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Research Article

Exact Solutions of the Vakhnenko-Parkes Equation with Complex Method

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We derive exact solutions to the Vakhnenko-Parkes equation by means of the complex method, and then we illustrate our main results by some computer simulations. We can apply the idea of this study to related nonlinear differential equation.

1. Introduction and Main Results

Nonlinear differential equations are widely used as models to describe many important dynamical systems in various fields of science, especially in nonlinear optics, plasma physics, solid state physics, and fluid mechanics. It has aroused extensive interest in the study of nonlinear differential equations [1–15].

In 1992, Vakhnenko [16] first presented the nonlinear differential equation

$$\frac{\partial}{\partial x}\left(u\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)+u=0 \tag{1}$$

and obtained solitary wave solutions to (1). The equation above gives a description of high-frequency waves in the relaxation medium [17].

In 1998, Vakhnenko and Parkes [18] found the soliton solution to the transformed form of (1) as follows:

$$uu_{xxt}-u_xu_{xt}+u^2u_t=0. \tag{2}$$

Hereafter (2) is called the Vakhnenko-Parkes equation (VPE). In recent years, many powerful methods for constructing the solutions of VPE have been used, for instance, the Hirota-Backlund transformations method [19], the inverse scattering method [20, 21], the exp-function method [22], and the (G'/G)-expansion method [23]. In this article, we would like to use the complex method [24–26] to obtain traveling wave solutions of VPE.

Substituting traveling wave transform

$$u(x,t)=w(z),\quad z=\nu x-\delta t, \tag{3}$$

into Vakhnenko-Parkes (2), we get

$$\nu^2ww'''-\nu^2w'w''+w^2w'=0, \tag{4}$$

Integrating (4) yields

$$3\nu^2ww''-3\nu^2\left(w'\right)^2+w^3-\lambda=0, \tag{5}$$

where ν and λ are constants.

If a meromorphic function g is a rational function of z, or a rational function of $e^{\mu z}$, $\mu\in\mathbb{C}$, or an elliptic function, then we say that g belongs to the class W [27].

Theorem 1. *If $\nu\neq 0$, then the meromorphic solutions w of (5) belong to the class W. In addition, (5) has the following classes of solutions.*

(I) The Rational Function Solutions

$$W_r(z)=\frac{-6\nu^2}{\left(z-z_0\right)^2}, \tag{6}$$

where $\lambda=0$, $z_0\in\mathbb{C}$.

(II) The Simply Periodic Solutions

$$W_s(z)=-\frac{3\nu^2\mu^2}{2}\coth^2\frac{\mu\left(z-z_0\right)}{2}+\frac{\left(3\mu^2-1\right)\nu^2}{2}, \tag{7}$$

where $\lambda=\nu^6/8$, $z_0\in\mathbb{C}$.

(III) The Elliptic Function Solutions

$$W_d(z) = -6\nu^2\left(-\wp(z) + \frac{1}{4}\left(\frac{\wp'(z)+D}{\wp(z)-C}\right)^2\right) + 6\nu^2 C, \quad (8)$$

where $D^2 = 4C^3 - c_3$, $c_2 = 0$, *and* $c_3 = -\lambda/108\nu^6$.

2. Preliminaries

At first, we give some notations and definitions, and then we introduce some lemmas.

Let $m \in \mathbb{N}^* := \{1,2,3,\ldots\}$, $r_j \in \mathbb{N} = \mathbb{N}^* \cup \{0\}$, $r = (r_0, r_1, \ldots, r_m)$, $j = 0, 1, \ldots, m$, and

$$\begin{aligned} &K_r[w](z) \\ &\quad := [w(z)]^{r_0}\left[w'(z)\right]^{r_1}\left[w''(z)\right]^{r_2}\cdots\left[w^{(m)}(z)\right]^{r_m}; \end{aligned} \quad (9)$$

then $d(r) := \sum_{j=0}^{m} r_j$ is the degree of $K_r[w]$. Let the differential polynomial be defined by

$$F\left(w, w', \ldots, w^{(m)}\right) := \sum_{r\in J} a_r K_r[w], \quad (10)$$

where J is a finite index set and a_r are constants; then $\deg F(w, w', \ldots, w^{(m)}) := \max_{r\in J}\{d(r)\}$ is the degree of $F(w, w', \ldots, w^{(m)})$.

Consider the following differential equation:

$$F\left(w, w', \ldots, w^{(m)}\right) = cw^n + d, \quad (11)$$

where $c \neq 0$, d are constants, $n \in \mathbb{N}^*$.

Let $p, q \in \mathbb{N}^*$, and assume that the meromorphic solutions w of (11) have at least one pole. If (11) has exactly p distinct meromorphic solutions, and their multiplicity of the pole at $z = 0$ is q, then (11) is said to satisfy the $\langle p, q\rangle$ condition. It is not easy to verify that the $\langle p, q\rangle$ condition of (11) holds, so we need the weak $\langle p, q\rangle$ condition as follows.

By substituting the Laurent series

$$w(z) = \sum_{\tau=-q}^{\infty} \beta_\tau z^\tau, \quad \beta_{-q} \neq 0,\ q > 0, \quad (12)$$

into (11), we determine exactly p different Laurent singular parts:

$$\sum_{\tau=-q}^{-1} \beta_\tau z^\tau; \quad (13)$$

then (11) is said to satisfy the weak $\langle p, q\rangle$ condition.

Given two complex numbers l_1, l_2 such that $\mathrm{Im}(l_1/l_2) > 0$, and let L be the discrete subset $L[2l_1, 2l_2] := \{l \mid l = 2al_1 + 2bl_2,\ a, b \in \mathbb{Z}\}$, and L is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Let the discriminant $\Delta = \Delta(b_1, b_2) := b_1^3 - 27b_2^2$ and

$$H_n = H_n(L) := \sum_{l\in L\setminus\{0\}} \frac{1}{l^n}. \quad (14)$$

A meromorphic function $\wp(z) := \wp(z, c_2, c_3)$ with double periods $2l_1$, $2l_2$, which satisfies the equation

$$\left(\wp'(z)\right)^2 = 4\wp(z)^3 - c_2\wp(z) - c_3, \quad (15)$$

where $c_2 = 60H_4$, $c_3 = 140H_6$, and $\Delta(c_2, c_3) \neq 0$, is called the Weierstrass elliptic function.

In 2009, Eremenko et al. [28] studied the m-order Briot-Bouquet equation (BBEq)

$$F\left(w, w^{(m)}\right) = \sum_{j=0}^{n} F_j(w)\left(w^{(m)}\right)^j = 0, \quad (16)$$

where $F_j(w)$ are constant coefficients polynomials, $m \in \mathbb{N}^*$. For the m-order BBEq, we have the following lemma.

Lemma 2 (see [26, 29, 30]). *Let* $m, n, p, s \in \mathbb{N}^*$, $\deg F(w, w^{(m)}) < n$. *If a m-order BBEq*

$$F\left(w, w^{(m)}\right) = cw^n + d \quad (17)$$

satisfies the weak $\langle p, q\rangle$ *condition, then the meromorphic solutions* w *belong to the class* W. *Suppose that for some values of parameters such solution* w *exists; then other meromorphic solutions form a one-parametric family* $(z - z_0)$, $z_0 \in \mathbb{C}$. *Furthermore, each elliptic solution with pole at* $z = 0$ *can be written as*

$$\begin{aligned} w(z) &= \sum_{i=1}^{s-1}\sum_{j=2}^{q} \frac{(-1)^j \beta_{-ij}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\left(\frac{1}{4}\left[\frac{\wp'(z)+C_i}{\wp(z)-D_i}\right]^2 - \wp(z)\right) \\ &\quad + \sum_{i=1}^{s-1}\frac{\beta_{-i1}}{2}\frac{\wp'(z)+C_i}{\wp(z)-D_i} \\ &\quad + \sum_{j=2}^{q}\frac{(-1)^j\beta_{-sj}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\wp(z) + \beta_0, \end{aligned} \quad (18)$$

where β_{-ij} *are determined by (12),* $\sum_{i=1}^{s}\beta_{-i1} = 0$, *and* $C_i^2 = 4D_i^3 - c_2D_i - c_3$.

Each rational function solution has $s(\leq p)$ distinct poles of multiplicity q and is expressed as

$$R(z) = \sum_{i=1}^{s}\sum_{j=1}^{q}\frac{\beta_{ij}}{(z-z_i)^j} + \beta_0. \quad (19)$$

Each simply periodic solution has $s(\leq p)$ distinct poles of multiplicity q and is expressed as

$$R(\eta) = \sum_{i=1}^{s}\sum_{j=1}^{q}\frac{\beta_{ij}}{(\eta-\eta_i)^j} + \beta_0, \quad (20)$$

which is a rational function of $\eta = e^{\mu z}$ $(\mu \in \mathbb{C})$.

Lemma 3 (see [27, 30]). *Weierstrass elliptic functions $\wp(z) := \wp(z, c_2, c_3)$ have an addition formula as below:*

$$\wp(z - z_0) = \frac{1}{4}\left[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)}\right]^2 - \wp(z_0) - \wp(z). \quad (21)$$

When $c_2 = c_3 = 0$, Weierstrass elliptic functions can be degenerated to rational functions according to

$$\wp(z, 0, 0) = \frac{1}{z^2}. \quad (22)$$

When $\Delta(c_2, c_3) = 0$, Weierstrass elliptic functions can be degenerated to simple periodic functions according to

$$\wp\left(z, 3d^2, -d^3\right) = 2d - \frac{3d}{2}\coth^2\sqrt{\frac{3d}{2}}z. \quad (23)$$

3. Proof of Theorem 1

Substituting (12) into (5) we obtain $p = 1$, $q = 2$, $\beta_{-2} = -6\nu^2$, $\beta_{-1} = 0$, β_0 is an arbitrary constant, $\beta_1 = 0$, $\beta_2 = -\beta_0^2/10\nu^2$, and $\beta_3 = 0$.

Multiplying (5) by w'/w^3, we get

$$\frac{3\nu^2 w' w''}{w^2} - \frac{3\nu^2\left(w'\right)^3}{w^3} + w' - \frac{\lambda w'}{w^3} = 0. \quad (24)$$

Integrating (24) yields

$$3\nu^2\left(w'\right)^2 + 2w^3 + 2\gamma w^2 - \lambda = 0, \quad (25)$$

where ν is an arbitrary constant and γ is the integrable constant.

Therefore, (25) is a first-order BBEq and satisfies the weak $\langle 1, 2\rangle$ condition. Hence, by Lemma 2, the meromorphic solutions of (25) $w \in W$. It means that the meromorphic solutions of (5) $w \in W$. The forms of the meromorphic solutions to (5) will be given in the following.

By (19), we infer that the indeterminate rational solutions of (5) are

$$R_1(z) = \frac{\beta_{11}}{z^2} + \frac{\beta_{12}}{z} + \beta_{10}, \quad (26)$$

with pole at $z = 0$.

Substituting $R_1(z)$ into (5), we have

$$\begin{aligned}
&\frac{\beta_{11}^2\left(6\nu^2 + \beta_{11}\right)}{z^6} + \frac{3\beta_{11}\beta_{12}\left(4\nu^2 + \beta_{11}\right)}{z^5} \\
&\quad + \frac{3\left(6\beta_{11}\beta_{10}\nu^2 + \beta_{12}^2\nu^2 + \beta_{11}^2\beta_{10} + \beta_{11}\beta_{12}^2\right)}{z^4} \\
&\quad + \frac{\beta_{12}\left(6\beta_{10}\nu^2 + 6\beta_{11}\beta_{10} + \beta_{12}^2\right)}{z^3} \\
&\quad + \frac{3\beta_{10}\left(\beta_{11}\beta_{10} + \beta_{12}^2\right)}{z^2} + \frac{3\beta_{12}\beta_{10}^2}{z} + \beta_{10}^3 + \lambda \\
&= 0;
\end{aligned} \quad (27)$$

then we get $\beta_{11} = -6\nu^2$, $\beta_{12} = \beta_{10} = 0$.

Therefore, we can determine that

$$R_1(z) = \frac{-6\nu^2}{z^2}, \quad (28)$$

where $\lambda = 0$.

So the rational solutions of (5) are

$$W_r(z) = \frac{-6\nu^2}{\left(z - z_0\right)^2}, \quad (29)$$

where $\lambda = 0$, $z_0 \in \mathbb{C}$.

To obtain simply periodic solutions, let $\eta = e^{\mu z}$, and substitute $w = R(\eta)$ into (5); then

$$3\nu^2\mu^2 R\left(\eta R' + \eta^2 R''\right) - 3\nu^2\mu^2\left(R'\eta\right)^2 + R^3 + \lambda = 0. \quad (30)$$

Substituting

$$R_2(z) = \frac{\beta_{21}}{(\eta - 1)^2} + \frac{\beta_{22}}{(\eta - 1)} + \beta_{20} \quad (31)$$

into (30), we obtain that

$$R_2(z) = -\frac{6\nu^2\mu^2}{(\eta - 1)^2} - \frac{6\nu^2\mu^2}{(\eta - 1)} - \frac{\nu^2}{2}, \quad (32)$$

where $\lambda = \nu^6/8$.

Substituting $\eta = e^{\mu z}$ into (32), we can get simply periodic solutions to (5) with pole at $z = 0$

$$\begin{aligned}
W_{s0}(z) &= -\frac{6\nu^2\mu^2}{(e^{\mu z} - 1)^2} - \frac{6\nu^2\mu^2}{(e^{\mu z} - 1)} - \frac{\nu^2}{2} \\
&= -6\nu^2\mu^2\frac{e^{\mu z}}{(e^{\mu z} - 1)^2} - \frac{\nu^2}{2} \\
&= -\frac{3\nu^2\mu^2}{2}\coth^2\frac{\mu z}{2} + \frac{\left(3\mu^2 - 1\right)\nu^2}{2},
\end{aligned} \quad (33)$$

where $\lambda = \nu^6/8$.

So simply periodic solutions of (5) are

$$W_s(z) = -\frac{3\nu^2\mu^2}{2}\coth^2\frac{\mu(z - z_0)}{2} + \frac{\left(3\mu^2 - 1\right)\nu^2}{2}, \quad (34)$$

where $\lambda = \nu^6/8$, $z_0 \in \mathbb{C}$.

From (18) of Lemma 2, we can express the elliptic solutions of (5) as

$$w_{d0}(z) = \beta_{-2}\wp(z) + \beta_{30}, \quad (35)$$

with pole at $z = 0$.

Putting $w_{d0}(z)$ into (5), we obtain that

$$w_{d0}(z) = -6\nu^2\wp(z), \quad (36)$$

where $c_2 = 0$ and $c_3 = -\lambda/108\nu^6$.

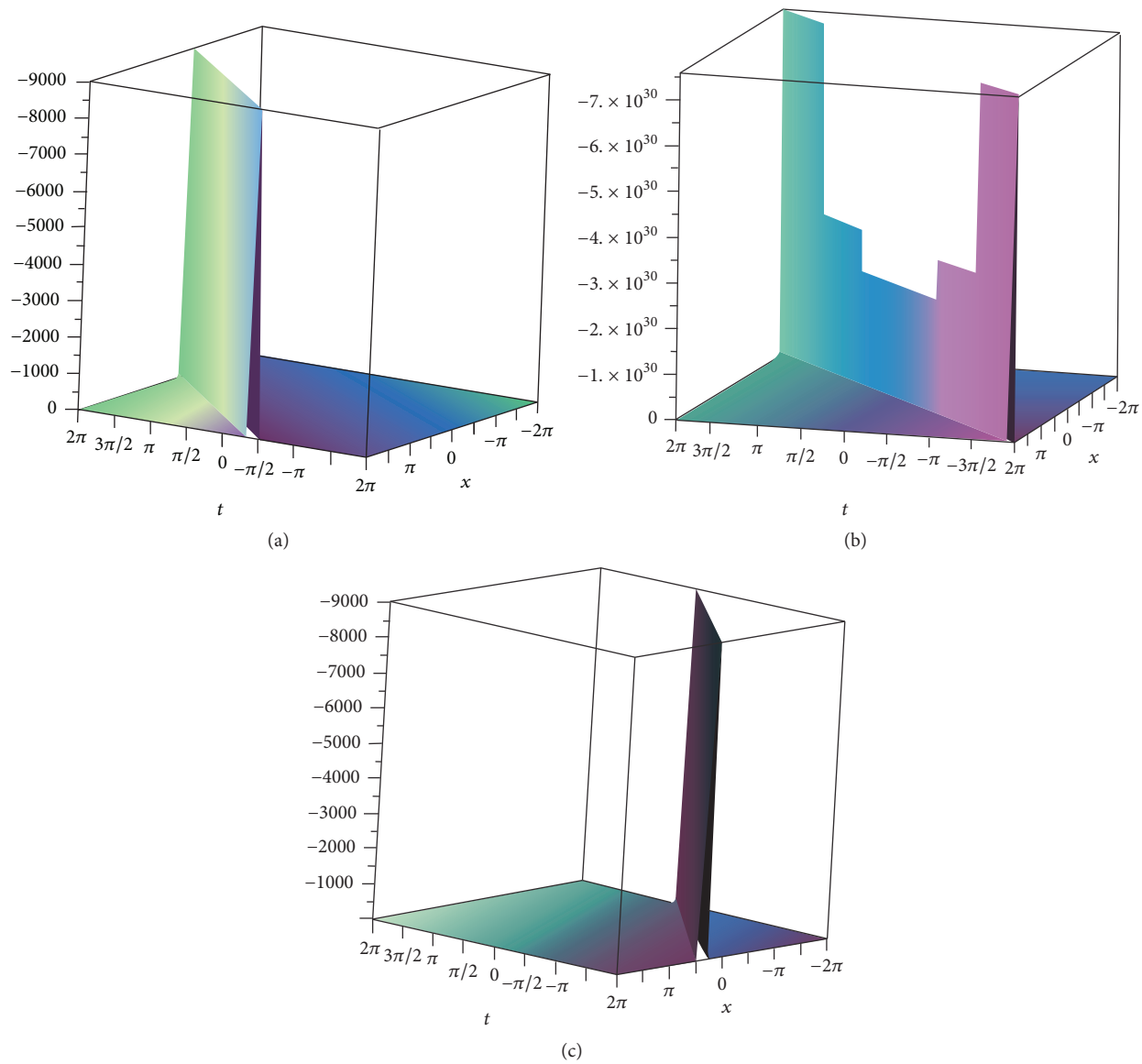

Figure 1: The solution of the Vakhnenko-Parkes equation corresponding to $W_r(z)$, (a) $z_0 = 5$, (b) $z_0 = 0$, and (c) $z_0 = -5$.

Therefore, the elliptic solutions of (5) are

$$W_d(z) = -6\nu^2 \wp(z - z_0), \tag{37}$$

where $z_0 \in \mathbb{C}$.

Applying the addition formula, we can rewrite it as

$$W_d(z) = -6\nu^2 \left(-\wp(z) + \frac{1}{4} \left(\frac{\wp'(z) + D}{\wp(z) - C} \right)^2 \right) + 6\nu^2 C, \tag{38}$$

where $D^2 = 4C^3 - c_3$, $c_2 = 0$, and $c_3 = -\lambda/108\nu^6$.

4. Computer Simulations

In this section, we illustrate our main results by some computer simulations. We carry out further analysis to the properties of simply periodic solutions $W_s(z)$ and the rational solutions $W_r(z)$ as in Figures 1 and 2.

(1) For $W_s(z)$, take $\nu = 1$, $\delta = 1$, and $\mu = \sqrt{2}/2$.

(2) For $W_r(z)$, take $\nu = 1$ and $\delta = 1$.

5. Conclusions

Employing the complex method, we can easily find exact solutions to some nonlinear differential equation. By this method, we get the meromorphic exact solutions of VPE, and then we obtain the traveling wave solutions to VPE. In $W_r(z)$ of our solutions, let $z = kx + \omega t$ and $z_0 = -c_1/c_2$; then it will be equivalent to Eq. (20) in [23]. Simply periodic solutions $W_s(z)$ are new and cannot be degenerated through elliptic function solutions. Our results demonstrate that the complex method

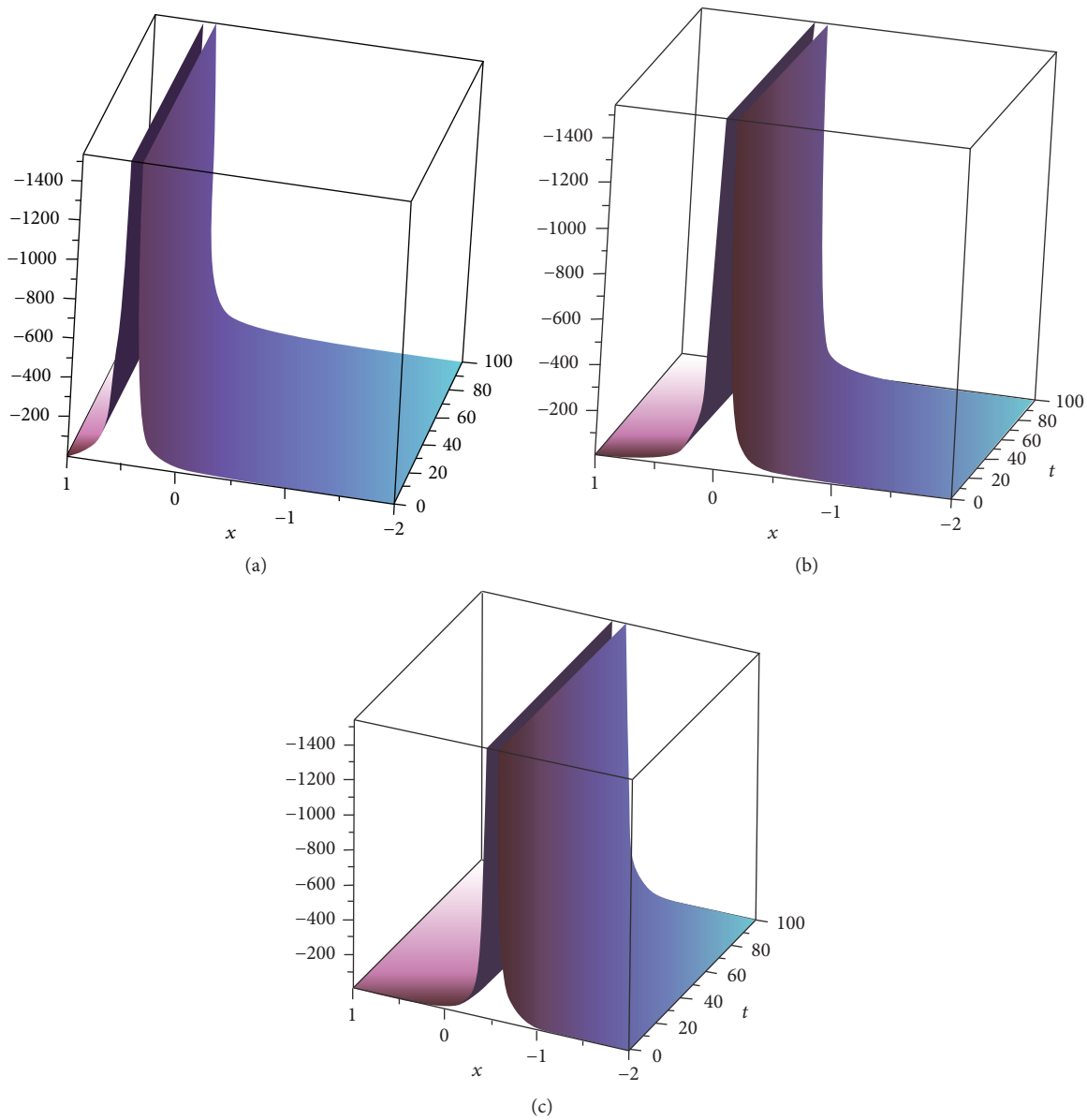

Figure 2: The solution of the Vakhnenko-Parkes equation corresponding to $W_r(z)$, (a) $z_0 = 0.5$, (b) $z_0 = 0$, and (c) $z_0 = -0.5$.

is more simpler, and we can apply the idea of this study to related nonlinear evolution equation.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

Yongyi Gu and Wenjun Yuan carried out the design of this paper and performed the analysis. Najva Aminakbari and Qinghua Jiang participated in the calculations and computer simulations. All authors typed, read, and approved the final manuscript.

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