q;\alpha,\beta)$ are functions $f_p(z) = z^p$ and*

$$f_n(z) = z^p + \frac{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)}{\Phi(n,p,q,m,\alpha,\beta)} z^n, \tag{41}$$

for each $n = p+1, p+2, \ldots$.

5. Integral Means Inequalities

Assume that two functions $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$. We say that $f(z)$ is subordinate to $g(z)$, written as $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ with $w(0) = 1$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$.

Lemma 13 (see [11]). *If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$ with $f(z) \prec g(z)$, then, for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^\mu \, d\theta \le \int_0^{2\pi} |g(z)|^\mu \, d\theta. \tag{42}$$

Theorem 14. *Let $f(z) \in \mathcal{MD}_m^*(p,q;\alpha,\beta)$ and $f_n(z)$ be given by (35). Suppose that*

$$\sum_{n=p+1}^{\infty} a_n \le \frac{\Phi(n,p,q,m,\alpha,\beta)}{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)}. \tag{43}$$

If there exists function $w(z)$, $z \in \mathbb{U}$, that satisfied the condition

$$w(z) = \left(\frac{\Phi(n,p,q,m,\alpha,\beta)}{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)} \cdot \sum_{n=p+1}^{\infty} a_n z^{n-p} \right)^{1/(n-p)}, \tag{44}$$

then, for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\mu \, d\theta \le \int_0^{2\pi} |f_n(z)|^\mu \, d\theta. \tag{45}$$

Proof. In order to obtain the result, it is necessary to prove the following inequality:

$$\int_0^{2\pi} \left| 1 + \sum_{n=p+1}^{\infty} a_n z^{n-p} \right|^\mu d\theta \le \int_0^{2\pi} \left| 1 + \frac{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)}{\Phi(n,p,q,m,\alpha,\beta)} z^{n-p} \right|^\mu d\theta. \tag{46}$$

From Lemma 13, it is sufficient to verify the subordination:

$$1 + \sum_{n=p+1}^{\infty} a_n z^{n-p} \prec 1 + \frac{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)}{\Phi(n,p,q,m,\alpha,\beta)} z^{n-p}. \tag{47}$$

Thus, there exists an analytic function $w(z)$ in $\mathbb{C}$ such that

$$1 + \sum_{n=p+1}^{\infty} a_n z^{n-p} = 1 + \frac{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)}{\Phi(n,p,q,m,\alpha,\beta)} w(z)^{n-p}. \tag{48}$$

We find that

$$w(z) = \left(\frac{\Phi(n,p,q,m,\alpha,\beta)}{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)} \cdot \sum_{n=p+1}^{\infty} a_n z^{n-p} \right)^{1/(n-p)}, \tag{49}$$

which readily yields $w(0) = 0$ and

$$\begin{aligned}
&\left| w(z)^{1/(n-p)} \right| \\
&\quad= \left| \frac{\Phi(n,p,q,m,\alpha,\beta)}{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)} \sum_{n=p+1}^{\infty} a_n z^{n-p} \right| \\
&\quad\le \frac{\Phi(n,p,q,m,\alpha,\beta)}{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)} \sum_{n=p+1}^{\infty} a_n |z^{n-p}| \\
&\quad\le \frac{\Phi(n,p,q,m,\alpha,\beta)}{\delta(p,q)(\alpha-|2\delta(p-q,m)-\alpha|)} |z| \sum_{n=p+1}^{\infty} a_n \\
&\quad\le |z| < 1.
\end{aligned} \tag{50}$$

This means that the hypotheses of $w(z)$ are satisfied and the theorem is proved. □

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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