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Research Article

Existence of Solutions for a Class of Coupled Fractional Differential Systems with Nonlocal Boundary Conditions

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Applying Schauder fixed point theorem and Leray-Schauder nonlinear alternative theory, this paper is concerned with the existence of solutions to coupled fractional differential systems with fractional integral boundary value conditions. Meanwhile, two examples are worked out to illustrate the application of the main results.

1. Introduction

Fractional differential equations have a wide range of applications in many science and engineering, such as in physics, chemistry, biology, and electrodynamics. We refer the reader to see [1–5]. The main reason is that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. Therefore, this topic has attracted much attention of scientists and engineers. More and more good results are obtained. See [6–10] and references therein.

In recent years, fractional differential equations with the nonlinear terms involving fractional derivative $D_{0^+}^{\alpha}$ of unknown functions have been investigated by some authors. See [11–16] and references therein. For example, in [13], Su studied the following nonlinear coupled fractional differential systems:

$$\begin{gathered} D_{0^+}^{\alpha} u(t) = f\left(t, v(t), D_{0^+}^{\mu} v(t)\right), \quad t \in (0,1), \\ D_{0^+}^{\beta} v(t) = g\left(t, u(t), D_{0^+}^{\delta} u(t)\right), \quad t \in (0,1), \\ u(0) = u(1) = v(0) = v(1) = 0, \end{gathered} \tag{1}$$

where $1 < \alpha, \beta < 2$, $\mu, \delta > 0$, $\alpha - \delta \geq 1$, $\beta - \mu \geq 1$, and $f, g : [0,1] \times R \times R \to R$ are given continuous functions. $D_{0^+}^{\rho}$ $(\rho = \alpha, \beta, \delta, \mu)$ is the standard Riemann-Liouville fractional derivative. Applying Schauder fixed point theorem, the existence of solution was studied.

On the other hand, integral boundary conditions have various applications in applied fields such as chemical engineering, underground water flow, blood flow problems, thermoelasticity, population dynamics, and finite element method approaches with the minimization of constitutive error. In consequence, the integral boundary value problem of fractional differential equations is gaining much importance and attention. See [17–21] and references therein. For instance, in [19], Zhao and Liu studied the following coupled fractional differential systems with integral boundary conditions:

$$\begin{gathered} {}^{C}D_{0^+}^{\alpha} u(t) + f(t, v(t)) = 0, \quad t \in (0,1), \\ {}^{C}D_{0^+}^{\alpha} v(t) + g(t, u(t)) = 0, \quad t \in (0,1), \\ u^{(j)}(0) = v^{(j)}(0) = 0, \\ 0 \leq j \leq n-1, \ j \neq 1, \\ u'(1) = \lambda \int_0^1 u(t)\, dt, \\ v'(1) = \lambda \int_0^1 v(t)\, dt, \end{gathered} \tag{2}$$

where $n-1<\alpha\le n$, $n\ge 3$, $0\le\lambda<2$. ${}^{C}D_{0^+}^{\alpha}$ is the Caputo fractional derivate and $f,g:[0,1]\times[0,+\infty)\to[0,+\infty)$ are given continuous functions. By using the monotone method and the theory of fixed point index on cone, they investigated the existence and uniqueness of solution for this coupled system.

In [21], Ahmad et al. investigated the following coupled fractional differential system with nonlocal and integral boundary value conditions:

$$\begin{aligned}
&{}^{C}D_{0^+}^{\alpha}x(t)=f\left(t,x(t),y(t),{}^{C}D_{0^+}^{\gamma}y(t)\right),\\
&\qquad\qquad t\in[0,T],\\
&{}^{C}D_{0^+}^{\beta}y(t)=g\left(t,x(t),{}^{C}D_{0^+}^{\delta}x(t),y(t)\right),\\
&\qquad\qquad t\in[0,T],\\
&x(0)=h(y),\\
&\int_0^T y(s)\,ds=\mu_1x(\eta),\\
&y(0)=\phi(x),\\
&\int_0^T x(s)\,ds=\mu_2y(\xi),
\end{aligned}\tag{3}$$

where $1<\alpha,\beta\le 2$, $0<\gamma\le\delta<1$, $\eta,\xi\in(0,T)$. $f,g:[0,T]\times R\times R\times R\to R$, $h,\phi:C([0,T],R)\to R$ are given continuous functions and μ_1, μ_2 are real number. Applying Banach contraction mapping principle and Leray-Schauder nonlinear alternative theory, the existence and uniqueness of solution were studied.

To the best of our knowledge, there are fewer results for coupled fractional differential systems with nonlocal and fractional integral boundary value conditions. Motivated by the above-mentioned references, we consider the existence of solutions of the following systems:

$$\begin{aligned}
&{}^{C}D_{0^+}^{\alpha}x(t)=f\left(t,y(t),{}^{C}D_{0^+}^{\gamma}y(t)\right),\quad t\in[0,1],\\
&{}^{C}D_{0^+}^{\beta}y(t)=g\left(t,x(t),{}^{C}D_{0^+}^{\delta}x(t)\right),\quad t\in[0,1],\\
&x(0)=h(x),\\
&x(1)=aI_{0^+}^{\theta_1}x(\eta),\\
&y(0)=\varphi(y),\\
&y(1)=bI_{0^+}^{\theta_2}y(\xi),
\end{aligned}\tag{4}$$

where $I_{0^+}^{\theta_i}$ $(i=1,2)$ is Riemann-Liouville fractional integral, $1<\alpha,\beta<2$, $0<\gamma,\delta<1$, $0\le\xi,\eta\le 1$, $\theta_i>0$ $(i=1,2)$, $a,b\in R$. $f,g\in C([0,1]\times R\times R\to R)$, and $h,\varphi:C([0,1],R)\to R$ are given continuous functionals. Applying Schauder fixed point theorem and Leray-Schauder nonlinear alternative theory, some existence results of solutions to this systems are obtained. Finally, two examples are worked out to illustrate the application of our results.

The main features of this paper are as follows. (1) The coupled fractional differential systems with nonlocal and fractional integral boundary value conditions are first studied. (2) The nonlinear term here involves fractional derivative of unknown functions. (3) Nonlocal conditions such as $x(0)=h(x)$ and $y(0)=\varphi(y)$ and fractional integral boundary conditions such as $x(1)=aI_{0^+}^{\theta_1}x(\eta)$ and $y(1)=bI_{0^+}^{\theta_2}y(\xi)$ are more extensive and superior to local conditions.

The rest of this paper is organized as follows. Section 2 introduces some basic definitions and lemmas. In Section 3, the main results are presented. Finally, in Section 4, some examples are given to illustrate the effectiveness of the main results.

2. Preliminary Results

In this section, we first introduce some definitions and lemmas for fractional calculus. For details, please refer to [1, 22].

Definition 1. The Caputo fractional derivative of order $\alpha>0$ of a function $u:(0,\infty)\to R$ is given by

$$ {}^{C}D_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds,\tag{5}$$

where $n=[\alpha]+1$, $[\alpha]$ denotes the integer part of the real number α.

Definition 2. The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $u:(0,\infty)\to R$ is given by

$$I_{0^+}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)\,ds,\tag{6}$$

where Γ is the gamma function.

Lemma 3. *Let $\alpha>0$. Then the fractional differential equation ${}^{C}D_{0^+}^{\alpha}u(t)=0$ has solution*

$$u(t)=c_1+c_2t+c_3t^2+\cdots+c_nt^{n-1},\tag{7}$$

where $c_i\in R$, $i=1,2,\ldots,n$, $n=[\alpha]+1$.

Lemma 4. *Let $n-1<\alpha\le n$ $(n\in N)$. Then*

$$I_{0^+}^{\alpha}\,{}^{C}D_{0^+}^{\alpha}u(t)=u(t)+c_1+c_2t+c_3t^2+\cdots+c_nt^{n-1},\tag{8}$$

where $c_i\in R$, $i=1,2,\ldots,n$, $n=[\alpha]+1$.

Lemma 5. *Let $p>q\ge 0$, and $g\in L[a,b]$. Then*

$$\begin{aligned}
&I_{0^+}^{p}I_{0_+}^{q}g(t)=I_{0^+}^{p+q}g(t)=I_{0^+}^{q}I_{0^+}^{p}g(t),\\
&{}^{C}D_{0^+}^{p}I_{0^+}^{p}g(t)=g(t),\\
&{}^{C}D_{0^+}^{q}I_{0^+}^{p}g(t)=I_{0^+}^{p-q}g(t),\\
&\qquad\qquad t\in[a,b].
\end{aligned}\tag{9}$$

Lemma 6. *Suppose $h, \varphi : C[0,1] \to R$ are given continuous functionals and $\omega, u \in L[0,1]$. Then the boundary value problem*

$$\begin{aligned} {}^{C}D_{0^+}^{\alpha} x(t) &= \omega(t), \quad t \in [0,1], \\ {}^{C}D_{0^+}^{\beta} y(t) &= u(t), \quad t \in [0,1], \\ x(0) &= h(x), \\ x(1) &= aI_{0^+}^{\theta_1} x(\eta), \\ y(0) &= \varphi(y), \\ y(1) &= bI_{0^+}^{\theta_2} y(\xi) \end{aligned} \tag{10}$$

is equivalent to system

$$\begin{aligned} x(t) = {} & (1+\tau_1 t) h(x) + \frac{at}{1-a\triangle_1} I_{0^+}^{\alpha+\theta_1} \omega(\eta) \\ & - \frac{t}{1-a\triangle_1} I_{0^+}^{\alpha} \omega(1) + I_{0^+}^{\alpha} \omega(t), \end{aligned} \tag{11}$$

$$\begin{aligned} y(t) = {} & (1+\tau_2 t) \varphi(y) + \frac{bt}{1-b\triangle_2} I_{0^+}^{\beta+\theta_2} u(\xi) \\ & - \frac{t}{1-b\triangle_2} I_{0^+}^{\beta} u(1) + I_{0^+}^{\beta} u(t), \end{aligned} \tag{12}$$

where $1 < \alpha, \beta < 2$, $0 \le \xi$, $\eta \le 1$, $\theta_i > 0$ $(i = 1,2)$, $a, b \in R$, $1 - a\triangle_1 \ne 0$, $1 - b\triangle_2 \ne 0$,

$$\begin{aligned} \triangle_1 &= \frac{\eta^{\theta_1+1}}{\Gamma(\theta_1+2)}, \\ \sigma_1 &= \frac{\eta^{\theta_1}}{\Gamma(\theta_1+1)}, \\ \tau_1 &= \frac{a\sigma_1 - 1}{1-a\triangle_1}, \\ \triangle_2 &= \frac{\xi^{\theta_2+1}}{\Gamma(\theta_2+2)}, \\ \sigma_2 &= \frac{\xi^{\theta_2}}{\Gamma(\theta_2+1)}, \\ \tau_2 &= \frac{b\sigma_2 - 1}{1-b\triangle_2}. \end{aligned} \tag{13}$$

Proof. In view of Lemmas 3 and 4, the solution of ${}^{C}D_{0^+}^{\alpha} x(t) = \omega(t)$ is

$$x(t) = c_1 + c_2 t + I_{0^+}^{\alpha} \omega(t), \tag{14}$$

where $c_i \in R$, $i = 1,2$. From $x(0) = h(x)$, one can get $c_1 = h(x)$. By means of Lemma 5 and (14), we have

$$I_{0^+}^{\theta_1} x(t) = \frac{c_1 t^{\theta_1}}{\Gamma(\theta_1+1)} + \frac{c_2 t^{\theta_1+1}}{\Gamma(\theta_1+2)} + I_{0^+}^{\alpha+\theta_1} \omega(t). \tag{15}$$

Since $x(1) = aI_{0^+}^{\theta_1} x(\eta)$, one has

$$\begin{aligned} c_1 + c_2 + I_{0^+}^{\alpha} \omega(1) = {} & \frac{ac_1}{\Gamma(\theta_1+1)} \eta^{\theta_1} + \frac{ac_2}{\Gamma(\theta_1+2)} \eta^{\theta_1+1} \\ & + aI_{0^+}^{\alpha+\theta_1} \omega(\eta), \end{aligned} \tag{16}$$

which implies

$$\begin{aligned} c_2(1 - a\triangle_1) = {} & (a\sigma_1 - 1) h(x) + aI_{0^+}^{\alpha+\theta_1} \omega(\eta) \\ & - I_{0^+}^{\alpha} \omega(1). \end{aligned} \tag{17}$$

Thus,

$$\begin{aligned} c_1 &= h(x), \\ c_2 &= \frac{a\sigma_1 - 1}{1-a\triangle_1} h(x) + \frac{a}{1-a\triangle_1} I_{0^+}^{\alpha+\theta_1} \omega(\eta) \\ & \quad - \frac{1}{1-a\triangle_1} I_{0^+}^{\alpha} \omega(1). \end{aligned} \tag{18}$$

Substituting c_1 and c_2 to (14), we obtain that (11) holds. Similarly, one can prove that (12) holds. □

Let $X = \{x \in C[0,1] : {}^{C}D_{0^+}^{\delta} x(t) \in C[0,1]\}$ be endowed with the norm $\|x\|_X = \|x\|_c + \|{}^{C}D_{0^+}^{\delta} x\|_c = \max_{t\in[0,1]}|x(t)| + \max_{t\in[0,1]}|{}^{C}D_{0^+}^{\delta} x(t)|$. Then we have the following conclusions.

Lemma 7. *$(X, \|\cdot\|_X)$ is a Banach space.*

Let $Y = \{y \in C[0,1] : {}^{C}D_{0^+}^{\gamma} y(t) \in C[0,1]\}$ be endowed with $\|y\|_Y = \|y\|_c + \|{}^{C}D_{0^+}^{\gamma} y\|_c = \max_{t\in[0,1]}|y(t)| + \max_{t\in[0,1]}|{}^{C}D_{0^+}^{\gamma} y(t)|$. Then $(Y, \|\cdot\|_Y)$ is a Banach space. Let $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$. Clearly $(X \times Y, \|\cdot\|_{X\times Y})$ is also a Banach space.

Defining an operator $T : X \times Y \to X \times Y$ by

$$T(x,y)(t) = (T_1(x,y)(t), T_2(x,y)(t)), \quad \forall x \in X,\ y \in Y, \tag{19}$$

where

$$\begin{aligned} & T_1(x,y)(t) \\ & \quad = (1+\tau_1 t) h(x) \\ & \qquad + \frac{at}{1-a\triangle_1} I_{0^+}^{\alpha+\theta_1} f\left(t, y(t), {}^{C}D_{0^+}^{\gamma} y(t)\right)\Big|_{t=\eta} \\ & \qquad - \frac{t}{1-a\triangle_1} I_{0^+}^{\alpha} f\left(t, y(t), {}^{C}D_{0^+}^{\gamma} y(t)\right)\Big|_{t=1} \\ & \qquad + I_{0^+}^{\alpha} f\left(t, y(t), {}^{C}D_{0^+}^{\gamma} y(t)\right), \\ & T_2(x,y)(t) \\ & \quad = (1+\tau_2 t) \varphi(y) \\ & \qquad + \frac{bt}{1-b\triangle_2} I_{0^+}^{\beta+\theta_2} g\left(t, x(t), {}^{C}D_{0^+}^{\delta} x(t)\right)\Big|_{t=\xi} \end{aligned}$$

$$- \frac{t}{1-b\triangle_2} I_{0^+}^{\beta} g\left(t, x(t), {}^C D_{0^+}^{\delta} x(t)\right)\Big|_{t=1} + I_{0^+}^{\beta} g\left(t, x(t), {}^C D_{0^+}^{\delta} x(t)\right), \tag{20}$$

$\triangle_i, \sigma_i, \tau_i$ $(i = 1, 2)$ *are described as in Lemma 6.*

Lemma 8. *Suppose $f, g \in C([0,1] \times R^2, R)$. Then $(x, y) \in X \times Y$ is a solution of problem (4) if and only if $(x, y) \in X \times Y$ is a fixed point of the operator T.*

Proof. By Lemma 6, the necessity is obvious. Now we show sufficiency.

Suppose $(x, y) \in X \times Y$ is a fixed point of the operator T. This together with (19) indicates

$$\begin{aligned} x(t) &= (1+\tau_1 t) h(x) \\ &\quad + \frac{at}{1-a\triangle_1} I_{0^+}^{\alpha+\theta_1} f\left(t, y(t), {}^C D_{0^+}^{\gamma} y(t)\right)\Big|_{t=\eta} \\ &\quad - \frac{t}{1-a\triangle_1} I_{0^+}^{\alpha} f\left(t, y(t), {}^C D_{0^+}^{\gamma} y(t)\right)\Big|_{t=1} \\ &\quad + I_{0^+}^{\alpha} f\left(t, y(t), {}^C D_{0^+}^{\gamma} y(t)\right), \\ y(t) &= (1+\tau_2 t) \varphi(y) \\ &\quad + \frac{bt}{1-b\triangle_2} I_{0^+}^{\beta+\theta_2} g\left(t, x(t), {}^C D_{0^+}^{\delta} x(t)\right)\Big|_{t=\xi} \\ &\quad - \frac{t}{1-b\triangle_2} I_{0^+}^{\beta} g\left(t, x(t), {}^C D_{0^+}^{\delta} x(t)\right)\Big|_{t=1} \\ &\quad + I_{0^+}^{\beta} g\left(t, x(t), {}^C D_{0^+}^{\delta} x(t)\right). \end{aligned} \tag{21}$$

Notice that ${}^C D_{0^+}^{\rho} t^{\rho-m} = 0$, $m = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to ρ. Therefore,

$$\begin{aligned} {}^C D_{0^+}^{\alpha} x(t) &= {}^C D_{0^+}^{\alpha} \left[I_{0^+}^{\alpha} f\left(t, y(t), {}^C D_{0^+}^{\gamma} y(t)\right)\right] \\ &= f\left(t, y(t), {}^C D_{0^+}^{\gamma} y(t)\right). \end{aligned} \tag{22}$$

Similarly, one has

$$\begin{aligned} {}^C D_{0^+}^{\beta} y(t) &= {}^C D_{0^+}^{\beta} \left[I_{0^+}^{\beta} g\left(t, x(t), {}^C D_{0^+}^{\delta} x(t)\right)\right] \\ &= g\left(t, x(t), {}^C D_{0^+}^{\delta} x(t)\right). \end{aligned} \tag{23}$$

By direct computation, we can easily get $x(0) = h(x)$, $y(0) = \varphi(y)$, $x(1) = aI_{0^+}^{\theta_1} x(\eta)$, $y(1) = bI_{0^+}^{\theta_2} y(\xi)$. Therefore, $(x, y) \in X \times Y$ is a solution of (4). □

Lemma 9 ([23] (Leray-Schauder nonlinear alternative)). *Let F be a Banach space, Ω be a bounded open subset of F, and $0 \in \Omega$, $T : \overline{\Omega} \to F$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $Tx = \lambda x$ or there exists a fixed point $x^* \in \overline{\Omega}$.*

3. Main Results

For convenience, some notations and assumptions are stated as follows:

$$\begin{aligned} A_1 &= \left|\frac{a}{1-a\triangle_1}\right| \frac{1}{\Gamma(\theta_1+\alpha+1)} + \left|\frac{1}{1-a\triangle_1}\right| \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}, \\ A_2 &= \left|\frac{b}{1-b\triangle_2}\right| \frac{1}{\Gamma(\theta_2+\beta+1)} + \left|\frac{1}{1-b\triangle_2}\right| \frac{1}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta+1)}, \\ B_1 &= |1+\tau_1| + \frac{|\tau_1|}{\Gamma(2-\delta)}, \\ B_2 &= |1+\tau_2| + \frac{|\tau_2|}{\Gamma(2-\gamma)}, \\ C_1 &= \left|\frac{a}{1-a\triangle_1}\right| \frac{1}{\Gamma(\theta_1+\alpha+1)} + \left|\frac{1}{1-a\triangle_1}\right| \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}, \\ C_2 &= \left|\frac{b}{1-b\triangle_2}\right| \frac{1}{\Gamma(\theta_2+\beta+1)} + \left|\frac{1}{1-b\triangle_2}\right| \frac{1}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta)}, \\ D_1 &= A_1 + \frac{C_1}{\Gamma(2-\delta)}, \\ D_2 &= A_2 + \frac{C_2}{\Gamma(2-\gamma)}, \\ P_1 &= a_1 + b_1\phi_1(Q) + c_1\psi_1(Q), \\ P_2 &= a_2 + b_2\phi_2(Q) + c_2\psi_2(Q). \end{aligned} \tag{24}$$

(H_1) $f, g : [0,1] \times R^2 \to R$ are continuous functions and there exist constants $k_i > 0$, $l_i > 0$, and $m_i, n_i \in (0,1)$, $i = 1, 2$, such that

$$\begin{aligned} |f(t,u,v)| &\le k_0 + k_1 |u|^{m_1} + k_2 |v|^{m_2}, \quad \forall t \in [0,1],\ u, v \in R; \\ |g(t,u,v)| &\le l_0 + l_1 |u|^{n_1} + l_2 |v|^{n_2}, \quad \forall t \in [0,1],\ u, v \in R. \end{aligned} \tag{25}$$

(H_2) $h, \varphi : C[0,1] \to R$ are continuous functionals, $h(0) = \varphi(0) = 0$, and there exist constants $K, L > 0$, $\mu, \lambda \in (0,1)$ such that, for all $x, y \in C[0,1]$,

$$\begin{aligned} |h(x)| &\le K \|x\|_c^{\mu}, \\ |\varphi(y)| &\le L \|y\|_c^{\lambda}. \end{aligned} \tag{26}$$

(H_3) $f, g : [0,1] \times R^2 \to R$ are continuous functions, and there exist $a_i, b_i, c_i, d_i \geq 0$ $(i = 1,2)$ and nondecreasing functions $\phi_i, \psi_i \in C([0,+\infty),[0,+\infty))$ $(i = 1,2)$, such that

$$\left|f(t,u,v)\right| \leq a_1 + b_1\phi_1(|u|) + c_1\psi_1(|v|), \quad \forall t \in [0,1],\ u,v \in R; \tag{27}$$

$$\left|g(t,u,v)\right| \leq a_2 + b_2\phi_2(|u|) + c_2\psi_2(|v|), \quad \forall t \in [0,1],\ u,v \in R. \tag{28}$$

(H_4) There exists $Q > 0$ such that $B_1KQ^{\mu} + D_1P_1 + B_2LQ^{\lambda} + D_2P_2 < Q$, where K, L, μ, λ are described as in (H_2).

Theorem 10. *Assume that (H_1)-(H_2) hold. Then FDE (4) has at least one solution.*

Proof. Let

$$B_R = \left\{(x,y) \in X \times Y \mid \|(x,y)\|_{X\times Y} \leq R\right\}, \tag{29}$$

where

$$\begin{aligned} R \geq \max\Big\{&(8B_1K)^{1/(1-\mu)}, (8B_2L)^{1/(1-\lambda)}, 8k_0D_1, 8l_0D_2, \\ &(8k_1D_1)^{1/(1-m_1)}, (8l_1D_2)^{1/(1-n_1)}, (8k_2D_1)^{1/(1-m_2)}, \\ &(8l_2D_2)^{1/(1-n_2)}\Big\}. \end{aligned} \tag{30}$$

Now we prove that $T : B_R \to B_R$. In fact, for any $(x,y) \in B_R$, it follows from Definition 2 and (H_1) that

$$\begin{aligned} \left|T_1(x,y)(t)\right| &\leq \left|1+\tau_1\right| |h(x)| \\ &+ \left|\frac{a}{1-a\triangle_1} I_{0^+}^{\alpha+\theta_1} f\left(t, y(t), {}^{C}D_{0^+}^{\gamma} y(t)\right)\Big|_{t=\eta}\right| \\ &+ \left|\frac{1}{1-a\triangle_1} I_{0^+}^{\alpha} f\left(t, y(t), {}^{C}D_{0^+}^{\gamma} y(t)\right)\Big|_{t=1}\right| \\ &+ \left|I_{0^+}^{\alpha} f\left(t, y(t), {}^{C}D_{0^+}^{\gamma} y(t)\right)\right| \leq \left|1+\tau_1\right| KR^{\mu} \\ &+ \left|\frac{a}{1-a\triangle_1}\right| \frac{1}{\Gamma(\theta_1+\alpha+1)} \left(k_0 + k_1R^{m_1} + k_2R^{m_2}\right) \\ &+ \left|\frac{1}{1-a\triangle_1}\right| \frac{1}{\Gamma(\alpha+1)} \left(k_0 + k_1R^{m_1} + k_2R^{m_2}\right) \\ &+ \frac{1}{\Gamma(\alpha+1)} \left(k_0 + k_1R^{m_1} + k_2R^{m_2}\right) = \left|1+\tau_1\right| KR^{\mu} \\ &+ \left(k_0 + k_1R^{m_1} + k_2R^{m_2}\right) \left[\left|\frac{a}{1-a\triangle_1}\right| \frac{1}{\Gamma(\theta_1+\alpha+1)}\right. \\ &\left. + \left|\frac{1}{1-a\triangle_1}\right| \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\right] \leq \left|1+\tau_1\right| KR^{\mu} \\ &+ A_1\left(k_0 + k_1R^{m_1} + k_2R^{m_2}\right). \end{aligned} \tag{31}$$

On the other hand, by Definition 1, we can obtain

$${}^{C}D_{0^+}^{\delta} T_1(x,y)(t) = \frac{1}{\Gamma(1-\delta)} \int_0^t \frac{T_1(x,y)'(s)}{(t-s)^{\delta}} ds. \tag{32}$$

Repeating a process similar to that of (31), it is easy to see

$$\begin{aligned} \left|T_1(x,y)'(t)\right| &\leq \left|\tau_1\right| KR^{\mu} + \left|\frac{a}{1-a\triangle_1}\right| \\ &\cdot \frac{1}{\Gamma(\alpha+\theta_1+1)} \left(k_0 + k_1R^{m_1} + k_2R^{m_2}\right) \\ &+ \left|\frac{1}{1-a\triangle_1}\right| \frac{1}{\Gamma(\alpha+1)} \left(k_0 + k_1R^{m_1} + k_2R^{m_2}\right) \\ &+ \frac{1}{\Gamma(\alpha)} \left(k_0 + k_1R^{m_1} + k_2R^{m_2}\right) = \left|\tau_1\right| KR^{\mu} + \left(k_0\right. \\ &\left. + k_1R^{m_1} + k_2R^{m_2}\right) \left[\left|\frac{a}{1-a\triangle_1}\right| \frac{1}{\Gamma(\alpha+\theta_1+1)}\right. \\ &\left. + \left|\frac{1}{1-a\triangle_1}\right| \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\right] \leq \left|\tau_1\right| KR^{\mu} \\ &+ C_1\left(k_0 + k_1R^{m_1} + k_2R^{m_2}\right). \end{aligned} \tag{33}$$

It follows from (32) and (33) that

$$\begin{aligned} \left|{}^{C}D_{0^+}^{\delta} T_1(x,y)(t)\right| &\leq \frac{1}{\Gamma(1-\delta)} \int_0^t \frac{\left|T_1(x,y)'(s)\right|}{(t-s)^{\delta}} ds \\ &\leq \frac{1}{\Gamma(1-\delta)} \\ &\cdot \int_0^t \frac{\left|\tau_1\right| KR^{\mu} + C_1\left(k_0 + k_1R^{m_1} + k_2R^{m_2}\right)}{(t-s)^{\delta}} ds \\ &\leq \frac{1}{\Gamma(2-\delta)} \left[\left|\tau_1\right| KR^{\mu} + C_1\left(k_0 + k_1R^{m_1} + k_2R^{m_2}\right)\right]. \end{aligned} \tag{34}$$

By virtue of (31) and (34), we have

$$\begin{aligned} \|T_1(x,y)\|_X &\leq B_1KR^{\mu} + \left(k_0 + k_1R^{m_1} + k_2R^{m_2}\right) D_1 \\ &\leq \frac{R}{8} + \frac{R}{8} + \frac{R}{8} + \frac{R}{8} = \frac{R}{2}. \end{aligned} \tag{35}$$

Similarly, we can get

$$\begin{aligned} \|T_2(x,y)\|_Y &\leq B_2LR^{\lambda} + \left(l_0 + l_1R^{n_1} + l_2R^{n_2}\right) D_2 \\ &\leq \frac{R}{8} + \frac{R}{8} + \frac{R}{8} + \frac{R}{8} = \frac{R}{2}. \end{aligned} \tag{36}$$

Therefore,

$$\begin{aligned} \|T(x,y)\|_{X\times Y} &= \|T_1(x,y)\|_X + \|T_2(x,y)\|_Y \\ &\leq \frac{R}{2} + \frac{R}{2} \leq R, \end{aligned} \tag{37}$$

which means that $T : B_R \to B_R$.

From the continuity of the functions f, g, h, and φ, it is not difficult to see that T is continuous.

Next we show that $T(B_R)$ are equicontinuous. For this sake, let

$$M_1 = \max_{t\in[0,1],|x|\le R,|y|\le R} |f(t,x,y)|,$$
$$M_2 = \max_{t\in[0,1],|x|\le R,|y|\le R} |g(t,x,y)|. \tag{38}$$

For any $t_1, t_2 \in [0,1]$ $(t_1 < t_2)$, one has

$$\begin{aligned}
&|T_1(x,y)(t_2) - T_1(x,y)(t_1)| \le |\tau_1| KR^{\mu} |t_2 - t_1| \\
&\quad + \left|\frac{a}{1-a\triangle_1}\right| \frac{M_1}{\Gamma(\alpha+\theta_1+1)} |t_2 - t_1| + \frac{M_1}{|1-a\triangle_1|} \\
&\quad \cdot \frac{1}{\Gamma(\alpha+1)} |t_2 - t_1| \\
&\quad + \left|\frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} f\left(s, y(s), {}^C D_{0^+}^{\gamma} y(s)\right) ds \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} f\left(s, y(s), {}^C D_{0^+}^{\gamma} y(s)\right) ds \right| \\
&\quad \le |\tau_1| KR^{\mu} |t_2 - t_1| + \left|\frac{a}{1-a\triangle_1}\right| \frac{M_1}{\Gamma(\alpha+\theta_1+1)} |t_2 \\
&\quad - t_1| + \frac{M_1}{|1-a\triangle_1|} \frac{1}{\Gamma(\alpha+1)} |t_2 - t_1| + \frac{M_1}{\Gamma(\alpha+1)} |t_2^{\alpha} \\
&\quad - t_1^{\alpha}|.
\end{aligned} \tag{39}$$

On the other hand, we have

$$\begin{aligned}
&\left|{}^C D_{0^+}^{\delta} T_1(x,y)(t_2) - {}^C D_{0^+}^{\delta} T_1(x,y)(t_1)\right| = \left|\frac{1}{\Gamma(1-\delta)}\right. \\
&\quad \cdot \int_0^{t_2} \frac{T_1(x,y)'(s)}{(t_2-s)^{\delta}} ds - \frac{1}{\Gamma(1-\delta)} \\
&\quad \left. \cdot \int_0^{t_1} \frac{T_1(x,y)'(s)}{(t_1-s)^{\delta}} ds \right| \\
&\quad \le \frac{1}{\Gamma(1-\delta)} \left[\int_0^{t_1} \frac{(t_2-s)^{\delta} - (t_1-s)^{\delta}}{(t_1-s)^{\delta}(t_2-s)^{\delta}} \left|T_1(x,y)'\right. \right. \\
&\quad \left. \left. \cdot (s)\right| ds + \int_{t_1}^{t_2} \frac{1}{(t_2-s)^{\delta}} \left|T_1(x,y)'(s)\right| ds \right].
\end{aligned} \tag{40}$$

Notice that

$$\left|T_1(x,y)'(s)\right| \le |\tau_1| KR^{\mu} + M_1 C_1. \tag{41}$$

This together with (39)-(40) guarantees that

$$\begin{aligned}
&\left|{}^C D_{0^+}^{\delta} T_1(x,y)(t_2) - {}^C D_{0^+}^{\delta} T_1(x,y)(t_1)\right| \\
&\quad \le \frac{1}{\Gamma(1-\delta)} \left(|\tau_1| KR^{\mu} + M_1 C_1\right) \\
&\quad \cdot \left[\int_0^{t_1} \frac{(t_2-s)^{\delta} - (t_1-s)^{\delta}}{(t_1-s)^{\delta}(t_2-s)^{\delta}} ds + \int_{t_1}^{t_2} \frac{1}{(t_2-s)^{\delta}} ds \right] \\
&\quad \le \frac{1}{\Gamma(2-\delta)} \left(|\tau_1| KR^{\mu} + M_1 C_1\right) \\
&\quad \cdot \left[t_2^{1-\delta} - t_1^{1-\delta} + 2(t_2 - t_1)^{1-\delta} \right].
\end{aligned} \tag{42}$$

Similarly, we can get

$$\begin{aligned}
&|T_2(x,y)(t_2) - T_2(x,y)(t_1)| \le |\tau_2| LR^{\lambda} |t_2 - t_1| \\
&\quad + \left|\frac{b}{1-b\triangle_2}\right| \frac{M_2}{\Gamma(\beta+\theta_2+1)} |t_2 - t_1| + \frac{M_2}{|1-b\triangle_2|} \\
&\quad \cdot \frac{1}{\Gamma(\beta+1)} |t_2 - t_1| + \frac{M_2}{\Gamma(\beta+1)} \left|t_2^{\beta} - t_1^{\beta}\right|. \\
&\left|{}^C D_{0^+}^{\gamma} T_2(x,y)(t_2) - {}^C D_{0^+}^{\gamma} T_2(x,y)(t_1)\right| \\
&\quad \le \frac{1}{\Gamma(2-\gamma)} \left(|\tau_2| LR^{\lambda} + M_2 C_2\right) \\
&\quad \cdot \left[t_2^{1-\gamma} - t_1^{1-\gamma} + 2(t_2 - t_1)^{1-\gamma} \right].
\end{aligned} \tag{43}$$

This together with (39), (42), and (43) implies that $T(B_R)$ are equicontinuous. By virtue of the Arzela-Ascoli theorem, we can infer that the operator T is completely continuous. Hence, applying Schauder fixed theorem, FDE (4) has at least one solution (x, y) in B_R. □

Theorem 11. *Assume that (H_2)–(H_4) hold. Then FDE (4) has at least one solution.*

Proof. Firstly, we show that T is completely continuous operator. From the continuity of the functions f, g, h, and φ, it follows that the operator T is continuous. Let $B_Q = \{(x,y) \in X \times Y \mid \|(x,y)\|_{X\times Y} \le Q\}$, where Q is described as in (H_4); we now prove that $T(B_Q)$ is relatively compact.

For any $(x, y) \in B_Q$, by similar computation as (31) and (34), we obtain that

$$\begin{aligned}
&|T_1(x,y)(t)| \le |1 + \tau_1| KQ^{\mu} + A_1 (a_1 + b_1\phi_1(Q) \\
&\quad + c_1\psi_1(Q)), \\
&\left|{}^C D_{0^+}^{\delta} T_1(x,y)(t)\right| \le \frac{1}{\Gamma(2-\delta)} \left[|\tau_1| KQ^{\mu}\right. \\
&\quad \left. + C_1 (a_1 + b_1\phi_1(Q) + c_1\psi_1(Q))\right],
\end{aligned} \tag{44}$$

which imply

$$\|T_1(x,y)(t)\|_X \le \left(|1+\tau_1| + \frac{|\tau_1|}{\Gamma(2-\delta)}\right) KQ^{\mu} + D_1\left(a_1 + b_1\phi_1(Q) + c_1\psi_1(Q)\right). \tag{45}$$

Similarly, we have

$$\|T_2(x,y)(t)\|_Y \le \left(|1+\tau_2| + \frac{|\tau_2|}{\Gamma(2-\gamma)}\right) LQ^{\lambda} + D_2\left(a_2 + b_2\phi_2(Q) + c_2\psi_2(Q)\right). \tag{46}$$

Combining (45) and (46), one can get

$$\begin{aligned}\|T(x,y)\|_{X\times Y} &\le \left(|1+\tau_1| + \frac{|\tau_1|}{\Gamma(2-\delta)}\right) KQ^{\mu} \\ &\quad + D_1\left(a_1 + b_1\phi_1(Q) + c_1\psi_1(Q)\right) \\ &\quad + \left(|1+\tau_2| + \frac{|\tau_2|}{\Gamma(2-\gamma)}\right) LQ^{\lambda} \\ &\quad + D_2\left(a_2 + b_2\phi_2(Q) + c_2\psi_2(Q)\right) \\ &= B_1KQ^{\mu} + D_1P_1 + B_2LQ^{\lambda} + D_2P_2.\end{aligned} \tag{47}$$

By (47) and (H_4) we know $T(B_Q) \subset B_Q$. Using similar computation as in (39), (40), (42), and (43), we can obtain $T(B_Q)$ are equicontinuous. Thus, T is completely continuous by Arzela-Ascoli theorem.

We now claim that T has at least one solution in $X \times Y$.

Suppose there exists $(x,y) \in \partial B_Q$ such that $(x,y) = \lambda T(x,y)$ for some $\lambda \in (0,1)$. This together with (44) implies that

$$\begin{aligned}|x(t)| &= \lambda\left|T_1(x,y)(t)\right| \le \left|T_1(x,y)(t)\right|, \\ \left|{}^{C}D_{0^+}^{\delta}x(t)\right| &= \lambda\left|{}^{C}D_{0^+}^{\delta}T_1(x,y)(t)\right| \\ &\le \left|{}^{C}D_{0^+}^{\delta}T_1(x,y)(t)\right|.\end{aligned} \tag{48}$$

By virtue of (44)–(47), it is easy to see

$$\|(x,y)\|_{X\times Y} \le B_1KQ^{\mu} + D_1P_1 + B_2LQ^{\lambda} + D_2P_2 < Q. \tag{49}$$

This is a contradiction with $(x,y) \in \partial B_Q$. Therefore, it follows from Lemma 9 that the operator T has a fixed point $(x,y) \in \overline{B}_Q$. Thus FDE (4) has at least one solution in $X \times Y$. □

Remark 12. Comparing (H_1) with (H_3), we know that condition (H_3) is more extensive than (H_1). However, the assumptions of Theorem 10 are easier to verify than that of Theorem 11.

4. Examples

Example 1. Consider the following coupled system:

$$\begin{aligned}{}^{C}D_{0^+}^{5/4}x(t) &= \frac{1}{\sqrt{25+t^2}}\cos t + \frac{1}{2}\left(y(t)\right)^{1/5} \\ &\quad + \frac{1}{4(1+t)}\left({}^{C}D_{0^+}^{1/4}y(t)\right)^{1/3}, \\ &\qquad t \in [0,1], \\ {}^{C}D_{0^+}^{3/2}y(t) &= \frac{e^{-t}}{3+t^2} + \frac{1}{4}\left(x(t)\right)^{1/7} \\ &\quad + \frac{1}{4(2+t)}\sin\left({}^{C}D_{0^+}^{1/2}x(t)\right)^{1/5}, \\ &\qquad t \in [0,1], \\ x(0) &= \frac{1}{4}\left(\int_0^1 x(t)\,dt\right)^{1/7}, \\ x(1) &= 3I_{0^+}^{1/4}x\left(\frac{1}{3}\right), \\ y(0) &= \frac{1}{2}\left(\int_0^1 y(t)\,dt\right)^{1/5}, \\ y(1) &= 2I_{0^+}^{1/2}y\left(\frac{1}{2}\right),\end{aligned} \tag{50}$$

where $\alpha = 5/4$, $\beta = 3/2$, $\gamma = 1/4$, $\delta = 1/2$, $a = 3$, $b = 2$, $\theta_1 = 1/4$, $\theta_2 = 1/2$, $\eta = 1/3$, $\xi = 1/2$.

(50) can be regarded as the form of (4), where

$$\begin{aligned}f(t,u,v) &= \frac{1}{\sqrt{25+t^2}}\cos t + \frac{1}{2}\left(u(t)\right)^{1/5} \\ &\quad + \frac{1}{4(1+t)}\left(v(t)\right)^{1/3}, \\ g(t,u,v) &= \frac{e^{-t}}{3+t^2} + \frac{1}{4}\left(u(t)\right)^{1/7} \\ &\quad + \frac{1}{4(2+t)}\sin\left(v(t)\right)^{1/5}, \\ h(x) &= \frac{1}{4}\left(\int_0^1 x(t)\,dt\right)^{1/7}, \\ \varphi(y) &= \frac{1}{2}\left(\int_0^1 y(t)\,dt\right)^{1/5}.\end{aligned} \tag{51}$$

It is not difficult to see

$$\begin{aligned}\left|f(t,u,v)\right| &\le \frac{1}{5} + \frac{1}{2}\left(|u|\right)^{1/5} + \frac{1}{4}\left(|v|\right)^{1/3}, \\ \left|g(t,u,v)\right| &\le \frac{1}{3} + \frac{1}{4}\left(|u|\right)^{1/7} + \frac{1}{8}\left(|v|\right)^{1/5},\end{aligned}$$

$$|h(x)| \le \frac{1}{4}\|x\|_c^{1/7},$$
$$|\varphi(y)| \le \frac{1}{2}\|y\|_c^{1/5}. \tag{52}$$

Choose $k_0 = 1/5$, $k_1 = 1/2$, $k_2 = 1/4$, $l_0 = 1/3$, $l_1 = 1/4$, $l_2 = 1/8$, $m_1 = 1/5$, $m_2 = 1/3$, $n_1 = 1/7$, $n_2 = 1/5$, $K = 1/4$, $L = 1/2$, $\mu = 1/7$, $\lambda = 1/5$. Obviously, $h(0) = \varphi(0) = 0$. Therefore, (H_1) and (H_2) are satisfied. Immediately, we can conclude that FDE (4) has at least one solution by Theorem 10.

Example 2. Consider the following problem:

$$\begin{aligned}
{}^{C}D_{0^+}^{3/2}x(t) &= \frac{(1-t)^2}{3000} + \frac{y^2(t)}{7000(1+|y|)} \\
&\quad + \frac{t}{5000}\left({}^{C}D_{0^+}^{1/2}y(t)\right)^2, \quad t \in [0,1], \\
{}^{C}D_{0^+}^{5/4}y(t) &= \frac{(1+t)^2}{4000} + \frac{(1-t)x^4(t)}{4000(1+x^2)} \\
&\quad + \frac{t^2}{1000}\left({}^{C}D_{0^+}^{1/2}x(t)\right)^3, \quad t \in [0,1], \\
x(0) &= \frac{1}{200}\left(\max_{t\in[0,1]}|x(t)|\right)^{1/3}, \\
x(1) &= 4I_{0^+}^{1/4}x\left(\frac{1}{4}\right), \\
y(0) &= \frac{1}{400}\left(\max_{t\in[0,1]}|y(t)|\right)^{1/2}, \\
y(1) &= 2I_{0^+}^{1/2}y\left(\frac{1}{2}\right),
\end{aligned} \tag{53}$$

where $\alpha = 3/2$, $\beta = 5/4$, $\gamma = 1/2$, $\delta = 1/2$, $a = 4$, $b = 2$, $\theta_1 = 1/4$, $\theta_2 = 1/2$, $\eta = 1/4$, $\xi = 1/2$.

(53) can be regarded as the form of (4), where

$$\begin{aligned}
f(t,u,v) &= \frac{(1-t)^2}{3000} + \frac{u^2}{7000(1+|u|)} + \frac{t}{5000}v^2, \\
g(t,u,v) &= \frac{(1+t)^2}{4000} + \frac{(1-t)u^4}{4000(1+u^2)} + \frac{t^2}{1000}v^3, \\
h(x) &= \frac{1}{200}\left(\max_{t\in[0,1]}|x(t)|\right)^{1/3}, \\
\varphi(y) &= \frac{1}{400}\left(\max_{t\in[0,1]}|y(t)|\right)^{1/2}.
\end{aligned} \tag{54}$$

It is not difficult to see

$$\begin{aligned}
|f(t,u,v)| &\le \frac{1}{3000} + \frac{|u|^2}{7000} + \frac{|v|^2}{5000}, \\
|g(t,u,v)| &\le \frac{1}{1000} + \frac{1}{4000}|u|^2 + \frac{1}{1000}|v|^3, \\
|h(x)| &\le \frac{1}{200}\|x\|_c^{1/3}, \\
|\varphi(y)| &\le \frac{1}{400}\|y\|_c^{1/2}.
\end{aligned} \tag{55}$$

Choose $a_1 = 1/3000$, $b_1 = 1/7000$, $c_1 = 1/5000$, $\phi_1(|u|) = |u|^2$, $\psi_1(|v|) = |v|^2$, $a_2 = 1/1000$, $b_2 = 1/4000$, $c_2 = 1/1000$, $\phi_2(|u|) = |u|^2$, $\psi_2(|v|) = |v|^3$, $K = 1/200$, $L = 1/400$, $\mu = 1/3$, $\lambda = 1/2$. Now we need to verify (H_4). By direct calculation, it is easy to see

$$\begin{aligned}
\triangle_1 &= 0.0781, \\
\sigma_1 &= 0.7801, \\
\tau_1 &= 3.0838, \\
\triangle_2 &= 0.2660, \\
\sigma_2 &= 0.1995, \\
\tau_2 &= -1.2842, \\
A_1 &= 11.2022, \\
B_1 &= 4.9537, \\
C_1 &= 10.7321, \\
D_1 &= 14.2296, \\
P_1 &= 0.0051, \\
A_2 &= 5.9608, \\
B_2 &= 0.6447, \\
C_2 &= 5.9846, \\
D_2 &= 7.6490, \\
P_2 &= 0.0690.
\end{aligned} \tag{56}$$

Thus, we can get $B_1KQ^{\mu} + D_1P_1 + B_2LQ^{\lambda} + D_2P_2 = 0.0394 + 0.0726 + 0.0032 + 0.5278 = 0.7582 < 4$ for $Q = 4$, which means that (H_4) holds. Consequently, FDE (4) has at least one solution by Theorem 11.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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