Research Article

Oscillation Criteria for Nonlinear Third-Order Neutral Dynamic Equations with Damping on Time Scales

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We establish several oscillation criteria for a class of third-order nonlinear dynamic equations with a damping term and a nonpositive neutral coefficient by using the Riccati transformation. Two illustrative examples are presented to show the significance of the results obtained.

1. Introduction

In this paper, we are concerned with the oscillation of a class of third-order damped dynamic equations of neutral type

$$\begin{aligned} & \left(r(t) \varphi_{\gamma} \left(z^{\Delta\Delta}(t) \right) \right)^{\Delta} + d(t) \varphi_{\gamma} \left(z^{\Delta\Delta}(t) \right) \\ & + f(t, x(h(t))) = 0 \end{aligned} \quad (1)$$

on a time scale $\mathbb{T}$ satisfying $\sup \mathbb{T} = \infty$, where $t \in [t_0, \infty)_{\mathbb{T}}$, $\varphi_{\gamma}(u) = |u|^{\gamma-1}u$, and $z(t) = x(t) - p(t)x(g(t))$. Throughout, we suppose that the following conditions are satisfied:

(C₁) $\gamma \geq 1$ is a constant.

(C₂) $r \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$, $d \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $-d/r$ is positively regressive (i.e., $r(t) - (\sigma(t) - t)d(t) > 0$), and

$$\int_{t_0}^{\infty} \left(\frac{e_{-d/r}(t, t_0)}{r(t)} \right)^{1/\gamma} \Delta t = \infty. \quad (2)$$

(C₃) $p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [0, \infty))$ and $\lim_{t \rightarrow \infty} p(t) = p_0$, where $0 \leq p_0 < 1$.

(C₄) $g, h \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$, $g(t) \leq t$, $h(t) \leq t$ or $h(t) \geq t$, $\lim_{t \rightarrow \infty} g(t) = \lim_{t \rightarrow \infty} h(t) = \infty$, and there exists

a sequence $\{c_k\}_{k \geq 0}$ such that $\lim_{k \rightarrow \infty} c_k = \infty$ and $g(c_{k+1}) = c_k$.

(C₅) $f \in C([t_0, \infty)_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$ and there exists a function $q \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ such that $uf(t, u) > 0$ and $f(t, u)/(|u|^{\gamma-1}u) \geq q(t)$ for $u \neq 0$.

The theory of time scales, which was firstly introduced by Hilger in [1, 2], has been enriched by researchers; see, for instance, [3, 4], monographs [5, 6], and the references cited therein. During the past decade, a great deal of interest in oscillation of solutions to different classes of dynamic equations on time scales has been shown; we refer the reader to [7–23].

Yu and Wang [23] studied a third-order dynamic equation

$$\begin{aligned} & \left(\frac{1}{a_2(t)} \left(\left(\frac{1}{a_1(t)} \left(x^{\Delta}(t) \right)^{\alpha_1} \right)^{\Delta} \right)^{\alpha_2} \right)^{\Delta} + q(t) f(x(t)) \\ & = 0, \quad \alpha_1 \alpha_2 = 1. \end{aligned} \quad (3)$$

Agarwal et al. [8, 10], Candan [12], Erbe et al. [13], Hassan [15], and Li et al. [18] considered a third-order retarded dynamic equation

$$\left(a(t) \left[\left(r(t) x^{\Delta}(t) \right)^{\Delta} \right]^{\gamma} \right)^{\Delta} + f(t, x(\tau(t))) = 0. \quad (4)$$

Saker et al. [21] studied a second-order damped dynamic equation

$$(a(t)x^\Delta(t))^\Delta + p(t)x^{\Delta^\sigma}(t) + q(t)(f \circ x^\sigma) = 0, \quad (5)$$

whereas Qiu and Wang [20] considered a second-order damped dynamic equation

$$(r(t)\varphi_\gamma(x^\Delta(t)))^\Delta + p(t)\varphi_\gamma(x^\Delta(t)) + f(t, x(g(t))) = 0, \quad (6)$$

where $\gamma > 0$, $\varphi_\gamma(u) = |u|^{\gamma-1}u$, and

$$g(t) \geq \begin{cases} \sigma(t), & 0 < \gamma < 1, \\ t, & \gamma \geq 1. \end{cases} \quad (7)$$

Han et al. [14] and Qiu [19] investigated the third-order dynamic equations with nonpositive neutral coefficients

$$(r(t)z^{\Delta\Delta}(t))^\Delta + q(t)x^\gamma(h(t)) = 0, \quad (8)$$

$$r^\Delta \geq 0,$$

$$(r_1(t)((r_2(t)(z^\Delta(t))^{\gamma_2})^\Delta)^{\gamma_1})^\Delta + f(t, x(h(t))) = 0, \quad (9)$$

respectively, where $z(t) = x(t) - p(t)x(g(t))$.

In this paper, using the Riccati transformation, we obtain some sufficient conditions which ensure that every solution x of (1) either is oscillatory or converges to a finite number asymptotically. We do not impose restrictive assumption $d \geq 0$ in our results. To illustrate the significance of new results, two examples are provided in the last section. In what follows, all functional inequalities are assumed to hold for all sufficiently large t . Without loss of generality, we can deal only with eventually positive solutions of (1).

Definition 1. A solution x of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is termed nonoscillatory.

Definition 2. Equation (1) is said to be almost oscillatory if all its solutions either are oscillatory or converge to zero asymptotically.

2. Auxiliary Results

Lemma 3 (see [19, Lemma 2.1]). *Suppose that x is an eventually positive solution of (1) and there exists a constant $a \geq 0$ such that $\lim_{t \rightarrow \infty} z(t) = a$. Then*

$$\lim_{t \rightarrow \infty} x(t) = \frac{a}{1 - p_0}. \quad (10)$$

Lemma 4. *If x is an eventually positive solution of (1), then there exists a sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$ such that, for $t \in [T, \infty)_{\mathbb{T}}$,*

$$\left[\frac{r(t)(z^{\Delta\Delta}(t))^\gamma}{e_{-d/r}(t, T)} \right]^\Delta < 0, \quad z^{\Delta\Delta}(t) > 0, \quad (11)$$

$$z^\Delta(t) > 0$$

$$\text{or } z^\Delta(t) < 0.$$

Proof. Let x be an eventually positive solution of (1). From (C_3) and (C_4) , there exist a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ and a constant p_1 such that $p_0 < p_1 < 1$, $x(t) > 0$, $x(g(t)) > 0$, $x(h(t)) > 0$, and $p(t) \leq p_1$ for $t \in [t_1, \infty)_{\mathbb{T}}$. By virtue of (1) and (C_5) , we conclude that

$$\left[\frac{r(t)\varphi_\gamma(z^{\Delta\Delta}(t))}{e_{-d/r}(t, t_1)} \right]^\Delta = -\frac{f(t, x(h(t)))}{e_{-d/r}(\sigma(t), t_1)} < 0. \quad (12)$$

We claim that there exists a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $z^{\Delta\Delta}(t) > 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$. Otherwise, assume that $z^{\Delta\Delta}(t) < 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$. Then, by (12) and (C_2) , $\lim_{t \rightarrow \infty} z^\Delta(t) = -\infty$. It follows from $z^{\Delta\Delta}(t) < 0$ and $z^\Delta(t) < 0$ that $\lim_{t \rightarrow \infty} z(t) = -\infty$. Hence, there exists a $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $z(t) < 0$ for $t \in [t_3, \infty)_{\mathbb{T}}$; that is,

$$x(t) < p(t)x(g(t)) \leq p_1x(g(t)), \quad t \in [t_3, \infty)_{\mathbb{T}}. \quad (13)$$

By (C_4) , there exists a sufficiently large integer k_0 such that $c_k \in [t_3, \infty)_{\mathbb{T}}$ for $k \geq k_0$. For $k \geq k_0 + 1$, we have

$$\begin{aligned} x(c_k) &< p_1x(g(c_k)) = p_1x(c_{k-1}) < p_1^2x(g(c_{k-1})) \\ &= p_1^2x(c_{k-2}) < \cdots < p_1^{k-k_0}x(g(c_{k_0+1})) \\ &= p_1^{k-k_0}x(c_{k_0}), \end{aligned} \quad (14)$$

which yields $\lim_{k \rightarrow \infty} x(c_k) = 0$ and so $\lim_{k \rightarrow \infty} z(c_k) = 0$, which contradicts the fact that $\lim_{t \rightarrow \infty} z(t) = -\infty$. Therefore, $z^{\Delta\Delta}(t) > 0$ and hence $z^\Delta(t) > 0$ or $z^\Delta(t) < 0$. The proof is complete. $\square$

Lemma 5 (see [19, Lemma 2.3]). *If x is an eventually positive solution of (1), then z is eventually positive or $\lim_{t \rightarrow \infty} x(t) = 0$.*

Lemma 6. *Let x be an eventually positive solution of (1) and suppose that z and z^Δ are eventually positive. Assume also that*

$$\int_{t_0}^{\infty} \frac{q(t)h^\gamma(t)}{e_{-d/r}(\sigma(t), t_0)} \Delta t = \infty. \quad (15)$$

Then there exists a sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$ such that, for $t \in [T, \infty)_{\mathbb{T}}$,

$$\frac{z(h(t))}{z(t)} \geq \min \left\{ \frac{h(t)}{t}, 1 \right\}. \quad (16)$$

Proof. Let x be an eventually positive solution of (1) and assume that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $z(t) > 0$ and $z^\Delta(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Define

$$y(t) = z(t) - tz^\Delta(t), \quad t \in [t_1, \infty)_{\mathbb{T}}. \quad (17)$$

Then, by Lemma 4, for $t \in [t_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned} y^\Delta(t) &= z^\Delta(t) - (z^\Delta(t) + \sigma(t) z^{\Delta\Delta}(t)) \\ &= -\sigma(t) z^{\Delta\Delta}(t) < 0. \end{aligned} \quad (18)$$

We can prove that y is eventually positive. If not, then there exists a $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $y(t) < 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$. Hence, we conclude that

$$\begin{aligned} \left(\frac{z(t)}{t}\right)^\Delta &= \frac{tz^\Delta(t) - z(t)}{t\sigma(t)} = -\frac{y(t)}{t\sigma(t)} > 0, \\ t &\in [t_2, \infty)_{\mathbb{T}}, \end{aligned} \quad (19)$$

which implies that $z(t)/t$ is strictly increasing on $[t_2, \infty)_{\mathbb{T}}$. Since $\lim_{t \rightarrow \infty} h(t) = \infty$, there exists a $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that, for $t \in [t_3, \infty)_{\mathbb{T}}$, $h(t) \geq h(t_3) \geq t_2$, and so

$$\frac{z(h(t))}{h(t)} \geq \frac{z(h(t_3))}{h(t_3)}. \quad (20)$$

Using (C_5) , we have $f(t, x(h(t))) \geq q(t)x^\gamma(h(t)) \geq q(t)z^\gamma(h(t))$. Therefore, by virtue of (1) and Lemma 4, for $t \in [t_3, \infty)_{\mathbb{T}}$,

$$\begin{aligned} \left[\frac{r(t)(z^{\Delta\Delta}(t))^\gamma}{e_{-d/r}(t, t_3)} \right]^\Delta &= -\frac{f(t, x(h(t)))}{e_{-d/r}(\sigma(t), t_3)} \\ &\leq -\frac{q(t)z^\gamma(h(t))}{e_{-d/r}(\sigma(t), t_3)}. \end{aligned} \quad (21)$$

Integrating (21) from t_3 to t , we get

$$\begin{aligned} \frac{r(t)(z^{\Delta\Delta}(t))^\gamma}{e_{-d/r}(t, t_3)} - r(t_3)(z^{\Delta\Delta}(t_3))^\gamma \\ \leq - \int_{t_3}^t \frac{q(s)z^\gamma(h(s))}{e_{-d/r}(\sigma(s), t_3)} \Delta s \\ \leq - \frac{z^\gamma(h(t_3))}{h^\gamma(t_3)} \int_{t_3}^t \frac{q(s)h^\gamma(s)}{e_{-d/r}(\sigma(s), t_3)} \Delta s. \end{aligned} \quad (22)$$

It follows from (15) that

$$\begin{aligned} r(t_3)(z^{\Delta\Delta}(t_3))^\gamma &\geq \frac{z^\gamma(h(t_3))}{h^\gamma(t_3)} \int_{t_3}^t \frac{q(s)h^\gamma(s)}{e_{-d/r}(\sigma(s), t_3)} \Delta s \rightarrow \\ \infty, \quad t &\rightarrow \infty, \end{aligned} \quad (23)$$

which is a contradiction. Hence, y is eventually positive. Then, there exists a sufficiently large $T \in [t_1, \infty)_{\mathbb{T}}$ such that, for $t \in [T, \infty)_{\mathbb{T}}$,

$$\left(\frac{z(t)}{t}\right)^\Delta = -\frac{y(t)}{t\sigma(t)} < 0, \quad (24)$$

so $z(t)/t$ is strictly decreasing on $[T, \infty)_{\mathbb{T}}$. If $h(t) \geq t$, then $z(h(t))/z(t) \geq 1$. If $h(t) \leq t$, then $z(h(t))/z(t) \geq h(t)/t$. Therefore, we arrive at (16). This completes the proof. $\square$

Lemma 7. Assume that all assumptions of Lemma 6 are satisfied. For $t \in [t_1, \infty)_{\mathbb{T}}$, define

$$u(t) = A(t) \frac{r(t)(z^{\Delta\Delta}(t))^\gamma}{e_{-d/r}(t, t_1) z^\gamma(t)}, \quad (25)$$

where $A \in C_{rd}^1([t_1, \infty)_{\mathbb{T}}, (0, \infty))$. Then, u satisfies

$$u^\Delta(t) + \frac{A(t)\alpha(t)q(t)}{e_{-d/r}(\sigma(t), t_1)} - \Phi(t) \leq 0, \quad (26)$$

where

$$\begin{aligned} \Phi(t) &= A^\Delta(t) \left(\frac{u}{A}\right)^\sigma(t) \\ &\quad - \gamma A(t) \delta(t) \left[\left(\frac{u}{A}\right)^\sigma\right]^{(1+\gamma)/\gamma}(t), \\ \alpha(t) &= \min \left\{ \left(\frac{h(t)}{t}\right)^\gamma, 1 \right\}, \\ \delta(t) &= \int_{t_1}^t \left(\frac{e_{-d/r}(s, t_1)}{r(s)}\right)^{1/\gamma} \Delta s. \end{aligned} \quad (27)$$

Proof. Suppose that all assumptions of Lemma 6 hold. Differentiating (25) and using (1), we have

$$\begin{aligned} u^\Delta(t) &= \frac{A(t)}{z^\gamma(t)} \left(\frac{r(z^{\Delta\Delta})^\gamma}{e_{-d/r}(\cdot, t_1)} \right)^\Delta(t) + \left(\frac{A}{z^\gamma}\right)^\Delta(t) \\ &\quad \cdot \left(\frac{r(z^{\Delta\Delta})^\gamma}{e_{-d/r}(\cdot, t_1)} \right)^\sigma(t) = -\frac{A(t)}{z^\gamma(t)} \frac{f(t, x(h(t)))}{e_{-d/r}(\sigma(t), t_1)} \\ &\quad + \frac{A^\Delta(t) z^\gamma(t) - A(t) (z^\gamma)^\Delta(t)}{z^\gamma(t) z^\gamma(\sigma(t))} \left(\frac{r(z^{\Delta\Delta})^\gamma}{e_{-d/r}(\cdot, t_1)} \right)^\sigma \\ &\quad \cdot (t). \end{aligned} \quad (28)$$

If $h(t) \geq t$, then

$$\begin{aligned} f(t, x(h(t))) &\geq q(t)x^\gamma(h(t)) \geq q(t)z^\gamma(h(t)) \\ &\geq q(t)z^\gamma(t). \end{aligned} \quad (29)$$

Assume now that $h(t) \leq t$. It follows from Lemma 6 that

$$\begin{aligned} f(t, x(h(t))) &\geq q(t)z^\gamma(h(t)) \\ &\geq \left(\frac{h(t)}{t}\right)^\gamma q(t)z^\gamma(t). \end{aligned} \quad (30)$$

Hence, we have

$$\begin{aligned} u^\Delta(t) &\leq -\frac{A(t)\alpha(t)q(t)}{e_{-d/r}(\sigma(t), t_1)} + A^\Delta(t) \left(\frac{u}{A}\right)^\sigma(t) \\ &\quad - A(t) \frac{(z^\gamma)^\Delta(t)}{z^\gamma(t)} \left(\frac{u}{A}\right)^\sigma(t). \end{aligned} \quad (31)$$

Using Pötzsche chain rule (see [5, Theorem 1.90] for details), we obtain

$$(z^\gamma)^\Delta(t) \geq \gamma z^{\gamma-1}(t) z^\Delta(t), \quad (32)$$

which yields

$$\frac{(z^\gamma)^\Delta(t)}{z^\gamma(t)} \geq \frac{\gamma z^{\gamma-1}(t) z^\Delta(t)}{z^\gamma(t)} = \frac{\gamma z^\Delta(t)}{z(t)}. \quad (33)$$

From Lemma 4, we conclude that

$$\begin{aligned} z^\Delta(t) &= z^\Delta(t_1) + \int_{t_1}^t z^{\Delta\Delta}(s) \Delta s \\ &= z^\Delta(t_1) \\ &\quad + \int_{t_1}^t \left(\frac{r(s) (z^{\Delta\Delta}(s))^\gamma}{e_{-d/r}(s, t_1)} \right)^{1/\gamma} \left(\frac{e_{-d/r}(s, t_1)}{r(s)} \right)^{1/\gamma} \Delta s \\ &\geq \left(\frac{r(t) (z^{\Delta\Delta}(t))^\gamma}{e_{-d/r}(t, t_1)} \right)^{1/\gamma} \int_{t_1}^t \left(\frac{e_{-d/r}(s, t_1)}{r(s)} \right)^{1/\gamma} \Delta s \\ &= \delta(t) \left(\frac{r(t) (z^{\Delta\Delta}(t))^\gamma}{e_{-d/r}(t, t_1)} \right)^{1/\gamma}, \end{aligned} \quad (34)$$

which implies that

$$\begin{aligned} \frac{z^\Delta(t)}{z(t)} &\geq \delta(t) \left(\frac{r(t) (z^{\Delta\Delta}(t))^\gamma}{e_{-d/r}(t, t_1)} \right)^{1/\gamma} \frac{1}{z(t)} \\ &\geq \delta(t) \left[\left(\frac{r(z^{\Delta\Delta})^\gamma}{e_{-d/r}(\cdot, t_1)} \right)^{1/\gamma} \right]^\sigma(t) \frac{1}{z^\sigma(t)} \\ &= \delta(t) \left[\left(\frac{u}{A} \right)^\sigma \right]^{1/\gamma}(t). \end{aligned} \quad (35)$$

It follows now from (31) that

$$\begin{aligned} u^\Delta(t) &\leq -\frac{A(t) \alpha(t) q(t)}{e_{-d/r}(\sigma(t), t_1)} + A^\Delta(t) \left(\frac{u}{A} \right)^\sigma(t) \\ &\quad - \gamma A(t) \delta(t) \left[\left(\frac{u}{A} \right)^\sigma \right]^{(1+\gamma)/\gamma}(t). \end{aligned} \quad (36)$$

The proof is complete. $\square$

Lemma 8. Assume that x is an eventually positive solution of (1) and z^Δ is eventually negative. If

$$\int_{t_0}^\infty \frac{q(t)}{e_{-d/r}(\sigma(t), t_0)} \Delta t = \infty, \quad (37)$$

then $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof. Since z^Δ is eventually negative, z is either eventually positive or eventually negative. If z is eventually negative,

then there exist a constant $c < 0$ and a $T \in [t_0, \infty)_\mathbb{T}$ such that $z(t) < c$ for $t \in [T, \infty)_\mathbb{T}$, which causes a contradiction as in the proof of Lemma 4. Thus, z is eventually positive.

Taking into account the fact that $x > 0$, by Lemma 4, there exists a $t_1 \in [t_0, \infty)_\mathbb{T}$ such that $z^{\Delta\Delta}(t) > 0$ for $t \in [t_1, \infty)_\mathbb{T}$. We prove that $\lim_{t \rightarrow \infty} z(t) = b \geq 0$. Otherwise, there exists a $t_2 \in [t_1, \infty)_\mathbb{T}$ such that $z(t) < 0$ for $t \in [t_2, \infty)_\mathbb{T}$, and a similar contradiction can be obtained. Suppose that $b > 0$. It follows from the proof of Lemma 4 and $z(h(t)) > b$ that

$$\begin{aligned} \left[\frac{r(t) (z^{\Delta\Delta}(t))^\gamma}{e_{-d/r}(t, t_1)} \right]^\Delta &\leq -\frac{q(t) z^\gamma(h(t))}{e_{-d/r}(\sigma(t), t_1)} \\ &< -\frac{b^\gamma q(t)}{e_{-d/r}(\sigma(t), t_1)}. \end{aligned} \quad (38)$$

Integrating (38) from t_1 to t , $t \in [\sigma(t_1), \infty)_\mathbb{T}$, we have

$$\begin{aligned} \frac{r(t) (z^{\Delta\Delta}(t))^\gamma}{e_{-d/r}(t, t_1)} &\leq r(t_1) (z^{\Delta\Delta}(t_1))^\gamma \\ &\quad - b^\gamma \int_{t_1}^t \frac{q(s)}{e_{-d/r}(\sigma(s), t_1)} \Delta s \rightarrow \\ &\quad -\infty, \quad t \rightarrow \infty, \end{aligned} \quad (39)$$

which contradicts the fact that $z^{\Delta\Delta}(t) > 0$. Hence, $b = 0$ and so $\lim_{t \rightarrow \infty} x(t) = 0$ when using Lemma 3. This completes the proof. $\square$

3. Main Results

Let $D = \{(t, s) \in \mathbb{T}^2 : t \geq s \geq t_0\}$. Define

$$\begin{aligned} \mathcal{H} &= \{H(t, s) \in C_{rd}(D, [0, \infty)) : H(t, t) \\ &= 0, H(t, s) > 0, H_2^\Delta(t, s) \leq 0, t > s \geq t_0\}, \end{aligned} \quad (40)$$

where H_2^Δ is the Δ -partial derivative of H with respect to s .

Theorem 9. Assume that (15) holds and there exist two functions $A \in C_{rd}^1([t_0, \infty)_\mathbb{T}, (0, \infty))$ and $H \in \mathcal{H}$ such that, for all sufficiently large $t_1 \in [t_0, \infty)_\mathbb{T}$ and for some $t_2 \in [t_1, \infty)_\mathbb{T}$,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_2)} \int_{t_2}^t \left[\frac{H(t, s) A(s) \alpha(s) q(s)}{e_{-d/r}(\sigma(s), t_1)} \right. \\ \left. - \frac{1}{(1+\gamma)^{1+\gamma}} \right. \\ \left. \cdot \frac{(H_2^\Delta(t, s) A^\sigma(s) + H(t, s) A^\Delta(s))^{1+\gamma}}{(H(t, s) A(s) \delta(s))^\gamma} \right] \Delta s \\ = \infty, \end{aligned} \quad (41)$$

where α and δ are as in Lemma 7. Then every solution x of (1) is oscillatory or $\lim_{t \rightarrow \infty} x(t)$ exists (finite).

Proof. Suppose that (1) has a nonoscillatory solution x . Without loss of generality, let x be eventually positive. From Lemma 5, it follows that z is eventually positive or $\lim_{t \rightarrow \infty} x(t) = 0$. Assume that z is eventually positive. By Lemma 4, there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that either $z^\Delta(t) > 0$ or $z^\Delta(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Let $z^\Delta(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Define u by (25). Then, by Lemma 7, (26) holds. It follows from (26) that, for some $t_2 \in [t_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned} \int_{t_2}^t \frac{H(t, s) A(s) \alpha(s) q(s)}{e_{-d/r}(\sigma(s), t_1)} \Delta s &\leq - \int_{t_2}^t H(t, s) u^\Delta(s) \Delta s \\ &+ \int_{t_2}^t H(t, s) \Phi(s) \Delta s = H(t, t_2) u(t_2) \\ &+ \int_{t_2}^t \left(H_2^\Delta(t, s) u^\sigma(s) + H(t, s) \Phi(s) \right) \Delta s \\ &= H(t, t_2) u(t_2) + \int_{t_2}^t \left(H_2^\Delta(t, s) A^\sigma(s) \left(\frac{u}{A} \right)^\sigma(s) \right. \\ &\quad \left. + H(t, s) \Phi(s) \right) \Delta s, \end{aligned} \quad (42)$$

where

$$\begin{aligned} H_2^\Delta(t, s) A^\sigma(s) \left(\frac{u}{A} \right)^\sigma(s) &+ H(t, s) \Phi(s) \\ &= \left(H_2^\Delta(t, s) A^\sigma(s) + H(t, s) A^\Delta(s) \right) \left(\frac{u}{A} \right)^\sigma(s) \\ &\quad - \gamma H(t, s) A(s) \delta(s) \left[\left(\frac{u}{A} \right)^\sigma \right]^{(1+\gamma)/\gamma}(s). \end{aligned} \quad (43)$$

Let $\lambda = (1 + \gamma)/\gamma$,

$$\begin{aligned} a^\lambda &= \gamma H(t, s) A(s) \delta(s) \left[\left(\frac{u}{A} \right)^\sigma \right]^{(1+\gamma)/\gamma}(s), \\ b^{\lambda-1} &= \frac{\gamma}{1 + \gamma} \frac{H_2^\Delta(t, s) A^\sigma(s) + H(t, s) A^\Delta(s)}{(\gamma H(t, s) A(s) \delta(s))^{\gamma/(1+\gamma)}}. \end{aligned} \quad (44)$$

Using the inequality (a variation of the well-known Young inequality)

$$\lambda a b^{\lambda-1} - a^\lambda \leq (\lambda - 1) b^\lambda, \quad a, b \geq 0, \quad (45)$$

we deduce that

$$\begin{aligned} H_2^\Delta(t, s) A^\sigma(s) \left(\frac{u}{A} \right)^\sigma(s) &+ H(t, s) \Phi(s) \\ &\leq \frac{1}{\gamma} \left[\frac{\gamma}{1 + \gamma} \frac{H_2^\Delta(t, s) A^\sigma(s) + H(t, s) A^\Delta(s)}{(\gamma H(t, s) A(s) \delta(s))^{\gamma/(1+\gamma)}} \right]^{1+\gamma} \\ &= \frac{1}{(1 + \gamma)^{1+\gamma}} \frac{\left(H_2^\Delta(t, s) A^\sigma(s) + H(t, s) A^\Delta(s) \right)^{1+\gamma}}{(H(t, s) A(s) \delta(s))^\gamma}. \end{aligned} \quad (46)$$

Therefore, we obtain

$$\begin{aligned} \int_{t_2}^t \frac{H(t, s) A(s) \alpha(s) q(s)}{e_{-d/r}(\sigma(s), t_1)} \Delta s &\leq H(t, t_2) u(t_2) \\ &+ \frac{1}{(1 + \gamma)^{1+\gamma}} \end{aligned} \quad (47)$$

$$\cdot \int_{t_2}^t \frac{\left(H_2^\Delta(t, s) A^\sigma(s) + H(t, s) A^\Delta(s) \right)^{1+\gamma}}{(H(t, s) A(s) \delta(s))^\gamma} \Delta s,$$

which implies that

$$\begin{aligned} \frac{1}{H(t, t_2)} \int_{t_2}^t \left[\frac{H(t, s) A(s) \alpha(s) q(s)}{e_{-d/r}(\sigma(s), t_1)} - \frac{1}{(1 + \gamma)^{1+\gamma}} \right. \\ \left. \cdot \frac{\left(H_2^\Delta(t, s) A^\sigma(s) + H(t, s) A^\Delta(s) \right)^{1+\gamma}}{(H(t, s) A(s) \delta(s))^\gamma} \right] \Delta s \\ \leq u(t_2). \end{aligned} \quad (48)$$

This contradicts (41). Thus, $z^\Delta(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$, and so $\lim_{t \rightarrow \infty} z(t)$ exists. By Lemma 3, $\lim_{t \rightarrow \infty} x(t)$ exists. The proof is complete. $\square$

From Lemma 8 and Theorem 9, we have the following corollary.

Corollary 10. Assume that (37) is satisfied and there exist two functions $A \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ and $H \in \mathcal{H}$ such that, for all sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$ and for some $t_2 \in [t_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_2)} \int_{t_2}^t \left[\frac{H(t, s) A(s) \alpha(s) q(s)}{e_{-d/r}(\sigma(s), t_1)} \right. \\ \left. - \frac{1}{(1 + \gamma)^{1+\gamma}} \right. \\ \left. \cdot \frac{\left(H_2^\Delta(t, s) A^\sigma(s) + H(t, s) A^\Delta(s) \right)^{1+\gamma}}{(H(t, s) A(s) \delta(s))^\gamma} \right] \Delta s \\ = \infty, \end{aligned} \quad (49)$$

where α and δ are as in Lemma 7. Then (1) is almost oscillatory.

Theorem 11. Assume that (15) holds and there exists a function $A \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ such that, for all sufficiently large $t_1 \in [t_0, \infty)_{\mathbb{T}}$ and for some $t_2 \in [t_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \int_{t_2}^t \left[\frac{A(s) \alpha(s) q(s)}{e_{-d/r}(\sigma(s), t_1)} \right. \\ \left. - \frac{1}{(1 + \gamma)^{1+\gamma}} \frac{\left(A^\Delta(s) \right)^{1+\gamma}}{(A(s) \delta(s))^\gamma} \right] \Delta s = \infty, \end{aligned} \quad (50)$$

where α and δ are as in Lemma 7. Then conclusion of Theorem 9 remains intact.

Proof. Proceeding as in the proof of Theorem 9, assume that $z^\Delta(t) > 0$ for $t \in [t_1, \infty)_\mathbb{T}$. Let u be defined by (25). By virtue of Lemma 7, we arrive at (36). Let $\lambda = (1 + \gamma)/\gamma$,

$$\begin{aligned} a^\lambda &= \gamma A(t) \delta(t) \left[\left(\frac{u}{A} \right)^\sigma \right]^{(1+\gamma)/\gamma} (t), \\ b^{\lambda-1} &= \frac{\gamma}{1+\gamma} \frac{A^\Delta(t)}{(\gamma A(t) \delta(t))^{\gamma/(1+\gamma)}}. \end{aligned} \quad (51)$$

Using (45), we conclude that

$$u^\Delta(t) \leq -\frac{A(t) \alpha(t) q(t)}{e_{-d/r}(\sigma(t), t_1)} + \frac{1}{(1+\gamma)^{1+\gamma}} \frac{(A^\Delta(t))^{1+\gamma}}{(A(t) \delta(t))^\gamma}, \quad (52)$$

which yields

$$\frac{A(t) \alpha(t) q(t)}{e_{-d/r}(\sigma(t), t_1)} - \frac{1}{(1+\gamma)^{1+\gamma}} \frac{(A^\Delta(t))^{1+\gamma}}{(A(t) \delta(t))^\gamma} \leq -u^\Delta(t). \quad (53)$$

Let $t_2 \in [t_1, \infty)_\mathbb{T}$. It follows from (53) that

$$\begin{aligned} \int_{t_2}^t \left[\frac{A(s) \alpha(s) q(s)}{e_{-d/r}(\sigma(s), t_1)} - \frac{1}{(1+\gamma)^{1+\gamma}} \frac{(A^\Delta(s))^{1+\gamma}}{(A(s) \delta(s))^\gamma} \right] \Delta s \\ \leq u(t_2), \end{aligned} \quad (54)$$

which contradicts (50). Therefore, $z^\Delta(t) < 0$ for $t \in [t_1, \infty)_\mathbb{T}$. Along the same lines as in Theorem 9, we complete the proof. $\square$

If (37) holds, then we have the following corollary on the basis of Lemma 8 and Theorem 11.

Corollary 12. Assume that (37) is satisfied and there exists a function $A \in C_{rd}^1([t_0, \infty)_\mathbb{T}, (0, \infty))$ such that, for all sufficiently large $t_1 \in [t_0, \infty)_\mathbb{T}$ and for some $t_2 \in [t_1, \infty)_\mathbb{T}$,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \int_{t_2}^t \left[\frac{A(s) \alpha(s) q(s)}{e_{-d/r}(\sigma(s), t_1)} \right. \\ \left. - \frac{1}{(1+\gamma)^{1+\gamma}} \frac{(A^\Delta(s))^{1+\gamma}}{(A(s) \delta(s))^\gamma} \right] \Delta s = \infty, \end{aligned} \quad (55)$$

where α and δ are as in Lemma 7. Then conclusion of Corollary 10 remains intact.

Remark 13. If $d \geq 0$, then it is not difficult to see that

$$0 < e_{-d/r}(\sigma(t), t_1) \leq 1, \quad (56)$$

and so (41) and (50) can be simplified to

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{H(t, t_2)} \int_{t_2}^t \left[H(t, s) A(s) \alpha(s) q(s) \right. \\ \left. - \frac{1}{(1+\gamma)^{1+\gamma}} \right. \\ \left. \cdot \frac{(H_2^\Delta(t, s) A^\sigma(s) + H(t, s) A^\Delta(s))^{1+\gamma}}{(H(t, s) A(s) \delta(s))^\gamma} \right] \Delta s \\ = \infty, \end{aligned} \quad (57)$$

$$\begin{aligned} \limsup_{t \rightarrow \infty} \int_{t_2}^t \left[A(s) \alpha(s) q(s) \right. \\ \left. - \frac{1}{(1+\gamma)^{1+\gamma}} \frac{(A^\Delta(s))^{1+\gamma}}{(A(s) \delta(s))^\gamma} \right] \Delta s = \infty, \end{aligned} \quad (58)$$

respectively.

Remark 14. If $h(t) \geq t$, then $\alpha(t) = 1$ and we do not impose restrictive condition (15) in our theorems and corollaries.

Remark 15. Our results complement and improve those obtained by Han et al. [14] since we do not impose specific restrictions on r^Δ .

4. Examples

The following examples are presented to show applications of the main results.

Example 1. Consider the third-order equation

$$\left(\frac{1}{t} \varphi_\gamma(z^{\Delta\Delta}(t)) \right)^\Delta + \frac{\lambda}{t^2} \varphi_\gamma(z^{\Delta\Delta}(t)) + \frac{1}{t^2} x^\gamma \left(\frac{t}{2} \right) = 0, \quad (59)$$

where $t \in [1, \infty)_\mathbb{R}$, $z(t) = x(t) - (t-1)x(t-1)/(2t)$, $\gamma \geq 1$, and $0 \leq \lambda < 2$. It is clear that $r(t) = 1/t$, $p(t) = (t-1)/(2t)$, $d(t) = \lambda/t^2$, $g(t) = t-1$, and $h(t) = t/2 < t$. Then, $p_0 = 1/2$ and $\alpha(t) = 1/2^\gamma$. Let $q(t) = 1/t^2$. Since $-d(t)/r(t) = -\lambda/t$, we have

$$e_{-d/r}(t, t_0) = \frac{1}{t^\lambda},$$

$$e_{-d/r}(t, t_1) = \left(\frac{t_1}{t} \right)^\lambda,$$

$$\int_{t_0}^\infty \left(\frac{e_{-d/r}(t, t_0)}{r(t)} \right)^{1/\gamma} \Delta t = \int_1^\infty \frac{1}{t^{(\lambda-1)/\gamma}} dt = \infty,$$

$$\int_{t_0}^\infty \frac{q(t) h^\gamma(t)}{e_{-d/r}(\sigma(t), t_0)} \Delta t = \frac{1}{2^\gamma} \int_1^\infty t^{\gamma+\lambda-2} dt = \infty,$$

$$\begin{aligned}\delta(t) &= \int_{t_1}^t \left(\frac{e_{-d/r}(s, t_1)}{r(s)} \right)^{1/\gamma} \Delta s = t_1^{\lambda/\gamma} \int_{t_1}^t \frac{ds}{s^{(\lambda-1)/\gamma}} \\ &= O\left(t^{(\gamma-\lambda+1)/\gamma}\right).\end{aligned}\quad (60)$$

Hence, assumptions (C_1) – (C_5) and (15) hold. Let $H(t, s) = (t-s)^2$ and $A(t) = t^2$. If

$$3\gamma - 4 - \lambda + \frac{\lambda - 1}{\gamma} < 0, \quad (61)$$

we obtain

$$\begin{aligned}&\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_2)} \int_{t_2}^t \left[\frac{H(t, s) A(s) \alpha(s) q(s)}{e_{-d/r}(\sigma(s), t_1)} \right. \\ &\quad \left. - \frac{1}{(1+\gamma)^{1+\gamma}} \cdot \frac{(H_2^\Delta(t, s) A^\sigma(s) + H(t, s) A^\Delta(s))^{1+\gamma}}{(H(t, s) A(s) \delta(s))^\gamma} \right] \Delta s \\ &= \limsup_{t \rightarrow \infty} \frac{1}{(t-t_2)^2} \int_{t_2}^t \left[\frac{(t-s)^2 s^\lambda}{2^\gamma t_1^\lambda} \right. \\ &\quad \left. - \frac{O(s^{3\gamma-2+(\lambda-1)/\gamma})}{(1+\gamma)^{1+\gamma}} \right] ds = \infty.\end{aligned}\quad (62)$$

That is, (41) is satisfied. By virtue of Theorem 9, we deduce that every solution of (59) either is oscillatory or converges to a finite number asymptotically. Furthermore, if (61) holds and $1 \leq \lambda < 2$, then

$$\int_{t_0}^{\infty} \frac{q(t)}{e_{-d/r}(\sigma(t), t_0)} \Delta t = \int_1^{\infty} t^{\lambda-2} dt = \infty, \quad (63)$$

which implies that (59) is almost oscillatory by using Corollary 10.

Example 2. Consider the third-order equation

$$\begin{aligned}(t\varphi_\gamma(z^{\Delta\Delta}(t)))^\Delta - \varphi_\gamma(z^{\Delta\Delta}(t)) + (t^\beta + t^\lambda)x^\gamma(t+1) \\ = 0,\end{aligned}\quad (64)$$

where $t \in [2, \infty)_{\mathbb{Z}}$, $z(t) = x(t) - x(t-2)/t$, $\beta, \gamma, \lambda \in \mathbb{R}$, and $\gamma \geq 1$. It is easy to see that $r(t) = t$, $p(t) = -1/t$, $d(t) = -1$,

$g(t) = t-2$, and $h(t) = t+1 > t$. Then, $p_0 = 0$ and $\alpha(t) = 1$. Since $-d(t)/r(t) = 1/t$, we get

$$\begin{aligned}e_{-d/r}(t, t_0) &= \frac{t}{2}, \\ e_{-d/r}(t, t_1) &= \frac{t}{t_1}, \\ \int_{t_0}^{\infty} \left(\frac{e_{-d/r}(t, t_0)}{r(t)} \right)^{1/\gamma} \Delta t &= \frac{1}{2^{1/\gamma}} \int_2^{\infty} \Delta t = \infty, \\ \delta(t) &= \int_{t_1}^t \left(\frac{e_{-d/r}(s, t_1)}{r(s)} \right)^{1/\gamma} \Delta s = \frac{1}{t_1^{1/\gamma}} \int_{t_1}^t \Delta s \\ &= \frac{t-t_1}{t_1^{1/\gamma}}.\end{aligned}\quad (65)$$

Obviously, conditions (C_1) – (C_5) are satisfied. Let $q(t) = t^\beta$ or t^λ and $A(t) = 1$. If $\beta \geq 0$ or $\lambda \geq 0$, then

$$\begin{aligned}\int_{t_0}^{\infty} \frac{q(t)}{e_{-d/r}(\sigma(t), t_0)} \Delta t &\geq \int_2^{\infty} \frac{2}{t+1} \Delta t = \infty, \\ \limsup_{t \rightarrow \infty} \int_{t_2}^t \left[\frac{A(s) \alpha(s) q(s)}{e_{-d/r}(\sigma(s), t_1)} \right. \\ &\quad \left. - \frac{1}{(1+\gamma)^{1+\gamma}} \frac{(A^\Delta(s))^{1+\gamma}}{(A(s) \delta(s))^\gamma} \right] \Delta s \geq t_1 \\ &\cdot \limsup_{t \rightarrow \infty} \int_{t_2}^t \frac{1}{s+1} \Delta s = \infty.\end{aligned}\quad (66)$$

That is, both (37) and (55) hold. By Corollary 12, we conclude that (64) is almost oscillatory for $\beta \geq 0$ or $\lambda \geq 0$.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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