Research Article

Bounded Subsets of Classes $M^p(X)$ of Holomorphic Functions

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Some characterizations of boundedness in $M^p(X)$ will be described, where $M^p(X)$ ($0 < p < \infty$) are F -algebras which consist of holomorphic functions defined by maximal functions.

1. Introduction

Let n be a positive integer. The space of n -complex variables $z = (z_1, \dots, z_n)$ is denoted by $\mathbb{C}^n$. The unit polydisk $\{z \in \mathbb{C}^n : |z_j| < 1, 1 \leq j \leq n\}$ is denoted by U^n and the distinguished boundary $\mathbb{T}^n$ is $\{\zeta \in \mathbb{C}^n : |\zeta_j| = 1, 1 \leq j \leq n\}$. The unit ball $\{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 < 1\}$ is denoted by B_n and $S_n = \{\zeta \in \mathbb{C}^n : \sum_{j=1}^n |\zeta_j|^2 = 1\}$ is its boundary. In this paper, X denotes the unit polydisk or the unit ball for $n \geq 1$ and ∂X denotes $\mathbb{T}^n$ for $X = U^n$ or S_n for $X = B_n$. The normalized (in the sense that $\sigma(\partial X) = 1$) Lebesgue measure on ∂X is denoted by $d\sigma$.

The Hardy space on X is denoted by $H^q(X)$ ($0 < q \leq \infty$). The Nevanlinna class $N(X)$ on X is defined as the set of all holomorphic functions f on X such that

$$\sup_{0 \leq r < 1} \int_{\partial X} \log^+ |f(r\zeta)| d\sigma(\zeta) < \infty \quad (1)$$

holds. It is known that $f \in N(X)$ has a finite nontangential limit, denoted by f^* , almost everywhere on ∂X .

The Smirnov class $N_+(X)$ is defined as the set of all $f \in N(X)$ which satisfy the equality

$$\begin{aligned} \sup_{0 \leq r < 1} \int_{\partial X} \log^+ |f(r\zeta)| d\sigma(\zeta) \\ = \int_{\partial X} \log^+ |f^*(\zeta)| d\sigma(\zeta). \end{aligned} \quad (2)$$

Define a metric

$$d_{N_+(X)}(f, g) = \int_{\partial X} \log(1 + |f^*(\zeta) - g^*(\zeta)|) d\sigma(\zeta) \quad (3)$$

for $f, g \in N_+(X)$. With the metric $d_{N_+(X)}(\cdot, \cdot)$ $N_+(X)$ is an F -algebra. Recall that an F -algebra is a topological algebra in which the topology arises from a complete metric.

The Privalov class $N^p(X)$, $1 < p < \infty$, is defined as the set of all holomorphic functions f on X such that

$$\sup_{0 \leq r < 1} \int_{\partial X} (\log^+ |f(r\zeta)|)^p d\sigma(\zeta) < \infty \quad (4)$$

holds. It is well-known that $N^p(X)$ is a subalgebra of $N_+(X)$; hence every $f \in N^p(X)$ has a finite nontangential limit almost everywhere on ∂X . Under the metric defined by

$$\begin{aligned} d_{N^p(X)}(f, g) \\ = \left(\int_{\partial X} (\log(1 + |f^*(\zeta) - g^*(\zeta)|))^p d\sigma(\zeta) \right)^{1/p} \end{aligned} \quad (5)$$

for $f, g \in N^p(X)$, $N^p(X)$ becomes an F -algebra (cf. [1]).

Now we define the class $M^p(X)$. For $0 < p < \infty$, the class $M^p(X)$ is defined as the set of all holomorphic functions f on X such that

$$\int_{\partial X} (\log^+ Mf(\zeta))^p d\sigma(\zeta) < \infty, \quad (6)$$

where $Mf(\zeta) := \sup_{0 \leq r < 1} |f(r\zeta)|$ is the maximal function. The class $M^p(X)$ with $p = 1$ in the case $n = 1$ was introduced by Kim in [2]. As for $p > 0$ and $n \geq 1$, the class was considered in [3, 4]. For $f, g \in M^p(X)$, define a metric

$$\begin{aligned} d_{M^p(X)}(f, g) \\ = \left\{ \int_{\partial X} (\log(1 + M(f - g)(\zeta)))^p d\sigma(\zeta) \right\}^{\alpha_p/p}, \end{aligned} \quad (7)$$

where $\alpha_p = \min(1, p)$. With this metric $M^p(X)$ is also an F -algebra (see [5]).

It is well-known that the following inclusion relations hold:

$$H^q(X) \subsetneq N^p(X) \subsetneq M^1(X) \subsetneq N_*(X) \quad (0 < q \leq \infty, p > 1). \quad (8)$$

Moreover, it is known that $N(X) \subsetneq M^p(X)$ ($0 < p < 1$) [6].

A subset L of a linear topological space A is said to be *bounded* if for any neighborhood U of zero in A there exists a real number α , $0 < \alpha < 1$, such that $\alpha L = \{\alpha f; f \in L\} \subset U$. Yanagihara characterized bounded subsets of $N_*(X)$ in the case $n = 1$ [7]. As for $M^p(X)$ with $p = 1$ in the case $n = 1$, Kim described some characterizations of boundedness (see [2]). For $p > 1$ and $n = 1$, these characterizations were considered by Meštrović [8]. As for $N^p(X)$ with $p > 1$ in the case $n \geq 1$, Subbotin investigated the properties of boundedness [1].

In this paper, we consider some characterizations of boundedness in $M^p(X)$ with $0 < p < \infty$ in the case $n \geq 1$.

2. The Results

Theorem 1. Let $0 < p < \infty$. $L \subset M^p(X)$ is bounded if and only if

(i) there exists $K < \infty$ such that

$$\int_{\partial X} (\log^+ Mf(\zeta))^p d\sigma(\zeta) < K \quad (9)$$

for all $f \in L$;

(ii) for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_E (\log^+ Mf(\zeta))^p d\sigma(\zeta) < \varepsilon, \quad \forall f \in L, \quad (10)$$

for any measurable set $E \subset \partial X$ with the Lebesgue measure $|E| < \delta$.

Proof.

Necessity. Let L be a bounded subset of $M^p(X)$. We put $\beta_p = \max(1, p) = p/\alpha_p$.

(i) For any $\eta > 0$, there is a number $\alpha_0 = \alpha_0(\eta)$ ($0 < \alpha_0 < 1$) such that

$$\begin{aligned} & (d_{M^p(X)}(\alpha f, 0))^{\beta_p} \\ &= \int_{\partial X} (\log(1 + |\alpha| Mf(\zeta)))^p d\sigma(\zeta) < \eta^{\beta_p} \end{aligned} \quad (11)$$

for all $f \in L$ and $|\alpha| \leq \alpha_0$. It follows that

$$\int_{\partial X} (\log^+ |\alpha| Mf(\zeta))^p d\sigma(\zeta) < \eta^{\beta_p} \quad (12)$$

for all $f \in L$ and $|\alpha| \leq \alpha_0$. Since

$$\log^+ Mf \leq \log^+ \alpha_0 Mf + \log \frac{1}{\alpha_0}, \quad (13)$$

using the elementary inequality

$$(a + b)^p \leq 2^p (a^p + b^p) \quad (a \geq 0, b \geq 0, p > 0), \quad (14)$$

we have

$$\begin{aligned} & \int_{\partial X} (\log^+ Mf(\zeta))^p d\sigma(\zeta) \\ & \leq 2^p \left(\int_{\partial X} (\log^+ \alpha_0 Mf(\zeta))^p d\sigma(\zeta) \right. \\ & \quad \left. + \int_{\partial X} \left(\log \frac{1}{\alpha_0} \right)^p d\sigma(\zeta) \right) = 2^p \left(\eta^{\beta_p} \right. \\ & \quad \left. + \left(\log \frac{1}{\alpha_0} \right)^p \right) = K = \text{constant}. \end{aligned} \quad (15)$$

Thus (i) is satisfied.

(ii) For given $\varepsilon > 0$, we take η as $\eta < (\varepsilon/2^{p+1})^{1/\beta_p}$ and $\alpha_0 = \alpha_0(\eta)$ as above. Next take $\delta > 0$ such that

$$\delta \left(\log \frac{1}{\alpha_0} \right)^p < \frac{\varepsilon}{2^{p+1}}. \quad (16)$$

Then, for each set $E \subset \partial X$ with $|E| < \delta$ and for every $f \in L$, we obtain

$$\begin{aligned} & \int_E (\log^+ Mf(\zeta))^p d\sigma(\zeta) \\ & \leq 2^p \left(\int_E (\log^+ \alpha_0 Mf(\zeta))^p d\sigma(\zeta) \right. \\ & \quad \left. + \int_E \left(\log \frac{1}{\alpha_0} \right)^p d\sigma(\zeta) \right) \leq 2^p \eta^{\beta_p} + 2^p |E| \\ & \quad \cdot \left(\log \frac{1}{\alpha_0} \right)^p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned} \quad (17)$$

Therefore, the condition (ii) is satisfied.

Sufficiency. Let

$$V = \{g \in M^p(X); d_{M^p(X)}(g, 0) < \eta\} \quad (18)$$

be a neighborhood of 0 in $M^p(X)$. Take $\varepsilon > 0$ such that

$$(\log(1 + \varepsilon))^p + 2^p (\log 2)^p \varepsilon + 2^p \varepsilon < \eta^{\beta_p}. \quad (19)$$

Then, there is δ ($0 < \delta < \varepsilon$) such that (ii) is satisfied. For $f \in L$, we can find $E_f \subset \partial X$ so that

$$\begin{aligned} & |\partial X \setminus E_f| < \delta, \\ & (\log^+ Mf(\zeta))^p \leq \frac{K}{\delta} \quad \text{on } E_f \end{aligned} \quad (20)$$

by Chebyshev's inequality. We have

$$Mf(\zeta) \leq \exp \left(\frac{K}{\delta} \right)^{1/p} = A(\delta) = A \quad \text{on } E_f. \quad (21)$$

Choose α such that $0 < \alpha < \varepsilon/A$. Then, using inequality (14) and

$$\log(1+x) \leq \log 2 + \log^+ x \quad (x > 0), \quad (22)$$

we obtain, for every $f \in L$,

$$\begin{aligned} & (d_{M^p(X)}(\alpha f, 0))^{\beta_p} \\ &= \int_{\partial X} (\log(1 + |\alpha| Mf(\zeta)))^p d\sigma(\zeta) = \int_{E_f} + \int_{\partial X \setminus E_f} \\ &\leq \int_{E_f} (\log(1 + \varepsilon))^p d\sigma(\zeta) \\ &+ 2^p \left(\int_{\partial X \setminus E_f} (\log 2)^p d\sigma(\zeta) \right. \\ &\left. + \int_{\partial X \setminus E_f} (\log^+ Mf(\zeta))^p d\sigma(\zeta) \right) \leq (\log(1 + \varepsilon))^p \\ &+ 2^p (\log 2)^p \delta + 2^p \varepsilon < \eta^{\beta_p}. \end{aligned} \quad (23)$$

Therefore we get $d_{M^p(X)}(\alpha f, 0) < \eta$, which shows L is a bounded subset of $M^p(X)$.

The proof of the theorem is complete. $\square$

Remark 2. We note that the characterization of boundedness in $M^p(X)$ ($0 < p < \infty$) has the same conditions as the characterization of boundedness in the Smirnov class $N_*(X)$ in the case $n = 1$ ([7], Theorem 1), the class $M^p(X)$ with $p = 1$ in the case $n = 1$ ([2], Theorem 4.1), and the Privalov class $N^p(X)$ ($1 < p < \infty$) ([1], Theorem 5). On the other hand, we see that (ii) implies (i) in Theorem 1. Suppose that (ii) holds. Then there is a positive integer K such that

$$\int_E (\log^+ Mf(\zeta))^p d\sigma(\zeta) < 1, \quad \forall f \in L, \quad (24)$$

for any measurable set $E \subset \partial X$ with $|E| \leq 1/K$. There are measurable sets $E_1, \dots, E_K \subset \partial X$ such that $\bigcup_{j=1}^K E_j = \partial X$, $E_i \cap E_j = \emptyset$ for $i \neq j$, and $|E_j| = 1/K$ for every j . Then

$$\begin{aligned} & \int_{\partial X} (\log^+ Mf(\zeta))^p d\sigma(\zeta) \\ &= \sum_{j=1}^K \int_{E_j} (\log^+ Mf(\zeta))^p d\sigma(\zeta) < K \end{aligned} \quad (25)$$

holds (cf. [8] (Theorem 19 and Remark 20)).

Next we show a standard example of a bounded set of $M^p(X)$. The following theorem is easily proved in the same way of [1] (p.236) and [2] (Theorem 4.6); therefore, we do not prove it here.

Theorem 3. Let $0 < p < \infty$. If $f \in M^p(X)$, then $f_\rho(z) = f(\rho z)$ ($z \in X$, $0 \leq \rho < 1$) form a bounded set in $M^p(X)$.

Let $p > 1$ and we set $|f|_{N^p(X)} := d_{N^p(X)}(f, 0)$. Subbotin proved an equivalent condition that a subset $L \subset N^p(X)$ ($1 < p < \infty$) is bounded. The following is a theorem by Subbotin.

Theorem 4 (see [1]). Let $p > 1$. A subset $L \subset N^p(X)$ is bounded if and only if the following two conditions are satisfied:

- (i) There exists $K < \infty$ such that $|f|_{N^p(X)} \leq K$ for all $f \in L$.
- (ii) For each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_E (\log^+ |f^*(\zeta)|)^p d\sigma(\zeta) < \varepsilon, \quad \forall f \in L, \quad (26)$$

for any measurable set $E \subset \partial X$ with the Lebesgue measure $|E| < \delta$.

As shown in [1, 3], for any $p > 1$ the class $M^p(X)$ coincides with the class $N^p(X)$ and the metrics $d_{M^p(X)}$ and $d_{N^p(X)}$ are equivalent. Therefore the topologies induced by these metrics are identical on the set $M^p(X) = N^p(X)$.

The following theorem is clear; therefore the proof may be omitted.

Theorem 5. Let $p > 1$. A subset $L \subset M^p(X)$ is bounded if and only if the following two conditions are satisfied:

- (i) There exists $K < \infty$ such that

$$\int_{\partial X} (\log^+ |f^*(\zeta)|)^p d\sigma(\zeta) < K \quad (27)$$

for all $f \in L$.

- (ii) For each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_E (\log^+ |f^*(\zeta)|)^p d\sigma(\zeta) < \varepsilon, \quad \forall f \in L, \quad (28)$$

for any measurable set $E \subset \partial X$ with the Lebesgue measure $|E| < \delta$.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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References

- [1] A. V. Subbotin, "Functional properties of privalov spaces of holomorphic functions in several variables," *Mathematical Notes*, vol. 65, no. 1-2, pp. 230–237, 1999.
- [2] H. O. Kim, "On an F -algebra of holomorphic functions," *Canadian Journal of Mathematics*, vol. 40, no. 3, pp. 718–741, 1988.
- [3] B. R. Choe and H. O. Kim, "On the boundary behavior of functions holomorphic on the ball," *Complex Variables, Theory and Application*, vol. 20, no. 1–4, pp. 53–61, 1992.
- [4] H. O. Kim and Y. Y. Park, "Maximal functions of plurisubharmonic functions," *Tsukuba Journal of Mathematics*, vol. 16, no. 1, pp. 11–18, 1992.
- [5] V. I. Gavrilov and A. V. Subbotin, " F -algebras of holomorphic functions in a ball containing the Nevanlinna class," *Mathematica Montisnigri*, vol. 12, pp. 17–31, 2000 (Russian).

- [6] A. V. Subbotin, "Groups of linear isometries of spaces M^q of holomorphic functions of several complex variables," *Mathematical Notes*, vol. 83, no. 3-4, pp. 437–440, 2008.
- [7] N. Yanagihara, "Bounded subsets of some spaces of holomorphic functions," *Scientific Papers of the College of General Education, University of Tokyo*, vol. 23, pp. 19–28, 1973.
- [8] R. Meštrović, "On F -algebras M^p ($1 < p < \infty$) of holomorphic functions," *The Scientific World Journal*, vol. 2014, Article ID 901726, 10 pages, 2014.

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