Research Article

On Fekete-Szegö Problems for Certain Subclasses Defined by $q$-Derivative

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We derive the Fekete-Szegö theorem for new subclasses of analytic functions which are $q$-analogue of well-known classes introduced before.

1. Introduction

Denote by $A$ the class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$.

For two analytic functions $f$ and $g$ in $U$, the subordination between them is written as $f \prec g$. Frankly, the function $f(z)$ is subordinate to $g(z)$ if there is a Schwarz function $w$ with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. Note that, if $g$ is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

In [1, 2], Jackson defined the $q$-derivative operator $D_q$ of a function as follows:

$$D_q f(z) = \frac{f(qz) - f(z)}{(q - 1)z} \quad (z \neq 0, \ q \neq 0)$$

and $D_q f(z) = f'(0)$. In case $f(z) = z^k$ for $k$ is a positive integer, the $q$-derivative of $f(z)$ is given by

$$D_q z^k = \frac{z^k - (zq)^k}{z(1 - q)} = [k]_q z^{k-1}.$$  

As $q \to 1^-$ and $k \in \mathbb{N}$, we have

$$[k]_q = \frac{1 - q^k}{1 - q} \to k.$$  

Quite a number of great mathematicians studied the concepts of $q$-derivative, for example, by Gasper and Rahman [3], Aral et al. [4], Li et al. [5], and many others (see [6–15]).

Making use of the $q$-derivative, we define the subclasses $\delta_q^*(\alpha)$ and $\psi_q(\alpha)$ of the class $A$ for $0 \leq \alpha < 1$ by

$$\delta_q^*(\alpha) = \left\{ f \in A : \text{Re}\left(\frac{z D_q (f(z))}{f(z)}\right) > \alpha, \ z \in U \right\},$$

$$\psi_q(\alpha) = \left\{ f \in A : \text{Re}\left(1 + \frac{zq D_q (D_q (f(z)))}{D_q f(z)}\right) > \alpha, \ z \in U \right\}.$$  

These classes are also studied and introduced by Seoudy and Aouf [16].
Noting that
\[ f \in \mathcal{C}_q(\alpha) \iff zD_q f \in \mathcal{S}^*_q(\alpha), \]
\[ \lim_{q \to 1} \mathcal{S}^*_q(\alpha) = \left\{ f \in \mathcal{A} : \lim_{q \to 1} \mathrm{Re} \left( \frac{zD_q f(z)}{f(z)} \right) > \alpha, z \in \mathbb{U} \right\} = \mathcal{S}^*(\alpha), \]
\[ \lim_{q \to 1} \mathcal{C}_q(\alpha) = \left\{ f \in \mathcal{A} : \lim_{q \to 1} \mathrm{Re} \left( 1 + \frac{zqD_q \left( D_q f(z) \right)}{D_q f(z)} \right) > \alpha, z \in \mathbb{U} \right\} = \mathcal{C}(\alpha), \]
where \( \mathcal{S}^*(\alpha) \) and \( \mathcal{C}(\alpha) \) are, respectively, the classes of starlike of order \( \alpha \) and convex of order \( \alpha \) in \( \mathbb{U} \) ([17, 18]).

Next, we state the \( q \)-analogue of Ruscheweyh operator given by Aldiweby and Darius [8] that will be used throughout.

**Definition 1** (see [8]). Let \( f \in \mathcal{A} \). Denote by \( R^\lambda_q \) the \( q \)-analogue of Ruscheweyh operator defined by
\[ R^\lambda_q f(z) = z + \sum_{k=1}^{\infty} \left[ \frac{(k+\lambda-1)q!}{[\lambda]_q!} - [k-1]q! \right] a_k z^k, \]
where \([k]_q!\) given by is as follows:
\[ [k]_q! = \begin{cases} [k]_q! [k-1]_q! \cdots [1]_q!, & k = 1, 2, \ldots; \\ 1, & k = 0. \end{cases} \]

From the definition we observe that if \( q \to 1 \), we have
\[ \lim_{q \to 1} R^\lambda_q f(z) = z + \sum_{k=1}^{\infty} \left( \frac{(k+\lambda-1)q!}{[\lambda]!} - [k-1]q! \right) a_k z^k = R^\lambda f(z), \]
where \( R^\lambda \) is Ruscheweyh differential operator defined in [19].

Using the principle of subordination and \( q \)-derivative, we define the classes of \( q \)-starlike and \( q \)-convex analytic functions as follows.

**Definition 2.** For \( \varphi \in \mathcal{P} \) and \( \lambda > -1 \), the class \( \mathcal{S}^*_{q,\lambda}(\varphi) \) which consists of all analytic functions \( f \in \mathcal{A} \) satisfies
\[ \frac{zD_q \left( R^\lambda_q (f(z)) \right)}{R^\lambda_q (f(z))} < \varphi(z), \quad |z| < 1. \]
Since \( w(z) \) is a Schwarz function, immediately \( \text{Re}(p(z)) > 0 \) and \( p(0) = 1 \). Let
\[
g(z) = zD_q \left( \mathcal{R}^q_1(f(z)) \right) = 1 + d_1 z + d_2 z^2 + \cdots. \tag{18}
\]
Then from (16), (17), and (18), obtain
\[
g(z) = q \left( \frac{p(z) - 1}{p(z) + 1} \right). \tag{19}
\]
Since
\[
\frac{p(z) - 1}{p(z) + 1} = 1 + \frac{1}{2} \left[ p_1 z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \left( p_3 + \frac{p_2^3}{4} - p_1 p_2 \right) z^3 + \cdots \right] \tag{20}
\]
we have
\[
q \left( \frac{p(z) - 1}{p(z) + 1} \right) = 1 + \frac{1}{2} B_1 p_1 z
+ \frac{1}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right) z^2
+ \cdots. \tag{21}
\]
From the last equation and (18), we obtain
\[
d_1 = \frac{1}{2} B_1 p_1, \quad d_2 = \frac{1}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right). \tag{22}
\]
A simple computation in (18) and knowing that \( [n]_q - 1 = q [n-1]_q \), we obtain
\[
zD_q \left( \mathcal{R}^q_1(f(z)) \right) \]
\[
= 1 + q [\lambda + 1]_q a_2 z
+ \left[ q [\lambda + 1]_q [\lambda + 2]_q a_3 - q [\lambda + 1]^2_2 a_2^2 \right] z^2
+ \cdots. \tag{23}
\]
Then, from last equation and (18), we see that
\[
d_1 = q [\lambda + 1]_q a_2, \quad d_2 = q [\lambda + 1]_q [\lambda + 2]_q a_3 - q [\lambda + 1]^2_2 a_2^2, \tag{24}
\]
or equivalently, we have
\[
a_2 = \frac{B_1 p_1}{2q [\lambda + 1]_q}, \quad a_3 = \frac{B_1}{2q [\lambda + 1]_q [\lambda + 2]_q} \left( p_2 - \frac{p_1^2}{2} \right)
+ \frac{B_2 p_3^2}{4q [\lambda + 1]_q [\lambda + 2]_q}
+ \frac{B_2^2 p_4^2}{8q^2 [\lambda + 1]_q [\lambda + 2]_q}. \tag{25}
\]
Therefore
\[
a_3 - \mu a_2^2 = \frac{B_1}{2q [\lambda + 1]_q [\lambda + 2]_q} \left( p_2 - \nu p_1^2 \right), \tag{26}
\]
where
\[
y = \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} \right]
- \frac{[\lambda]_q + q^4 - \mu \left( [\lambda]_q + q^4 (1 + q) \right)}{q ([\lambda]_q + q^4) B_1}. \tag{27}
\]
By an application of Lemma 4, our result follows. Again by Lemma 4, the equality in (15) is gained for
\[
p(z) = \frac{1 + z}{1 - z} \tag{28}
\]
or \( p(z) = \frac{1 + z^2}{1 - z^2} \).
Thus Theorem 6 is complete. \( \Box \)

Similarly, we can prove for the class \( \mathcal{C}_{q,\lambda}(\varphi) \). We omit the proofs.

**Theorem 7.** Let \( \varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots \in P \). If \( f \) given by (1) is in the class \( \mathcal{C}_{q,\lambda}(\varphi) \) and \( \mu \) is a complex number, then
\[
\left| a_3 - \mu a_2^2 \right| \leq \frac{B_1}{2q [\lambda + 1]_q [\lambda + 2]_q} \left. \left| p_2 - \nu p_1^2 \right| \right. \tag{29}
\]
\[
\cdot \max \left\{ 1, \frac{B_2}{B_1} \left[ \frac{[\lambda]_q + q^4 (1 + q) \left( [\lambda]_q + q^4 \right)}{q [\lambda + 1]_q [\lambda + 2]_q} \right] \right\}.
\]
The result is sharp.

Taking \( \lambda = 0 \) in Theorem 6, we have the corollary for the class \( \mathcal{S}^*_q(\varphi) \) as follows.

**Corollary 8.** Let \( \varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots \in P \). If \( f \) given by (1) is in the class \( \mathcal{S}^*_q(\varphi) \) and \( \mu \) is a complex number, then
\[
\left| a_3 - \mu a_2^2 \right| \leq \frac{B_1}{q (1 + q)} \max \left\{ 1, \frac{B_2}{B_1} \left[ 1 - \frac{\mu (1 + q)}{q} \right] B_1 \right\}. \tag{30}
\]
The result is sharp.
Taking $q \to 1$ and $\lambda = 0$ in Theorem 6, we obtain the following.

**Corollary 9.** Let $q(z) = 1 + B_1 z + B_2 z^2 + \cdots$, $B_1 \in P$. If $f$ given by (1) is in the class $C_{q, \lambda}(\phi)$ and $\mu$ is a complex number, then

$$|a_3 - \mu a_2^2| \leq \frac{B_2}{2} \max \left\{ 1, \frac{B_2}{B_1} + \frac{1 - 2\mu}{1} \right\}.$$  \hspace{1cm} (31)

By using Lemma 4, we have the following theorem.

**Theorem 10.** Let $q(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$ and $B_2 \geq 0$. Let

$$\xi_1 = \frac{\left[ \lambda \right]_q + q^2 \left( B_2 - B_1 \right)}{\left[ \lambda \right]_q + q^2 [2]_q B_1^2},$$

$$\xi_2 = \frac{\left[ \lambda \right]_q + q^2 \left( B_2 + B_1 \right)}{\left[ \lambda \right]_q + q^2 [2]_q B_1^2}.$$  \hspace{1cm} (32)

Let $f$ given by (1) be in the class $C_{q, \lambda}(\phi)$. Then

$$|a_3 - \mu a_2^2| \leq \frac{B_2}{q \left[ \lambda + 1 \right]_q \left[ \lambda + 2 \right]_q} \left[ \frac{\left[ \lambda \right]_q + q^2 - \left(\left[ \lambda \right]_q + q^2 [2]_q \right) \mu}{q \left(\left[ \lambda \right]_q + q^2 \right)} \right],$$  \hspace{1cm} if $\mu \leq \xi_1$;

$$- \frac{B_2}{q \left[ \lambda + 1 \right]_q \left[ \lambda + 2 \right]_q} \left( \frac{\left[ \lambda \right]_q + q^2 - \left(\left[ \lambda \right]_q + q^2 [2]_q \right) \mu}{q \left(\left[ \lambda \right]_q + q^2 \right)} \right).$$  \hspace{1cm} (35)

Now, let $\xi_1 \leq \mu \leq \xi_2$; then using the above calculation, we obtain

$$|a_3 - \mu a_2^2| \leq \frac{B_2}{q \left[ \lambda + 1 \right]_q \left[ \lambda + 2 \right]_q}.$$  \hspace{1cm} (36)

Similarly, we can prove for the class $C_{q, \lambda}(\phi)$ as follows.

**Theorem 11.** Let $q(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$ and $B_2 \geq 0$. Let

$$\xi_1 = \frac{\left[ \lambda \right]_q \left( B_2 - B_1 \right)}{\left[ \lambda \right]_q B_1^2},$$

$$\xi_2 = \frac{\left[ \lambda \right]_q \left( B_2 + B_1 \right)}{\left[ \lambda \right]_q B_1^2}.$$  \hspace{1cm} (37)

If $f$ given by (1) is in the class $C_{q, \lambda}(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \frac{B_2}{2q \left[ \lambda + 1 \right]_q \left[ \lambda + 2 \right]_q} \left[ \frac{B_2}{B_1} + \left( \frac{\left[ \lambda \right]_q + q^2 \left[ \lambda + 2 \right]_q \mu}{q \left[ 2 \right]_q} \right) B_1 \right],$$  \hspace{1cm} if $\mu \leq \xi_1$;

$$- \frac{B_2}{2q \left[ \lambda + 1 \right]_q \left[ \lambda + 2 \right]_q} \left( \frac{\left[ \lambda \right]_q + q^2 \left[ \lambda + 2 \right]_q \mu}{q \left[ 2 \right]_q} \right) B_1.$$  \hspace{1cm} (38)

Finally, if $\mu \geq \xi_2$, then

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2q \left[ \lambda + 1 \right]_q \left[ \lambda + 2 \right]_q} \left[ \frac{B_2}{B_1} - \left( \frac{\left[ \lambda \right]_q + q^2 \left[ \lambda + 2 \right]_q \mu}{q \left[ 2 \right]_q} \right) B_1 \right],$$  \hspace{1cm} if $\xi_1 \leq \mu \leq \xi_2$;
Taking $\lambda = 0$ in Theorem 10, we obtain next result for the class $\mathcal{S}_q^*(\phi)$.

**Corollary 12.** Let $q(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$ and $B_2 \geq 0$. Let

$$
\sigma_1 = \frac{B_2^2 + q(B_2 - B_1)}{[2]_q B_1^2},
$$

$$
\sigma_2 = \frac{B_2^2 + q(B_2 + B_1)}{[2]_q B_1^2}.
$$

If $f$ given by (1) is in the class $\mathcal{S}_q^*(\phi)$, then

$$
|d_3 - \mu_2| \leq \begin{cases} 
\frac{B_2}{q[2]_q} + \frac{B_1^2}{q[2]_q} \left( \frac{1 - [2]_q \mu}{q} \right), & \text{if } \mu \leq \sigma_1; \\
\frac{B_1}{q[2]_q}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\
-\frac{B_2}{q[2]_q} - \frac{B_1^2}{q[2]_q} \left( \frac{1 - [2]_q \mu}{q} \right), & \text{if } \mu \geq \sigma_2.
\end{cases}
$$

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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**References**


